

## ON THE STRUCTURE OF NON-COMMUTATIVE TORI

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ABSTRACT. The non-commutative torus  $A_\omega = C^*(\mathbb{Z}^n, \omega)$  may be realized as the  $C^*$ -algebra of sections of a locally trivial  $C^*$ -algebra bundle over  $\widehat{S}_\omega$  with fibres  $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$  for some totally skew multiplier  $\omega_1$  on  $\mathbb{Z}^n/S_\omega$ . It is shown that  $A_\omega \otimes M_l(\mathbb{C})$  has the trivial bundle structure if and only if  $\mathbb{Z}^n/S_\omega$  is torsion-free.

### 1. Introduction

Given a locally compact abelian group  $G$  and a multiplier  $\omega$  on  $G$ , one can associate to them the twisted group  $C^*$ -algebra  $C^*(G, \omega)$ , which is the universal object for unitary  $\omega$ -representations of  $G$ . Our problem is to understand the structure, especially the bundle structure, of such  $C^*$ -algebras. The twisted group  $C^*$ -algebra  $C^*(G, \omega)$  of a finitely generated discrete torsion-free abelian group  $G$  defined by a multiplier  $\omega$  is called a *non-commutative torus* and is denoted by  $A_\omega$ . It is a fundamental and an important example in non-commutative differential geometry [3].

The simplest non-trivial non-commutative tori arise when  $G = \mathbb{Z}^2$ . In this case we may assume  $\omega$  is antisymmetric and  $\omega((1, 0), (0, 1)) = e^{\pi i \theta}$ . When  $\theta$  is irrational, one obtains a simple  $C^*$ -algebra called an *irrational rotation algebra*, and denoted by  $A_\theta$ . When  $\theta = \frac{m}{k}$ , one obtains a *rational rotation algebra*, and denoted by  $A_{\frac{m}{k}}$ .

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Now the multiplier  $\omega$  determines a subgroup  $S_\omega$  of  $G$ , called its *symmetry group*. A multiplier  $\omega$  on an abelian group  $G$  is called *totally skew* if the symmetry group  $S_\omega$  is trivial. Baggett and Kleppner [1] showed that if  $G$  is a locally compact abelian group and  $\omega$  is a totally skew multiplier on  $G$ , then  $C^*(G, \omega)$  is a simple  $C^*$ -algebra. So  $C^*(\mathbb{Z}^n, \omega)$  is simple if  $\omega$  is totally skew.

Baggett and Kleppner [1] also showed that even when  $\omega$  is not totally skew on a locally compact abelian group  $G$ , the restriction of  $\omega$ -representations from  $G$  to  $S_\omega$  induces a canonical homeomorphism of  $\text{Prim}(C^*(G, \omega))$  with  $\widehat{S_\omega}$ . It was shown in [1] that there is a totally skew multiplier  $\omega_1$  on  $\mathbb{Z}^n/S_\omega$  such that  $\omega$  is similar to the pull-back of  $\omega_1$ . Furthermore, it is known (see [1, 5, 7, 10, 12]) that  $C^*(G, \omega)$  may be realized as the  $C^*$ -algebra  $\Gamma(\zeta)$  of sections of a locally trivial  $C^*$ -algebra bundle  $\zeta$  over  $\widehat{S_\omega} = \text{Prim}(C^*(G, \omega))$  with fibres  $C^*(G, \omega)/x$  for  $x \in \text{Prim}(C^*(G, \omega))$  and all  $C^*(G, \omega)/x$  turn out to be the simple twisted group  $C^*$ -algebra  $C^*(G/S_\omega, \omega_1)$ .

A natural question is when the locally trivial bundle  $\zeta$  is trivial. Brabanter [2] proved that the rational rotation algebra  $A_{\frac{m}{k}}$  has a non-trivial bundle structure and that  $A_{\frac{m}{k}} \otimes \mathcal{K}(\mathcal{H})$  has the trivial bundle structure.

