JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **13**, No.1, June 2000

THE PAN-DUAL GENERALIZED FUZZY INTEGRAL OF A COMMUTATIVE ISOTONIC SEMIGROUP-VALUED FUNCTION

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ABSTRACT. In this paper, we will introduce the pan-dual generalized fuzzy integral of a commutative isotonic semigroup-valued functions, which is generalized of the (DG) fuzzy integral and investigate the fundamental properties of this kind of fuzzy integral.

1. Introduction

In 1980, D. A. Ralescu and G. Adams generalized the concept of fuzzy integral due to M. Sugeno[4]. For convenience, we will call it (S) fuzzy integral. Following that D. A. Ralescu and G. Adams[1], and D. A. Ralescu[2] have investigated the basic properties of (S) fuzzy integral. Wang Zhenyuen obtained a series of (S) fuzzy integral convergent theorems in [5]. Meanwhile, Zhao Ruhuai introduced a new definition of fuzzy integral, viz. (N) fuzzy integral in [8]. Wu Congxin, Wang Shuli, and Ma Ming [6] introduced the (G) fuzzy integral using a generalized triangular norm which is a generalization of both (S) fuzzy integral and (N) fuzzy integral. In this paper, we introduce the pan-dual generalized fuzzy integral of a commutative isotonic semigroup-valued function, which is generalization of the (DG) fuzzy integral [3] , show some equivalent conditions of (PDG) fuzzy integral and investigate the fundamental properties of this kind of fuzzy integral.

Received by the editors on May 2, 2000.
1991 *Mathematics Subject Classifications* : Primary 28B05.
Key words and phrases: fuzzy measure, commutative isotonic semigroup, pan-dual generalized fuzzy integral.

2. Perliminaries

Definition 2.1. Let X be a nonempty set, $\mathcal{A}$ be a σ-algebra of a class of the subsets of X , the mapping $\mu : \mathcal{A} \to [0, \infty]$ is called a fuzzy measure provided

(1) $\mu(\emptyset) = 0$;
(2) if $A \subset B$, then $\mu(A) \leq \mu(B)$;
(3) if $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots, A_n \in \mathcal{A}$, then $\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty}(A_n)$;
(4) $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, A_n \in \mathcal{A}$ and there exists a natural number n_0 such that $\mu(A_{n_0}) < \infty$, then $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

If μ is a fuzzy measure, $(X, \mathcal{A}, \mu)$ is called a fuzzy measure space.

Definition 2.2. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, $f : X \to [0, \infty]$ is said to be $\mathcal{A}$-measurable function if $N_\alpha(f) \in \mathcal{A}$ for all $\alpha \in (-\infty, \infty)$, where $N_\alpha(f) = \{x : f(x) > \alpha\}$.

Definition 2.3. Let $\oplus$ be a binary operation on $\bar{R}_+$. The pair $(\bar{R}_+, \oplus)$ is called a commutative isotonic semigroup and $\oplus$ called pan-additive on $\bar{R}_+$ iff $\oplus$ satisfies the following requirements:

(PA1) $a \oplus b = b \oplus a$;
(PA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(PA3) $a \leq b$, then $a \oplus c \leq b \oplus c$ for any c;
(PA4) $a \oplus 0 = a$;
(PA5) if $\lim_n a_n$ and $\lim_n b_n$ exit, then $\lim_n(a_n \oplus b_n)$ exists, and $\lim_n(a_n \oplus b_n) = \lim_n a_n \oplus \lim_n b_n$.

Definition 2.4. Let $\odot$ be a binary operation on $\bar{R}_+$. The triple $(\bar{R}_+, \oplus, \odot)$, where $\oplus$ is a pan-addition on $\bar{R}_+$, is called a commutative isotonic semiring with respect to $\oplus$ and $\odot$, iff:

(PM1) $a \odot b = b \odot a$;
(PM2) $(a \odot b) \odot c = a \odot (b \odot c)$;

(PM3) $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$;
(PM4) if $a \leq b$, then $(a \odot c) \leq (b \odot c)$ for any c;
(PM5) $a \neq 0$ and $b \neq 0 \iff a \odot b \neq 0$;
(PM6) there exists $e \in \bar{R}_+$ such that $e \odot a = a$ for any $a \in \bar{R}_+$;
(PM7) if $\lim_n a_n$ and $\lim_n b_n$ exist and are finite, then $\lim_n (a_n \odot b_n) = \lim_n a_n \odot \lim_n b_n$.

