JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 13, No.1, June 2000

WEIGHT NASH EQUILIBRIA FOR GENERALIZED MULTIOBJECTIVE GAMES

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Abstract. The purpose of this paper is to give a new existence theorem of a generalized weight Nash equilibrium for generalized multiobjective games by using the quasi-variational inequality due to Yuan.

Recently the study of existence of weight Nash equilibria in game theory with vector payoffs has been extensively studied by a number of authors, e.g., see [2-4, 7-11] and the references therein. The motivation for the study of multicriteria models can be found in [2, 7] and the existence of Nash equilibria is one of the fundamental problem in the game theory. In a recent paper [10], Yu and Yuan proved some existence theorems of weight Nash equilibria by using Ky Fan's minimax inequality; and hence they provided an unified study for the existences of Pareto equilibria and Nash equilibria in multiobjective game under weaker conditions.

In this paper, we first introduce the new concepts of generalized multiobjective game and generalized weight Nash equilibrium. Next using the quasi-variational inequlity due to Yuan [11], we will prove the existence theorem of generalized weight Nash equilibria under general hypotheses.

Received by the editors on April 25, 2000.
1991 *Mathematics Subject Classifications*: Primary 90A15, 47N10 Secondary 52A07.
Key words and phrases: Generalized multiobjective game, generalized weight Nash equilibrium, quasi-variational inequality.

Let X be a non-empty convex subset of a topological vector space E and let $f : X \to \mathbb{R}$. We say that f is *quasi-concave* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is convex; and that f is *quasi-convex* if $-f$ is quasi-concave. A correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a correspondence $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. And we say that T is *continuous* if T is both upper semicontinuous and lower semicontinuous.

Next we recall the continuity definitions of the real-valued function. Let X be a non-empty subset of a topological space E and $f : X \to \mathbb{R}$. We say that f is *upper semicontinuous* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \geq t\}$ is closed in X, and f is *lower semicontinuous* if $-f$ is upper semicontinuous. And we say that f is *continuous* if f is both upper semicontinuous and lower semicontinuous. For the other standard notations and terminologies, we shall refer to [10, 11].

First, we shall introduce the *generalized multiobjective game* (or *generalized game with multicriterior*) in its strategic form of a finite (or infinite) number of players $G := (X_i, F^i, T_i)_{i \in I}$, where I is a (possibly uncountable) set of players, as follows : for each $i \in I$, X_i is the set of strategies in a Hausdorff topological vector space E_i for the player i, and $F^i : X = \Pi_{i \in I} X_i \to \mathbb{R}^{k_i}$, where $k_i \in \mathbb{N}$, which is called the *payoff function* (or called *multicriteria*) and $T_i : X \to 2^{X_i}$, which is called the *constraint correspondence* of the player i.

If an action $x := (x_1, \cdots, x_n) \in X$ is played, each player i is trying to find his/her payoff function $F^i(x) := (f^i_1(x), \cdots, f^i_{k_i}(x))$, which consists of noncommensurable outcomes under the possible constraint sets $T_i(x)$. In a generalized multiobjective game, the other players can influence the player j

(1) indirectly, by restricting j's feasible strategies to $T_j(x)$;

(2) directly, by affecting j's payoff function F^j.

Here we assume that the model of a game is a non-cooperative game, i.e., there is no replay communicating between players, and so players act as free agents, and each play is trying to minimize his/her own payoff according to his/her preferences and constraints.

For the games with vector payoff functions (or multicriteria), it is well-known that in general, there does not exist a strategy $\hat{x} \in X$ to minimize (or equivalently to say, maximize) all f^i_js for each player i in his/her constraint, e.g., see [11] and the references therein. Hence we shall need some solution concepts for generalized multicriteria games.

Throughout this paper, for each $m \in \mathbb{N}$, we shall denote by $\mathbb{R}^m_+$ the non-negative orthant of $\mathbb{R}^m$, i.e.,

$$\mathbb{R}^m_+ := \{u = (u_1, \cdots, u_m) \in \mathbb{R}^m \mid u_j \geq 0 \ \ \forall j = 1, \cdots, m\},$$

so that the non-negative orthant $\mathbb{R}^m_+$ of $\mathbb{R}^m$ has a non-empty interior with the topology induced in terms of convergence of vectors respect to the Euclidean metric. That is, we shall use the notation

$$int\ \mathbb{R}^m_+ := \{u = (u_1, \cdots, u_m) \in \mathbb{R}^m \mid u_j > 0 \ \ \forall j = 1, \cdots, m\}.$$

For each $i \in I$, denote $X_{\hat{i}} := \Pi_{j \in I \setminus \{i\}} X_j$. If $x = (x_1, \cdots, x_n) \in X$, we shall write $x_{\hat{i}} = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in X_{\hat{i}}$. If $x_i \in X_i$ and $x_{\hat{i}} \in X_{\hat{i}}$, we shall use the notation

$$(x_{\hat{i}}, x_i) := (x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) = x \in X.$$

For each $u,\, v \in \mathbb{R}^m$, $u \cdot v$ denote the standard Euclidean inner product.

