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ESTIMATES OF QUASICONFORMAL MAPPINGS NEAR THE BOUNDARY

BO-HYUN CHUNG AND SANG WOOK KIM

ABSTRACT. In [2], D. Gaier has given an estimate of conformal mappings near the boundary. In this paper, we generalize for the K-quasiconformal mapping the corresponding result.

1. Introduction and Results

Let G be a finite, simply connected domain with $0 \in G$ and $1 \in \partial G$. Let f be the conformal map of G onto the disc $D_w = \{w : |w| < 1\}$ with $f(0) = 0$, $f(1) = 1$.

DEFINITION 1.1 ([2]). We say that $z \in G$ is *visible* from a finite $z_0 \in \partial G$, if the $\ell = (z, z_0)$ connecting z to z_0 is contained in G.

DEFINITION 1.2 ([2]). We say that G is *starshaped* with respect to $z_0 \in \partial G$, if every $z \in G$ is visible from z_0.

Using those definitions, D. Gaier ([2]) established the following theorems for conformal mappings.

THEOREM 1.3 (D. Gaier, [2]). *Assume that $z \in G$ is visible from $z = 1$ and that the function f mapping G onto D_w is normalized by $f(0) = 0$ and $f(t) \to 1$ as $t \to 1$ on $\ell = (z, 1)$. Then, with $w = f(z)$, we have*

$$|w - 1| < 4\sqrt{|z - 1|}\,. \tag{1}$$

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COROLLARY 1.4 ([2]). *Let z, f and w be as in Theorem 1.3. If, in addition, $G \subset H$, where H is a half plane with $1 \in \partial H$, then we have*

$$|w-1| < 4|z-1|\,. \tag{2}$$

In this note we extend Gaier's theorems to the K-quasiconformal mappings. As is well known, a K-quasiconformal mapping of a plane domain can be defined in three apparently different but in fact equivalent ways.

DEFINITION 1.5 (Analytic definition, [6]). A homeomorphism $f(x)$ of a domain G is K-quasiconformal mapping if and only if it is absolutely continuous on lines, a.e. differentiable and

$$\frac{1}{K}\max|f'(x)| \le |J(x)| \le K\min|f'(x)|$$

a.e. in G, where $J(x)$ is the Jacobian of $f(x)$.

THEOREM 1.6. *Assume that $z_0 \in G$ can be connected to $z = 1$ by a Jordan are γ of length L lying in G except for its endpoint $z = 1$. Let f be a K-quasiconformal mapping of G onto the disc $D_w = \{w : |w| < 1\}$ with $f(0) = 0$ and $f(z) \to 1$ as $z \to 1$ on γ. Then we have*

$$|f(z_0)-1| < 4^{2-1/K}\cdot L^{1/(2K)}\,. \tag{3}$$

COROLLARY 1.7. *Let z_0, γ, L and f be as in Theorem 1.6. If, in addition, $G \subset H$, where H is a half plane with $1 \in \partial H$, then we have*

$$|f(z_0)-1| < 4^{2-1/K}\cdot L^{1/K}\,. \tag{4}$$

2. Capacity and an estimate of harmonic measure

Consider a positive mass distribution μ on the compact set E, i.e., a measure that vanishes on the complement of E. We define

$$p_N(z) = \int \min\left(N,\ \log\frac{1}{|z-\zeta|}\right)\, d\mu(\zeta)$$

and $p(z) = \lim_{N\to\infty} p_N(z)$. This is the *logarithmic potential* of μ. We set $V_\mu = \sup\limits_z p(z)$. It may be infinite.

DEFINITION 2.1 ([1]). If $\min V_\mu = V$, we call e^{-V} the *capacity* of E. It is denoted by $Cap(E)$.

The capacity is invariant under normalized conformal mappings. The double role of capacity as a conformal invariant and a geometric quantity permits us to gain relevant information about conformal mappings. Assume that the capacity of γ is $Cap(\gamma) < \frac{1}{4}$.

The proofs of our theorems depend on some estimates of the harmonic measure of arcs. This is a well known tool in the study of conformal mappings near the boundary.

Let G be a domain in the complex plane whose boundary ∂G consists of a finite number of disjoint Jordan curves. Suppose that the boundary ∂G is divided into two parts E and E', each consisting of a finite number of arcs and closed curves. There exists a unique bounded harmonic function $m(z)$ in G such that $m(z) \to 1$ when z tends to an interior point of E and $m(z) \to 0$ when z tends to an interior point of E'. The values of m lie strictly between 0 and 1.

DEFINITION 2.2 ([1]). The number $m(z)$ is called the *harmonic measure* of E at the point z with respect to the domain G. It is denoted by $m(z, E, G)$.

The following theorem are known result.

THEOREM 2.3 (Theorem of Nevanlinna, [4]). *Let F be a simply connected subdomain of $D_z = \{z : |z| < 1\}$ with $0 \notin F$. Let $\Gamma = \partial F \cap D_z$. Then*

$$m(z_0, \Gamma, F) \geq \frac{2}{\pi} \sin^{-1}\left(\frac{1 - |z_0|}{1 + |z_0|}\right)$$

for every $z_0 \in F$.