The operation $\odot$ is called a pan-multiplication on $\bar{R}_+$, and the number e is called the unit element of $(\bar{R}_+, \oplus, \odot)$.

Note 2.1 $\bar{R}_+$ with the common addition and the common multiplication of real numbers is a commutative isotonic semiring.

Note 2.2 $\bar{R}_+$ with the logical addition and the logical multiplication of real numbers is a commutative isotonic semiring. If $(X, \mathcal{A}, \mu)$

is a fuzzy measure space and $(\bar{R}_+, \oplus, \odot)$ is a commutative semiring, $(X, \mathcal{A}, \mu, \bar{R}_+, \oplus, \odot)$ is called a pan-space and if $E \subset X$,

$$\chi_E = \begin{cases} e, & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases}$$

is called the pan-characteristic function of E, where e is the unit element of $(\bar{R}_+, \oplus, \odot)$.

Definition 2.5. Let $(X, \mathcal{A}, \mu, \bar{R}_+, \oplus, \odot)$ be a pan-space. A function on X given by $s(x) = \oplus_{i=1}^{n} [a_i \odot \chi_{E_i}(x)]$ is called a pan-simple measurable function, where $a_i \in \bar{R}_+$, $i = 1, 2, \cdots, n$ and $\{E_i : i = 1, 2, \cdots, n\}$ is a measurable partition of X.

3. Definition and fundamental properties of (PDG) fuzzy integral

Definition 3.1. Denote $D = [0, \infty] \times [0, \infty]$, the mapping $T : D \to [0, \infty]$ is called a c-generalized triangular conorm provided

(1) $T[0,x]=x, \forall x \in [0,\infty]$ and there exists an $e \in [0,\infty]$ such that

$$T[x,e]=x, \forall x \in [0,\infty],$$

and e is called the unit element of T ;

(2) $T[x,y]=T[y,x], \forall (x,y) \in D$;

(3) $T[x_1,y_1] \leq T[x_2,y_2]$ whenever $x_1 \leq x_2, y_1 \leq y_2$;

(4) if $\{(x_n,y_n)\} \subset D$, $(x,y) \in D$, and $x_n \to x$, $y_n \to y$, then $T[x_n,y_n] \to T[x,y]$.

Note 3.1 From (1) and (3), $T[x,\infty]=\infty$ for any $x \in [0,\infty]$.

Note 3.2 Take $T_1[x,y]=max[x,y]$, $T_2[x,y]=x+y$, $T_3[x,y]=x \oplus y$, and

$$T_4[x,y]=\begin{cases}\infty, & \max\{x,y\}=\infty\\ x+y+k(xy)^p, & \max\{x,y\}<\infty\,(k>0,p>0),\end{cases}$$

then T_1, T_2, T_3, and T_4 are c-generalized triangular conorms.

Definition 3.2. Let $(X,\mathcal{A},\mu,\bar{R}_+,\oplus,\odot)$ be a pan-space, and let T be a c-generalized triangular conorm, and f be a nonnegative measurable function, $A \in \mathcal{A}$. (PDG) fuzzy integral of f on A is defined by

$$(PDG)\int_A f d\mu = \inf_{f \leq s} Q_A(s),$$

where $s=\bigoplus_{i=1}^{n}[\alpha_i \odot \chi_{A_i}]$, $\alpha_i \neq \alpha_j\,(i \neq j)$, $\alpha_i>0$, $A_i \in \mathcal{A}\ (i=1,2,\cdots,n)$, $A_i \cap A_j=\emptyset\,(i \neq j)$, $\cup_{i=1}^{n}A_i=X$, $A_i^c=X-A_i$ and χ_{A_i} denotes the characteristic function of A_i, and