Now we introduce the following general equilibrium concept of a generalized multiobjective game :

Definition. A strategy $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n) \in X$ is said to be a *generalized weight Nash equilibrium* for the game $G = (X_i, F^i, T_i)_{i \in I}$ respect to the weight vector $W := (W_1, \cdots, W_n)$ where $W_i \in \mathbb{R}^{k_i}_+ \setminus \{0\}$, if for each player $i \in I$,

(1) $\bar{x}_i \in T_i(\bar{x})$;

(2) $W_i \cdot F^i(\bar{x}_{\hat{i}}, \bar{x}_i) \leq W_i \cdot F^i(\bar{x}_{\hat{i}}, x_i)$ for each $x_i \in T_i(\bar{x})$.

In particular, when $W_i \in T^{k_i}_+$ for all $i \in I$, the strategy $\bar{x} \in X$ is said to be a *normalized form of generalized weight Nash equilibrium* respect to the weight W, where $T^{k_i}_+$ is the standard simplex of $\mathbb{R}^{k_i}$.

The above definition generalizes the corresponding definitions in [9, 10]. In fact, in the above definition, it is clear that every generalized weight Nash equilibrium is a weight Nash equilibrium when the constraint set is fixed with $T_i(x) = X_i$ for each $x \in X$ and $i \in I$.

For each $i \in I$, let $W_i \in \mathbb{R}^{k_i}_+ \setminus \{0\}$ be fixed. Then, from the above definition, it is easy to see that a strategy $\bar{x} \in X$ is a generalized weight Nash equilibrium of a game $G = (X_i, F^i, T_i)_{i \in I}$ respect to the weight vector $W = (W_1, \cdots, W_n)$ if and only if for each $i \in I$, $\bar{x}_{\hat{i}}$ is an optimal solution of the vector optimization problem

$$\min_{x_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_{\hat{i}}, x_i).$$

The following lemma is an easy consequence of the quasi-variational inequality due to Yuan [11], and it is the basic tool for proving the existence of generalized weight Nash equilibria :

Lemma 1. *Let X be a non-empty compact convex subset of a Hausdorff topological vector space E which has sufficiently many continuous linear functionals. Let $T : X \to 2^X$ be an upper semicontinuous correspondence such that each $T(x)$ is a non-empty closed convex subset of X. Let $\phi : X \times X \to \mathbb{R}$ be a function such that*

(1) for each fixed $y \in X,\ x \mapsto \phi(x, y)$ is lower semicontinuous ;
(2) for each fixed $x \in X,\ y \mapsto \phi(x, y)$ is quasi-concave ;
(3) the set $\{\, x \in X \mid \sup_{y \in T(x)} \phi(x, y) \le 0 \,\}$ is closed in X.

Then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in T(\hat{x}) \quad \text{and} \quad \sup_{y \in T(\hat{x})} \phi(\hat{x}, y) \ \le\ 0.$$

We also need the following lower semicontinuity property :

Lemma 2. *Let X, Y be Hausdorff topological vector spaces and X be compact. Let $T : X \to 2^Y$ be a lower semicontinuous correspondence such that each $T(x)$ is a non-empty subset of X, and let $f : X \times Y \to \mathbb{R}$ be a lower semicontinuous function on $X \times Y$. Then the function $\phi : X \to \mathbb{R}$, defined by*

$$\phi(x) := \sup_{y \in T(x)} f(x, y), \quad \text{for each } x \in X,$$

is a lower semicontinuous function on X.

