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THE STRONG PERRON AND MCSHANE INTEGRALS

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Abstract. In this paper, we define the strong Perron integral and study the relationship between the integrals of strong Perron and McShane.

1. Introduction

It is well-known [3] that the Perron integral is equivalent to the Henstock integral. In this paper, we define the strong upper derivate and strong lower derivate of the real valued funtiction F defined on an interval $[a, b]$. Using strong derivates, it is possible to define strong major and minor functions and then the strong Perron integral can be defined. We can show that this integral is equivalent to the McShane integral. The proof is similar to the proof that the Henstock and Perron integrals are equivalent.

2. The strong Perron integral

The upper and lower derivates of $F : [a, b] \to \mathbb{R}$ at $x \in [a, b]$ are defined by

$$\overline{D}F(x) = \lim_{\delta \to 0^+} \sup \{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \delta \} ;$$

$$\underline{D}F(x) = \lim_{\delta \to 0^+} \inf \{ \frac{F(y) - F(x)}{y - x} : 0 < |y - x| < \delta \}.$$

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The function F is differentiable at x if the two derivates are finite and equal.

We define strong derivates.

DEFINITION 2.1. Let $F : [a,b] \to \mathbb{R}$ and let $c \in [a,b]$ The *upper* and *lower strong derivates* of F at c are defined by

$$\overline{SD}F(c) = \lim_{\delta \to 0^+} \sup\{ \frac{F(y)-F(x)}{y-x} : [x,y] \subseteq (c-\delta, c+\delta) \cap [a,b]\} ;$$
$$\underline{SD}F(c) = \lim_{\delta \to 0^+} \inf\{ \frac{F(y)-F(x)}{y-x} : [x,y] \subseteq (c-\delta, c+\delta) \cap [a,b]\} .$$

The function F is strongly differentiable at c if $\underline{SD}F(c)$ and $\overline{SD}F(c)$ are finite and equal. Notice that the interval $[x,y]$ does not have to contain the point c. From definitions, it is clear that

$$\underline{SD}F \le \underline{D}F \le \overline{D}F \le \overline{SD}F .$$

Using the upper and lower strong derivates, we can define the strong major and strong minor functions.

DEFINITION 2.2. Let $f : [a,b] \to \mathbb{R}$ be a function.

(1) A measurable function $U : [a,b] \to \mathbb{R}$ is a *strong major function* of f on $[a,b]$ if

$$\underline{SD}U(x) > -\infty \ \text{ and } \ \underline{SD}U(x) \ge f(x) \qquad \text{for all } x \in [a,b].$$

(2) A measurable function $V : [a,b] \to \mathbb{R}$ is a *strong minor function* of f on $[a,b]$ if

$$\overline{SD}V(x) < \infty \ \text{ and } \ \overline{SD}V(x) \le f(x) \qquad \text{for all } x \in [a,b].$$

DEFINITION 2.3. A function $f : [a, b] \to \mathbb{R}$ is *s-Perron integrable* on $[a, b]$ if f has at least one strong major and one strong minor function on $[a, b]$ and the numbers

$$\inf \{ \ U_a^b : U \text{ is a strong major function of } f \text{ on } [a, b]\};$$
$$\sup \{ \ V_a^b : V \text{ is a strong minor function of } f \text{ on } [a, b]\}$$

are equal, where $U_a^b = U(b) - U(a)$ and $V_a^b = V(b) - V(a)$. This common value is the s-Perron integral of f on $[a, b]$ and will be denoted by $(SP) \int_a^b f$ to distinguish this integral from others.

The following theorem is an immediate consequence of the definition.

THEOREM 2.4. *A function $f : [a, b] \to \mathbb{R}$ is s-Perron integrable on $[a, b]$ if and only if for each $\epsilon > 0$ there exist a strong major function U and a strong minor function V of f on $[a, b]$ such that $U_a^b - V_a^b < \epsilon$.*

Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A *free tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [a, b]$. The free tagged interval $(x, [c, d])$ is *subordinate to* δ if

$$[c, d] \subseteq (x - \delta(x), x + \delta(x)).$$

A function $f : [a, b] \to \mathbb{R}$ is *McShane integrable* on $[a, b]$ if there exists a real number $\mathbb{L}$ with the following property ; for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that $|f(P) - \mathbb{L}| < \epsilon$ whenever P is a free tagged partition of $[a, b]$ that is subordinate to δ, where $f(P) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)$ if $P = \{(x_i, [c_i, d_i])\}$ is a free tagged partition. The real number $\mathbb{L}$ is the McShane integral of f on $[a, b]$ and will be denoted by $(M) \int_a^b f$.