$$Q_A(s)=\bigwedge_{i=1}^{n} T[\alpha_i,\mu(A \cap A_i^c)].$$

THEOREM 3.3. *For (PDG) fuzzy integrals we have the following equivalent forms:*

$$(PDG)\int_A f d\mu = \inf_{\alpha\geq 0} T[\alpha, \mu(A\cap N_\alpha^*(f))]$$
$$= \inf_{\alpha\geq 0} T[\alpha, \mu(A\cap N_\alpha(f))]$$
$$= \inf_{E\in\mathcal{A}} T[\sup_{x\in E} f(x), \mu(A\cap E^c)].$$

where $N_\alpha^*(f) = \{x : f(x) \geq \alpha\}$.

Proof. The above four expressions are denoted by (1),(2),(3),and(4) in proper order. Then we infer (1)≤(4): For any $E\in\mathcal{A}$, it is clear that

$$(\sup_{x\in E} f(x)) \odot \chi_E + \infty \odot \chi_{E^c} \geq f$$

from Definition 3.2, we know

$$(PDG)\int_A f d\mu \leq T[\sup_{x\in E} f(x), \mu(A\cap E^c)]$$

hence

$$(PDG)\int_A f d\mu \leq \inf_{E\in\mathcal{A}} T[\sup_{x\in E} f(x), \mu(A\cap E^c)].$$

(4)≤(3): By $\{x : f(x) \leq \alpha\}\in\mathcal{A}$ for any $\alpha\geq 0$, and

$$\sup_{x\in\{x:f(x)\leq\alpha\}} f(x) \leq \alpha,$$

we have

$$T[\alpha, \mu(A\cap N_\alpha(f))] \geq T[\sup_{x\in\{x:f(x)\leq\alpha\}} f(x), \mu(A\cap N_\alpha(f))]$$
$$\geq \inf_{E\in\mathcal{A}} T[\sup_{x\in E} f(x), \mu(A\cap E^c)]$$

Since α is arbitrary, we have

$$\inf_{\alpha\geq 0} T[\alpha, \mu(A\cap N_\alpha(f))] \geq \inf_{E\in\mathcal{A}} T[\sup_{x\in E} f(x), \mu(A\cap E^c)].$$

(3)$\leq$(2): $N_\alpha(f) \subset N^*_\alpha(f)$ and the monotonicity of fuzzy measure μ and c-generalized triangular conorm T follows (3)$\leq$(2).

(2)$\leq$(1): Suppose $s = \bigoplus_{i=1}^n [\alpha_i \odot \chi_{A_i}]$ is an arbitrary simple function such that $s \geq f$, then

$$Q_A(s) = \bigwedge_{i=1}^n T[\alpha_i, \mu(A \cap A_i^c)] = T[\alpha_{i0}, \mu(A \cap A_{i0}^c)]$$

by $A_{i_0} \subset \{x : f(x) \leq \alpha_{i0}\}$ and hence $A_{i0}^c \supset N_{\alpha_{i_0}}(f)$, it follows that

$$Q_A(s) = T[\alpha_{i0}, \mu(A \cap A_{i0}^c)] \geq T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))].$$

Further, we show that

$$T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))] \geq \inf_{\alpha \geq 0} T[\alpha, \mu(A \cap N^*_\alpha(f))].$$

In fact, if $T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))] = \infty$, then the inequality is trivial. If $T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))] < \infty$, then for any $\varepsilon > 0$, by Definition 3.1, there exists a natural number n_0 such that

$$\begin{aligned} T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))] + \varepsilon &> T[\alpha_{i_0} + 1/n_0, \mu(A \cap N_{\alpha_{i_0}}(f))] \\ &\geq T[\alpha_{i_0} + 1/n_0, \mu(A \cap N^*_{\alpha_{i_0}+1/n_0}(f))] \\ &\geq \inf_{\alpha \geq 0} T[\alpha, \mu(A \cap N^*_\alpha(f))] \end{aligned}$$

since ε is arbitrary, this implies that

$$T[\alpha_{i0}, \mu(A \cap N_{\alpha_{i_0}}(f))] \geq \inf_{\alpha \geq 0} T[\alpha, \mu(A \cap N^*_\alpha(f))].$$

From this, by Definition 3.2, it is known that (2)$\leq$(1). From the preceding proof we infer (1)=(2)=(3)=(4). □

In the following we give the simple properties of (PDG) fuzzy integrals.