Poguntke showed in [11] that any primitive quotient of the group  $C^*$ -algebra  $C^*(G)$  of a locally compact two step nilpotent group  $G$  is isomorphic to the tensor product of a completely irrational non-commutative torus  $A_\varphi$  and  $\mathcal{K}(\mathcal{H})$  for some (possibly finite-dimensional) Hilbert space  $\mathcal{H}$ . Since  $C^*(G/S_\omega, \omega_1)$  is the primitive quotient of  $C^*(G/S_\omega(\omega_1))$  where  $G/S_\omega(\omega_1)$  is the extension group of  $G/S_\omega$  by  $\mathbb{T}$  defined by  $\omega_1$ ,  $C^*(G/S_\omega, \omega_1)$  is isomorphic to  $A_\varphi \otimes \mathcal{K}(\mathcal{H})$ .

In this paper, we investigate the structure of the fibre of  $A_\omega$ . By a result of Poguntke we have a decomposition of the fibre of  $A_\omega$  as the

tensor product of a completely irrational non-commutative torus  $A_\varphi$  with a matrix algebra  $M_k(\mathbb{C})$ . So  $A_\omega$  is isomorphic to the  $C^*$ -algebra of sections of a locally trivial  $C^*$ -algebra bundle over  $\widehat{S}_\omega$  with fibres  $A_\varphi \otimes M_k(\mathbb{C})$ . We are going to show that  $A_\omega \otimes M_l(\mathbb{C})$  has a non-trivial bundle structure if and only if  $k > 1$ .

## 2. Preliminaries

Let  $G$  be a locally compact abelian group,  $\omega$  a multiplier on  $G$ . Denote by  $C^*(G, \omega)$  the twisted group  $C^*$ -algebra of  $G$  defined by  $\omega$ , which is the universal object for unitary  $\omega$ -representations of  $G$ . A non-commutative torus  $A_\omega$  of rank  $n$  is the twisted group  $C^*$ -algebra  $C^*(\mathbb{Z}^n, \omega)$  of  $\mathbb{Z}^n$  defined by a multiplier  $\omega$ . To fix notations, let

$\mathbb{Z}$  = the set of integers,

$\mathbb{C}$  = the set of complex numbers,

$\otimes$  = the minimal tensor product.

We start our investigations with a study of decomposition of multipliers on  $\mathbb{Z}^n/S_\omega$ . If  $\omega$  is a multiplier on  $G$  and  $H$  a closed subgroup of  $G$ , then we denote by  $\omega|_H$  the restriction of  $\omega$  to  $H$ . Furthermore, if  $G = G_1 \oplus G_2$ , and if  $\omega_1$  and  $\omega_2$  are multipliers on  $G_1$  and  $G_2$ , respectively, then we denote by  $\omega_1 \oplus \omega_2$  the multiplier on  $G$  defined by

$$(\omega_1 \oplus \omega_2)((x_1, x_2), (y_1, y_2)) = \omega_1(x_1, y_1)\omega_2(x_2, y_2),$$

$x_1, y_1 \in G_1$  and  $x_2, y_2 \in G_2$ .

For some groups  $G$ , each multiplier on  $G$  turns out to be a bicharacter.

**2.1 PROPOSITION.** ([8, Theorem 7.1.]) *Let  $G$  be a finitely generated discrete abelian group. Then every multiplier on  $G$  is similar to a bicharacter.*

Let  $\omega$  be a multiplier on a locally compact abelian group  $G$ . Define a homomorphism  $h_\omega : G \rightarrow \widehat{G}$  by  $h_\omega(x)(y) = \omega(x, y)\omega(y, x)^{-1}$ ,  $x, y \in G$  and let  $S_\omega := \ker(h_\omega)$  denote the symmetry group of  $\omega$ .

2.2 DEFINITION. The non-commutative torus  $A_\omega$  of rank  $n$  is said to be a *completely irrational non-commutative torus* if  $S_\omega \cong \{0\}$ .