The following two theorems show that the strong Perron integral is equivalent to the McShane integral.

THEOREM 2.5. *If $f : [a,b] \to \mathbb{R}$ is s-Perron integrable on $[a,b]$, then f is McShane integrable on $[a,b]$ and the integrals are equal.*

Proof. Let $\epsilon > 0$. By definition, there exists a strong major function U and a strong minor function V of f on $[a,b]$ such that

$$-\epsilon \ < \ V_a^b - (SP)\int_a^b f \ \leq \ 0 \ \leq \ U_a^b - (SP)\int_a^b f \ < \ \epsilon.$$

Since $\overline{SD}V \leq f \leq \underline{SD}U$ on $[a,b]$, for each $c \in [a,b]$ there exists $\delta(c) > 0$ such that

$$\frac{U(y)-U(x)}{y-x} \geq f(c) - \epsilon \ \text{ and } \ \frac{V(y)-V(x)}{y-x} \leq f(c) + \epsilon$$

whenever $[x,y] \subseteq (c-\delta(c), c+\delta(c)) \cap [a,b]$. Now let

$$P = \{ \ (x_i, [c_i, d_i]) \ : \ 1 \leq i \leq q\}$$

be a free tagged partition of $[a,b]$ that is subordinate to δ. Then we have

$$\begin{aligned} &\sum_{i=1}^{q} f(x_i)(d_i - c_i) - (SP)\int_a^b f \\ &\quad = \sum_{i=1}^{q} (f(x_i)(d_i - c_i) - U_{c_i}^{d_i}) + U_a^b - (SP)\int_a^b f \\ &\quad < \sum_{i=1}^{q} \epsilon(d_i - c_i) + \epsilon \\ &\quad = \epsilon(b - a + 1) \end{aligned}$$

Similarly, using the strong minor function V,

$$\sum_{i=1}^{q} f(x_i)(d_i - c_i) - (SP)\int_a^b f \ > \ -\epsilon(b-a+1).$$

Hence, f is McShane integrable on $[a,b]$ and $(M)\int_a^b f = (SP)\int_a^b f$. □

THEOREM 2.6. *If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$, then f is s-Perron integrable on $[a,b]$.*

Proof. Let $\epsilon > 0$. By definition, there exists a positive function δ on $[a,b]$ such that $|f(P) - (M)\int_a^b f| < \epsilon$ whenever P is a free tagged partition of $[a,b]$ that is subordinate to δ. For each $x \in (a,b]$, let

$$\begin{aligned} U(x) = \sup\{f(P) &: P \text{ is a free tagged partition of } [a,x] \\ &\quad \text{that is subordinate to } \delta\} \\ V(x) = \inf\{f(P) &: P \text{ is a free tagged partition of } [a,x] \\ &\quad \text{that is subordinate to } \delta\} \end{aligned}$$

and let $U(a) = 0 = V(a)$. By the Saks-Henstock lemma, the function U and V are finite vlaued on $[a,b]$.

We show that U is a strong major function of f on $[a,b]$. The proof that V is a strong minor function of f is quite similar.

Fix a point $c \in [a,b]$ and let $[x,y]$ be any interval with $[x,y] \subseteq (c-\delta(c), c+\delta(c)) \cap [a,b]$. For each free tagged partition P of $[a,x]$ that is subordinate to δ, we find that

$$U(y) \geq f(P) + f(c)(y-x)$$

and it follows that $U(y) \geq U(x) + f(c)(y-x)$.

It shows that $\frac{U(y)-U(x)}{y-x} \geq f(c)$. Hence $\underline{SD}U(c) \geq f(c) > -\infty$. Since $-\infty < \underline{SD}U \leq \underline{D}U$ on $[a,b]$, U is BVG_* on $[a,b]$ by [3, Theorem 6.21] and hence U is measurable on $[a,b]$ by [3, Corollary 6.9].

This shows that U is a strong major function of f. Since $|f(P_1) - f(P_2)| < 2\epsilon$ for any two free tagged partitions P_1 and P_2 of $[a,b]$ that are subrdinate to δ, it follows that $U_a^b - V_a^b \leq 2\epsilon$. By Theorem 2.4, the function f is s-Perron integrable on $[a, b]$. □

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