3. Proof of the theorem

For our proof we will need the following lemmas.

LEMMA 3.1. *The estimates of the harmonic measure of* γ *at the point* $z = 0$ *with respect to* $G \backslash \gamma$*:*

$$m(0, \gamma, G\backslash\gamma) \leq \frac{2}{\pi} \sin^{-1} 2\sqrt{Cap(\gamma)}\,. \tag{5}$$

Proof. Let $z = \varphi(w)$ map D_w conformally onto the complement of γ such that $\varphi(0) = \infty$ and $\varphi(1) = 1$. Since the diameter of γ is $\leq 4Cap(\gamma) < 1$, the point $z = 0$ is not on γ and therefore $w_0 = \varphi^{-1}(0) \in D_w$. The expansion of φ at $w = 0$ is of the form

$$\varphi(w) = \frac{a}{w} + a_0 + a_1 w + \cdots$$

with $|a| = Cap(\gamma)$. Since $\varphi(w) \neq 1$ for $w \in D_w$, we see that

$$F(w) = \frac{a}{\varphi(w) - 1}\,.$$

Then by the elementary distortion theorem (See Hayman [3]), we obtain

$$|F(w)| \geq \frac{|w|}{(1 + |w|)^2} \qquad i.e., \qquad \frac{Cap(\gamma)}{|\varphi(w) - 1|} \geq \frac{|w|}{(1 + |w|)^2}\,.$$

Put $w = w_0$. Then we have

$$\sqrt{|w_0|} + \frac{1}{\sqrt{|w_0|}} \geq \frac{1}{\sqrt{Cap(\gamma)}}\,. \tag{6}$$

Now, if $g = \varphi^{-1}(G\backslash\gamma)$ is the inverse image of $G\backslash\gamma$ not containing $w = 0$, and if $D'_w = D_w \backslash \{w : 0 \leq w < 1\}$ is the slit unit disc, then

$$m(w_0, \partial D_w, g) \leq m(-|w_0|, \partial D_w, D'_w)\,.$$

The theorem of Nevanlinna(Theorem 2.3) tells us that

$$m(w_0, \partial D_w, g) = m(0, \gamma, G\backslash\gamma)\,, \tag{7}$$

$$m(-|w_0|, \partial D_w, D'_w) = \frac{2}{\pi} \sin^{-1} \frac{2\sqrt{|w_0|}}{|w_0|+1}\,. \tag{8}$$

Hence by (6), (7) and (8), we obtain (5). □

LEMMA 3.2. *Let $L(< 1)$ be the length of γ, then*

$$m(0, \gamma, G\backslash\gamma) \leq \frac{2}{\pi} \sin^{-1} \sqrt{L}\,. \tag{9}$$

Proof. $Cap(\gamma)$ and L satisfy $L \geq 4Cap(\gamma)$ (see pemmerenke [5]). Hence by Lemma 3.1, we obtain (9). □

PROOF OF THOEREM 1.6. We may assume $L < 1$. Put $m_z = m(0, \gamma, G\backslash\gamma)$ and $m_w = m(0, f(\gamma), D_w\backslash f(\gamma))$. Since f is a K-quasiconformal mapping of G onto D_w the harmonic measures satisfy the well-known relation

$$\sin\left(\frac{\pi m_w}{2}\right) \leq 4^{1-1/K} \cdot \left[\sin\left(\frac{\pi m_z}{2}\right)\right]^{1/K}\,. \tag{10}$$

By virtue of (9), we have

$$\sin\left(\frac{\pi m_w}{2}\right) \leq \sqrt{L}\,. \tag{11}$$

And by Theorem 9 in [2], we obtain

$$\pi m_w > \sin^{-1}\left(\frac{|f(z_0 - 1|}{2}\right)\,.$$

But for $0 \leq x \leq \pi$, $\sin x \leq 2\sin\frac{x}{2}$ and we have

$$\sin\left(\frac{\pi m_w}{2}\right) > \frac{1}{4}|f(z_0) - 1|\,. \tag{12}$$

Hence by (10), (11) and (12), we obtain (3). □

PROOF OF COROLLARY 1.7. Let H be a half plane with $1 \in \partial H$. If $G \subset H$, the mapping $\xi = 1 - (1-z)^2$ carries G onto G_ξ so that 0, 1, z_0, γ correspond to 0, 1, ξ_0, γ_ξ, respectively. Now the length of γ is $\leq L^2$.

For if γ is represented in terms of arc length, $z = z(s)$ $(0 \le s \le L)$, the length of γ_ξ is

$$\frac{1}{2}\int |d\xi| = \int_0^L |1 - z(s)| \cdot |z'(s)|\, ds$$

where $|z'(s)| = 1$ almost everywhere and $|1 - z(s)| \le s$. We apply Theorem 1.6 to the mapping from the ξ-plane onto D_w. Hence we obtain (4). □

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Mathematics Section, College of Science and Technology
Hongik University
Chochiwon 339-701, Korea
E-mail: bohyun@wow.hongik.ac.kr

Department of Mathematics
Chungbuk National University
Cheongju 361–763, Korea
E-mail: swkim@cbucc.chungbuk.ac.kr