Next we introduce the concept of  $C^*$ -algebra bundle over a locally compact Hausdorff space. A multiplier  $\omega$  on an abelian group  $G$  is called *totally skew* if the symmetry group  $S_\omega$  is trivial. Let  $\text{Prim}(C^*(G, \omega))$  be the primitive ideal space of the twisted group  $C^*$ -algebra  $C^*(G, \omega)$  of a locally compact abelian group  $G$  defined by a multiplier  $\omega$ .

2.3 PROPOSITION. ([1, 7]) *Let  $G$  be a locally compact abelian group and  $\omega$  a multiplier on  $G$ . Then*

- i) *there is a multiplier  $\omega_1$  on  $G/S_\omega$  such that  $C^*(G, \omega)/P$  is isomorphic to  $C^*(G/S_\omega, \omega_1)$  for any  $P \in \text{Prim}(C^*(G, \omega))$  and  $\omega$  is similar to the pull-back of a totally skew multiplier  $\omega_1$ ;*
- ii) *the restriction of  $\omega$ -representations from  $G$  to  $S_\omega$  induces a canonical homeomorphism of  $\text{Prim}(C^*(G, \omega))$  with  $\widehat{S_\omega}$ .*

2.4 THEOREM. ([10]) *Let  $A$  be a  $C^*$ -algebra and  $T$  a locally compact Hausdorff space. Then  $A$  is isomorphic to the  $C^*$ -algebra of sections of a  $C^*$ -algebra bundle over  $T$  if and only if there is a continuous open surjection of  $\text{Prim}(A)$  onto  $T$ .*

Here the  $C^*$ -algebra bundle is not necessarily locally trivial.

By a trick similar to the proof of [5, Theorem 1], one can show that for a multiplier  $\omega$  on a locally compact abelian group  $G$ ,  $C^*(G, \omega)$  can be realized as the  $C^*$ -algebra of sections of a locally trivial  $C^*$ -algebra bundle. That is, if  $A$  is a twisted group  $C^*$ -algebra of a locally compact abelian group, its  $C^*$ -algebra bundle is locally trivial.

In particular,  $A_\omega \cong C^*(\mathbb{Z}^n, \omega)$  may be represented as the  $C^*$ -algebra of sections of a locally trivial  $C^*$ -algebra bundle over  $\widehat{S}_\omega$  with fibres  $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ . See [5, Theorem 1] for details.

A problem then is to decide when this locally trivial bundle is actually trivial. Brabanter [2] proved that the rational rotation algebra has a non-trivial bundle structure. We will present a new proof of this result in the next section.

Let  $G$  be a finitely generated discrete abelian group, e.g.,  $\mathbb{Z}^n/S_\omega$ ,  $\omega_1$  a totally skew multiplier on  $G$ , and  $T$  the maximal torsion subgroup of  $G$ . Then  $G \cong T \oplus F$  where  $F$  is a torsion-free subgroup. Note that  $\omega_1|_F$  is always totally skew, but  $\omega_1|_T$  need not be totally skew. A multiplier  $\omega$  on a group  $G$  is said to be *type I* if  $C^*(G, \omega)$  is a type I  $C^*$ -algebra.

**2.5 LEMMA.** ([5, Lemma 1.]) *Let  $\omega$  be a multiplier on a locally compact abelian group  $G$ . Suppose  $G$  has a closed subgroup  $H$  such that  $\omega|_H$  is totally skew and type I, and such that the group extension*

$$\{0\} \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow \{0\}$$

*splits. Then there is a complement  $L$  to  $H$  in  $G$  such that (after replacing  $\omega$  by a similar multiplier)  $\omega$  splits as  $\omega|_H \oplus \omega|_L$ .*