THEOREM 3.4. *For (PDG) fuzzy integrals, we have*

(1) *if* $f_1 \leq f_2$, *then* $(PDG) \int_A f_1 d\mu \leq (PDG) \int_A f_2 d\mu$;

(2) *if* $A_1 \subset A_2$, $(PDG)\int_{A_1} f d\mu \leq (PDG)\int_{A_2} f d\mu$

(3) *if* $\mu(A) = 0$, *then* $(PDG)\int_A f d\mu = 0$;

(4) $(PDG)\int_A f d\mu = (PDG)\int_X f \cdot \chi_A d\mu$;

(5) $(PDG)\int_A (f_1 \wedge f_2) d\mu \leq (PDG)\int_A f_1 d\mu \wedge (PDG)\int_A f_2 d\mu$;

(6) $(PDG)\int_A c d\mu = c \wedge \mu(A)$ *for any* $A \in \mathcal{A}$ *and constant* $c \in [0, \infty]$.

Proof. The proofs of (1)-(5) are deduced directly from Theorem 3.1. We will prove (6).

(6) For any $\alpha \geq 0$, we have

$$N_\alpha(c) = \begin{cases} X, & \alpha < c \\ \emptyset, & \alpha \geq c \end{cases}$$

Hence from Theorem 3.1 it is known that

$$\begin{aligned}(PDG)\int_A c d\mu &= \int_{0 \leq \alpha < c} T[\alpha, \mu(A \cap N_\alpha(c))] \wedge \int_{\alpha > c} T[\alpha, \mu(A \cap N_\alpha(c))] \\ &= \inf_{0 \leq \alpha < c} T[\alpha, \mu(A)] \wedge \inf_{\alpha \geq c} T[\alpha, 0] = \mu(A) \wedge c.\end{aligned}$$

□

THEOREM 3.5. *Let* f, g *be nonnegative measurable functions. Then* $(PDG)\int f d\mu = (PDG)\int g d\mu$ *whenever* $f = g$ a.e., *if and only if* μ *is null-additive.*

Proof. Sufficiency: Suppose that μ is null-additive and $f = g$ a.e. Put $B = \{x : f(x) \neq g(x)\}$ Then $\mu(B) = 0$ and $\mu(N_\alpha(g)) = \mu(N_\alpha(g) \cup B)$. So we have $\mu(N_\alpha(f)) \leq \mu(N_\alpha(g) \cup B) = \mu(N_\alpha(g))$ for any $\alpha > 0$. The converse inequality holds as well, we have $\mu(N_\alpha(f)) = \mu(N_\alpha(g))$. By Theorem 3.3, we have $(PDG)\int f d\mu = (PDG)\int g d\mu$

Necessity: For any $A \in \mathcal{A}$, $B \in \mathcal{A}$ with $\mu(B) = 0$, if $\mu(A) = \infty$, then by the monotonicity of μ, we have $\mu(A \cup B) = \infty = \mu(A)$. Now we assume that $\mu(A) < \infty$. Define

$$f(x) = \begin{cases} e, & x \in A \cup B \\ 0, & x \notin A \cup B \end{cases}, \text{ and } g(x) = \begin{cases} e, & x \in A \\ 0, & x \notin A \end{cases}$$

where e is the unit of T. Then $f = g$ *a.e.* So by the hypothesis,

$$\begin{aligned} (PDG) \int f d\mu &= \inf_{\alpha \geq 0} T[\alpha, \mu(N_\alpha(f))] \\ &= \inf_{\alpha \geq 0} T[\alpha, \mu(N_\alpha(g))] \\ &= (PDG) \int g d\mu \end{aligned}$$

Therefore $T[e, \mu(A \cup B)] = T[e, \mu(A)]$. It follows that $\mu(A \cap B) = \mu(A)$. Hence μ is null-additive. □

Corollary 3.6. *If μ is null-additive, then*

$$(PDG) \int_A f d\mu = (PDG) \int_A g d\mu$$

whenever $f = g$ a.e. on A.