Proof. By applying Theorem 2.5.2 in [1] to $-f$, we can obtain the conclusion. □ □

Now we will prove an existence theorem of a generalized weight Nash equilibrium as follows :

Theorem. *Let I be a set of finite number of players and let $G = (X_i, F^i, T_i)_{i \in I}$ be a generalized multiobjective game, where for* *each $i \in I$, X_i is a non-empty compact convex subset of a Hausdorff topological vector space E_i which has sufficiently many continuous linear functionals. Let $T_i : X \to 2^{X_i}$ be a continuous constraint correspondence such that each $T_i(x)$ is a non-empty closed convex subset of X_i. If there exists a weight vector $W = (W_1, \cdots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$,*

(1) for each $y_i \in X_i$, $x \mapsto \sum_{i \in I} W_i \cdot F^i(x_{\hat{i}}, y_i)$ is upper semicontinuous on X ;

(2) for each $x_{\hat{i}} \in X_{\hat{i}}$, $y \mapsto \sum_{i \in I} W_i \cdot F^i(x_{\hat{i}}, y_i)$ is quasi-convex on X ;

(3) $(x, y) \mapsto \sum_{i \in I} W_i \cdot F^i(x_{\hat{i}}, y_i)$ is jointly lower semicontinuous on $X \times X$.

Then there exists a generalized weight Nash equilibrium $\bar{x} \in X$ for the game G respect to the weight vector $W = (W_1, \cdots, W_n)$.

Proof. In order to apply the quasi-variational inequality, we first define a real-valued function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) := \sum_{i \in I} W_i \cdot (F^i(x_{\hat{i}}, x_i) - F^i(x_{\hat{i}}, y_i)), \quad \text{for each} \quad (x, y) \in X \times X.$$

Then by the assumptions (1) and (2) and the fact that finite sum of lower semicontinuous functions is also lower semicontinuous, we can have

(a) for each fixed $y \in X$, $x \mapsto \phi(x, y)$ is lower semicontinuous ;

(b) for each fixed $x \in X$, $y \mapsto \phi(x, y)$ is quasi-concave.

Since the correspondence $T(x) := \Pi_{i \in I} T_i(x)$ is lower semicontinuous and the map ϕ is jointly lower semicontinuous, by Lemma 2, the map $x \mapsto \sup_{y \in T(x)} \phi(x, y)$ is lower semicontinuous and hence the set $\{ x \in X \mid \sup_{y \in T(x)} \phi(x, y) \leq 0 \}$ is closed in X. Therefore the whole assumptions of Lemma 1 are satisfied, and so there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$ and $\phi(\bar{x}, y) = \sum_{i \in I} W_i \cdot (F^i(\bar{x}_{\hat{i}}, \bar{x}_i) -$

$F^i(\bar{x}_{\hat{i}}, y_i)) \leq 0$ for all $y \in T(\bar{x})$. Then for each $i \in I$ and every $(\hat{x}_{\hat{i}}, y_i) \in T(\bar{x}_{\hat{i}}, \bar{x}_i)$, we have $W_i \cdot F^i(\bar{x}_{\hat{i}}, \bar{x}_i) - W_i \cdot F^i(\bar{x}_{\hat{i}}, y_i) \leq 0$; which implies that for each $i \in I$, $\bar{x}_i \in T_i(\bar{x})$ and

$$W_i \cdot F^i(\bar{x}_{\hat{i}}, \bar{x}_i) = \min_{y_i \in T_i(\bar{x})} W_i \cdot F^i(\bar{x}_{\hat{i}}, y_i).$$

Thus $\bar{x}$ is a generalized weight Nash equilibrium point of the game G respect to the weight vector W. □ □

Remarks. (1) Our Theorem generalizes the corresponding results in [9, 10]. In fact, when the constraint correspondence T_i is constant, i.e., $T_i(x) = X_i$ for each $i \in I$ and $x \in X$, our Theorem reduces to the corresponding Theorem 1 in [10], and so the corresponding theorems in [9] can be obtained.

(2) We can obtain the existence of equilibria for generalized multiobjective games by using some coercive conditions as in [6]; and in this case, we can assure that the strategy set X_i need not be compact as in the corresponding theorems in [10] nor X_i need be a subset of a normed linear spaces as in [9].

(3) By following the methods in [6], as applications of generalized weight Nash equilibria, we can prove the existence of generalized Pareto equilibria in non-compact generalized multiobjective games. □

One final comment. It is well-known that fixed point technique has wide applications in the study of economics and optimizations, e.g., see [7-11]. On the other hand, in a recent paper [10], Yu and Yuan proved the existence of weight Nash equilibria and Pareto equilibria by using Ky Fan's minimax inequality, which would not be widely used before as an efficient tool for investigating the equilibria in economics and optimizations. Furthermore, in this paper, it is our main purpose to present how the quasi-variational inequality can be applied to the existence of generalized weight Nash equilibria, and this method can be considered as an efficient tool for the equilibrium theory.

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