### 3. The bundle structure of non-commutative tori

Let  $A_\omega$  be a non-commutative torus of rank  $n$ .  $A_\omega$  is isomorphic to the  $C^*$ -algebra of sections of a locally trivial  $C^*$ -algebra bundle over  $\widehat{S}_\omega$  with fibres the simple twisted group  $C^*$ -algebra  $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$  of a finitely generated discrete abelian group  $\mathbb{Z}^n/S_\omega$  defined by a totally skew multiplier  $\omega_1$  on  $\mathbb{Z}^n/S_\omega$ . Here  $\omega$  is similar to the pull-back of  $\omega_1$ . Then  $\mathbb{Z}^n/S_\omega \cong T \oplus F$  where  $T$  is the maximal torsion subgroup of  $\mathbb{Z}^n/S_\omega$  and  $F$  is a maximal torsion-free subgroup of  $\mathbb{Z}^n/S_\omega$ .

Assume  $T$  is trivial. Then by Lemma 2.5, after replacing  $\omega_1$  by a similar multiplier, we may write  $\mathbb{Z}^n/S_\omega = F$  and  $\omega_1 = \omega_1|_F$ . Let  $\tilde{F}$  be the pull-back of  $F$  under the canonical map of  $\mathbb{Z}^n$  to  $\mathbb{Z}^n/S_\omega$ . Then there is a subgroup  $F'$  such that  $\tilde{F} = F' \oplus S_\omega \cong \mathbb{Z}^n$ . And so  $C^*(\mathbb{Z}^n, \omega) \cong C^*(\tilde{F}, \omega|_{\tilde{F}}) \cong C^*(F', \omega|_{F'}) \otimes C^*(S_\omega) \cong C^*(F, \omega_1|_F) \otimes C^*(S_\omega) \cong C^*(\mathbb{Z}^n/S_\omega, \omega_1) \otimes C^*(S_\omega)$ . This implies that if  $\mathbb{Z}^n/S_\omega$  is torsion-free, then the non-commutative torus  $A_\omega$  has the trivial bundle structure.

**3.1 THEOREM.** ([6, Theorem 2.2.]) *Let  $A_\omega = C^*(u_1, \dots, u_n)$  be a non-commutative torus of rank  $n$ , where  $u_1, \dots, u_n$  are unitary generators satisfying the commutation relations  $u_i u_j u_i^{-1} u_j^{-1} = \exp(2\pi i \theta_{ij})$  (here  $\theta$  is a skew-symmetric  $n \times n$  matrix with real entries). Then  $K_0(A_\omega) \cong K_1(A_\omega) \cong \mathbb{Z}^{2^n - 1}$ , and  $[1_{A_\omega}] \in K_0(A_\omega)$  is primitive.*

*Proof.* The proof is by induction on  $n$ . If  $n = 1$ ,  $A_\omega = C(S^1)$  is abelian, and the result is obvious. So assume that the result is true for all non-commutative tori of rank  $n - 1$ . Write  $A_\omega = C^*(B, u_n)$ , where  $B = C^*(u_1, \dots, u_{n-1})$ . Then the inductive hypothesis applies to  $B$ . Also, we can think of  $A_\omega$  as the crossed product of  $B$  by an action  $\alpha$  of  $\mathbb{Z}$ , where the generator of  $\mathbb{Z}$  corresponds to  $u_n$  and acts on  $B$  by conjugation (sending  $u_j$  to  $u_n u_j u_n^{-1} = \lambda_j u_j$ ,  $\lambda_j = \exp(2\pi i \theta_{nj})$ ). Note that this action is homotopic to the trivial action, since we can homotope  $\theta_{nj}$  to 0. Hence  $\mathbb{Z}$  acts trivially on the  $K$ -theory of  $B$ . The Pimsner-Voiculescu exact sequence for a crossed product gives

$$K_0(B) \xrightarrow{1-\alpha_*} K_0(B) \xrightarrow{\Phi} K_0(A_\omega) \rightarrow K_1(B) \xrightarrow{1-\alpha_*} K_1(B)$$