Proof. If $f = g$ *a.e.* on A, then $f\chi_A = g\chi_A$, *a.e.* From Theorem 3.5 and Theorem 3.4 (4), we get the conclusion. □

Corollary 3.7. *If μ is null-additive, then*

$$(PDG) \int_{A \cup B} f d\mu = (PDG) \int_A f d\mu$$

whenever $A \in \mathcal{A}$, $B \in \mathcal{A}$ with $\mu(B) = 0$.

Proof. Since $f\chi_{A \cup B} = f\chi_A$ *a.e.* by Theorem 3.5 and Theorem 3.4 (4), we get the conclusion. □

Theorem 3.8. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space and f be a nonnegative measurable function. Then $A \in \mathcal{A}$,*

$$(PDG) \int_A f d\mu \geq (PDG) \int_0^\infty g_A(\alpha) dm$$

where m is the Lebesgue measure and $g_A(\alpha) = \mu(A \cap N_\alpha(f))$.

Proof. If $t > \alpha$, then $g_A(t) \le g_A(\alpha)$. Therefore we have $\{x : g_A(x) > g_A(\alpha)\} \subset [0, \alpha]$ for any $\alpha \in R^+$. Hence $m(N_{g_A(\alpha)}(g_A)) \le \alpha$ for any $\alpha \in R^+$. It follows that

$$\begin{aligned}(PDG) \int_A f d\mu &= \inf_{\alpha > 0} T[\alpha, \mu(A \cap N_\alpha(f))] \\ &= \inf_{\alpha > 0} T[\alpha, g_A(\alpha)] \\ &= \inf_{\alpha > 0} T[m(N_\alpha(g_A)), g_A(\alpha)] \\ &\ge (PDG) \int_0^\infty g_A dm\end{aligned}$$

Hence we have $(PDG) \int_A f d\mu \ge (PDG) \int_0^\infty g_A dm$. □

4. Integral equations

In this section, we always suppose that $\mu(X) = \infty$ and $T[0, x] = x$ for all $x \in [0, \infty]$.

THEOREM 4.1. *For $\beta \in [0, \infty)$, the almost everywhere finite nonnegative measurable function f satisfies the equation $(PDG) \int_A f d\mu = \beta$ if and only if $T[\alpha, \mu(A \cap N_\alpha(f))] \ge \beta$ for all $\alpha \ge 0$ and there exists $\alpha_0 \in [0, \infty)$ such that $T[\alpha_0, \mu(A \cap N_{\alpha_0}(f))] = \beta$.*

Proof. Let $(PDG) \int_A f d\mu = \beta$. By Theorem 3.3, $(PDG) \int_A f d\mu = \inf_{\alpha \ge 0} T[\alpha, \mu(A \cap N_\alpha(f))]$. We have $T[\alpha, \mu(A \cap N_\alpha(f))] \ge \beta$ for all $\alpha \ge 0$. In addition, by the equation, there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} T[x_n, \mu(N_{x_n}(f))] = \beta$. Take a subsequence of $\{x_n\}$ monotonely convergent to $x_0 \in [0, \infty]$. Without confusion, we also denote it as $\{x_n\}$. If $x_n \uparrow \infty$, then $\mu(A \cap N_{x_n}(f)) \downarrow 0$. By the hypothesis $\lim_{n\to\infty} T[\frac{1}{n}, \infty] = \infty$, we infer that $\beta = \lim_{n\to\infty} T[x_n, \mu(A \cap N_{x_n}(f))] = \infty$. This contradicts $\beta \in (0, \infty)$. If x_n converges to $x_0 \in [0, \infty)$ monotonely, by Theorem 3.3 and properties of c-generalized triangle norm, we infer that