and similarly for  $K_1$ , where the map  $\Phi$  is induced by inclusion. Since  $\alpha_* = 1$  and since the  $K$ -groups of  $B$  are free abelian, this reduces a split short exact sequence

$$\{0\} \rightarrow K_0(B) \xrightarrow{\Phi} K_0(A_\omega) \rightarrow K_1(B) \rightarrow \{0\}$$

and similarly for  $K_1$ . So  $K_j(A_\omega)$  is free abelian of rank  $2 \cdot 2^{n-2} = 2^{n-1}$ . Furthermore, since the inclusion  $B \rightarrow A_\omega$  sends  $1_B$  to  $1_{A_\omega}$ ,  $[1_{A_\omega}]$  is the image of  $[1_B]$ , which is primitive in  $K_0(B)$  by inductive hypothesis. Hence the image is primitive, since the Pimsner-Voiculescu exact sequence is a split short exact sequence of torsion-free groups.  $\square$

Now we investigate the structure of the fibre  $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$  of  $C^*(\mathbb{Z}^n, \omega)$ .

Let  $G$  be a compactly generated locally compact abelian group and  $\omega_1$  a totally skew multiplier on  $G$ . Then let  $E := G(\omega_1)$  be the extension group of  $G$  by  $\mathbb{T}^1$  defined by  $\omega_1$ . The following result of Poguntke clarifies the structure of the fibres of  $A_\omega$ .

**3.2 THEOREM.** ([11, Theorem 1.]) *Let  $G$  be a compactly generated locally compact abelian group and  $\omega_1$  a totally skew multiplier on  $G$ . Let  $K$  be the maximal compact subgroup of  $E$  and let  $E_\rho$  be the stabilizer of an irreducible unitary representation  $\rho$  of  $K$  restricting on  $\mathbb{T}^1$  to the identity. Then*

$$C^*(G, \omega_1) \cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}),$$

where  $m$  is the associated Mackey obstruction.

This theorem is applied to understand the structure of the twisted group  $C^*$ -algebra  $C^*(\mathbb{Z}^n/S_\omega, \omega_1)$ . Let  $G = \mathbb{Z}^n/S_\omega$ ,  $E = (\mathbb{Z}^n/S_\omega)(\omega_1)$ , and let  $E_\rho$  be the stabilizer of an irreducible unitary representation  $\rho$  of the extension  $K := T(\omega_1|_T)$  of  $T$  by  $\mathbb{T}^1$  defined by  $\omega_1|_T$ , which restricts to the identity on  $\mathbb{T}^1$ . The Mackey method says that  $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F \oplus T, \omega_1)$  is isomorphic to the primitive quotient of  $C^*(E)$  lying over  $\rho$ . Then by Theorem 3.2,

$$C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}).$$

Now by definition,  $E_\rho$  is of index  $|S_{\omega_1|_T}|$  in  $E$ , where  $S_{\omega_1|_T}$  is the symmetry group, a subgroup of  $T$ , of  $\omega_1|_T$ . So

$$\begin{aligned} [E : E_\rho] &= \# \text{ of irreducible } \omega_1|_T\text{-representations of } T \\ &= |S_{\omega_1|_T}|, \end{aligned}$$

and  $\dim(\rho) = \sqrt{|T|/|S_{\omega_1|_T}|}$ . And  $E_\rho/K$  is a subgroup of finite index  $[E : E_\rho]$  in  $E/K$ . Let  $F_\rho$  be the isomorphic image of  $E_\rho/K$  under the natural map of  $E/K$  to  $F$ . Then the  $\{x \in F \mid h_{\omega_1}(x)(y) = 1, \forall y \in S_{\omega_1|_T}\}$  is exactly the  $F_\rho$ , and  $F_\rho$  is a subgroup of finite index  $[E : E_\rho]$  in  $F$ . Let  $J_F = F/F_\rho, J = J_F \oplus S_{\omega_1|_T}$ , and  $T_t = T/S_{\omega_1|_T}$ . Then  $|J_F| = |S_{\omega_1|_T}|$ . Since  $F_\rho$  is a subgroup of  $F$ , we can consider  $J_F \oplus S_{\omega_1|_T}$  as a subgroup of  $(F \oplus T)/F_\rho$ . So  $(\mathbb{Z}^n/S_\omega)/F_\rho$  is isomorphic to  $J_F \oplus T$  and  $((\mathbb{Z}^n/S_\omega)/F_\rho)/J$  is isomorphic to  $T_t$ .

Next, we show that  $C^*(E_\rho/K, m)$  is isomorphic to  $C^*(F_\rho, \omega_1|_{F_\rho})$ . By Theorem 3.2,  $C^*(F_\rho, \omega_1|_{F_\rho}) \cong C^*(F_\rho(\omega_1|_{F_\rho})/\mathbb{T}^1, m_1)$ , where  $m_1$  is the associated Mackey obstruction. Let  $\omega_2$  be a totally skew multiplier on  $T_t$  whose pull-back to  $T$  is similar to  $\omega_1|_T$ . It is enough to show that the Mackey obstruction  $m_2$ , in the isomorphism

$$\begin{aligned} &C^*(F_\rho \oplus T_t, \omega_1|_{F_\rho} \oplus \omega_2) \\ &\cong C^*((F_\rho \oplus T_t)(\omega_1|_{F_\rho} \oplus \omega_2)/T_t(\omega_2), m_2) \otimes C^*(T_t, \omega_2) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C^*(T_t, \omega_2), \end{aligned}$$

is essentially the same as  $m_1$ . But for  $h \in F_\rho$ , the unitary operators  $E'_h$  in [4, XII.1.17] are the same for  $F_\rho$  and for  $F_\rho \oplus T_t$  up to scalar. They give the same Mackey obstructions. So

$$\begin{aligned} C^*((F_\rho \oplus T_t)(\omega_1|_{F_\rho} \oplus \omega_2)/T_t(\omega_2), m_2) &\cong C^*(F_\rho(\omega_1|_{F_\rho})/\mathbb{T}^1, m_1) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}). \end{aligned}$$

And  $C^*(E_\rho/K, m)$  is isomorphic to  $C^*(F_\rho, \omega_1|_{F_\rho})$ .



3.3 COROLLARY.  $C^*(\mathbb{Z}^n/S_\omega, \omega_1) \cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$ .

*Proof.* By Theorem 3.2,

$$\begin{aligned} C^*(\mathbb{Z}^n/S_\omega, \omega_1) &\cong C^*(E_\rho/K, m) \otimes \mathcal{K}(\mathcal{L}^2(E/E_\rho)) \otimes M_{\dim(\rho)}(\mathbb{C}) \\ &\cong C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C}). \end{aligned}$$

Here  $M_{[E:E_\rho]}(\mathbb{C}) \cong M_{|J_F|}(\mathbb{C})$  and  $M_{\dim(\rho)}(\mathbb{C}) \cong M_{\sqrt{|T_t|}}(\mathbb{C})$ . Hence one obtains the result.  $\square$

Note that  $C^*(F_\rho, \omega_1|_{F_\rho})$  is a completely irrational non-commutative torus.

Let  $A_\omega$  be a non-commutative torus. It follows from Corollary 3.3 that  $A_\omega$  is isomorphic to the  $C^*$ -algebra  $\Gamma(\eta)$  of sections of a locally trivial  $C^*$ -algebra bundle  $\eta$  over  $\widehat{S}_\omega$  with fibres  $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$ .

We are now ready to prove the main result.