$$\begin{aligned}\beta &= \lim_{n\to\infty} T[x_n, \mu(A\cap N_{x_n}(f))]\\ &= T[x_0, \mu(A\cap N_{x_0}(f))]\\ &\geq \inf_{\alpha\geq 0} T[\alpha, \mu(A\cap N_\alpha(f))]\\ &= \beta\end{aligned}$$

Hence we have $T[x_0, \mu(A\cap N_{x_0}(f))] = \beta$.

Conversely, by the hypothesis

$$T[\alpha, \mu(A\cap N_\alpha(f))] \geq \beta$$

for all $\alpha \geq 0$, we have

$$(PDG)\int_A f d\mu \geq \beta.$$

In addition,

$$\begin{aligned}\beta &= T[\alpha_0, \mu(A\cap N_{\alpha_0}(f))]\\ &\geq \inf_{\alpha\geq 0} T[\alpha, \mu(A\cap N_\alpha(f))]\\ &= (PDG)\int_A f d\mu\end{aligned}$$

Hence we have $\beta = (PDG)\int_A f d\mu$. □

THEOREM 4.2. *Let $k(x)$ be a nonnegative measurable function and $\beta \in [0,\infty)$. Then there exists a nonnegative measurable function f such that $(PDG)\int (k\vee f)d\mu = \beta$ if and only if there exists a nonnegative measurable function h with $k(x) \leq h(x)$, $x \in X$ such that $T[\alpha, \mu(N_\alpha(h))] \geq \beta$ for all $\alpha \geq 0$ and $T[\alpha_0, \mu(N_{\alpha_0}(h))] = \beta$ for some $\alpha_0 > 0$.*

Proof. Let $(PDG)\int (k\vee f)d\mu = \beta$ and $h(x) = (k\vee f)(x)$. Then $k(x) \leq h(x)$ for all $x \in X$, and $\int h(x)d\mu = \beta$. By Theorem 4.1,

$T[\alpha, \mu(N_\alpha(h))] \geq \beta$ for all $\alpha \geq 0$ and $T[\alpha_0, \mu(N_{\alpha_0}(h))] = \beta$ for some $\alpha_0 > 0$. Conversely, if $h(x)$ fulfills the condition, then $(PDG)\int(h \vee k)d\mu = (PDG)\int hd\mu$. By Theorem 4.1, $(PDG)\int(k \vee f)d\mu = \beta$. □

References

1. D. A. Ralescu and G. Adams, *Fuzzy integral*, J. Math. Anal. Application **75** (1980), 562-570
2. D. A. Ralescu, *Towards a general theory of fuzzy variables*, J. Math. Anal. Application **86** (1982), 176-193
3. Song Shiji, Shi Peilin and Wang Haiyan, *The convergence of (DG) fuzzy integrals*, Fuzzy Sets and Systems **90** (1997), 45-53
4. M. Sugeno, *Theory of fuzzy integrals and its applications*, *Ph.D. Dissertation*, Tokyo Institution of Technology (1974)
5. W. Zhenyuen, *The autocontinuity of set function and the fuzzy integral*, J. Math. Anal. Application **99** (1984), 195-218
6. W. Congxin and W. Shuli and M. Ming, *Generalized fuzzy integrals*, Fuzzy Sets and Systems **57** (1993), 219-226
7. W. Congxin, M. Ming and S. Z. Shaotai, *Generalized fuzzy integrals:Part 3 convergent theorems*, Fuzzy Sets and Systems **70** (1995), 75-87
8. Z. Ruhuai, *(N) fuzzy integral*, J. Math. Resear. Exp. **2** (1981), 55-72
9. Z. Wang and G. Klir, *Fuzzy Measure Theory*, Plenum Press, New York and London (1992)

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