3.4 THEOREM. *Let  $l$  be a positive integer. Then  $A_\omega \otimes M_l(\mathbb{C})$  is not isomorphic to  $A \otimes M_{kl}(\mathbb{C})$  for any  $C^*$ -algebra  $A$  if  $k \neq 1$ .*

*Proof.* Assume that  $A_\omega \otimes M_l(\mathbb{C})$  is isomorphic to  $A \otimes M_{kl}(\mathbb{C})$  for some integer  $k$  and some  $C^*$ -algebra  $A$ . Then the unit  $1_{A_\omega} \otimes I_l$  in  $A_\omega \otimes M_l(\mathbb{C})$  maps to the unit  $1_A \otimes I_{kl}$  where  $I_d$  denotes the  $d \times d$  identity matrix. Let  $J_{kl,i}$  be the  $kl \times kl$  matrix with all components 0 except for the  $(i, i)$ -component 1. Then the  $1_A \otimes J_{kl,i}$  are projections in  $A \otimes M_{kl}(\mathbb{C})$  such that  $1_A \otimes J_{kl,i} \sim 1_A \otimes J_{kl,j}$  for all  $i, j$ ,  $(1_A \otimes J_{kl,i})(1_A \otimes J_{kl,j}) = 0$  for all  $i, j$  with  $i \neq j$ , and  $\sum_{i=1}^{kl} 1_A \otimes J_{kl,i} = 1_A \otimes I_{kl}$ . By assumption, there are corresponding projections  $p_i$  in  $A_\omega \otimes M_l(\mathbb{C})$  such that  $p_i \sim p_j$  for all  $i, j$ ,  $p_i p_j = 0$  for all  $i, j$  with  $i \neq j$ , and  $\sum_{i=1}^{kl} p_i = 1_{A_\omega} \otimes I_l$ . So

$$[1_{A_\omega} \otimes I_l] = \left[ \sum_{i=1}^{kl} p_i \right] = \sum_{i=1}^{kl} [p_i] = kl[p_1].$$

On the other hand,  $[1_{A_\omega} \otimes I_l] = l[1_{A_\omega}]$ . So

$$l[1_{A_\omega}] = kl[p_1].$$

But by Theorem 3.1 the  $K$ -groups of  $A_\omega$  are torsion-free, so  $[1_{A_\omega}] = k[p_1]$ , which contradicts Theorem 3.1 if  $k \neq 1$ .

Therefore,  $A_\omega \otimes M_l(\mathbb{C})$  is not isomorphic to  $A \otimes M_{kl}(\mathbb{C})$  for any  $C^*$ -algebra  $A$  if  $k \neq 1$ .  $\square$

In particular, one obtains that no non-trivial matrix algebra can be factored out of any rational rotation algebra  $A_{\frac{m}{k}}$ . So every rational rotation algebra has a non-trivial bundle structure. This gives an alternative proof of a result of Brabanter.

Theorem 3.4 implies that if  $A_\omega \otimes M_p(\mathbb{C})$  is isomorphic to  $A_\rho \otimes M_q(\mathbb{C})$ , then  $p = q$ . However, there are non-isomorphic non-commutative tori  $A_\omega$  and  $A_\rho$  such that  $A_\omega \otimes M_p(\mathbb{C})$  is isomorphic to  $A_\rho \otimes M_p(\mathbb{C})$  for some integer  $p$ .

**3.5 COROLLARY.** *Let  $l$  be a positive integer. Then  $A_\omega \otimes M_l(\mathbb{C})$  has a non-trivial bundle structure unless  $\mathbb{Z}^n/S_\omega$  is torsion-free.*

*Proof.* Assume  $A_\omega \otimes M_l(\mathbb{C})$  has the trivial bundle structure, i.e.,  $A_\omega \otimes M_l(\mathbb{C})$  is isomorphic to  $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C(\widehat{S_\omega}) \otimes M_l(\mathbb{C}) \otimes M_k(\mathbb{C})$  where  $M_k(\mathbb{C}) := M_{[E:E_\rho]}(\mathbb{C}) \otimes M_{\dim(\rho)}(\mathbb{C})$ . If  $\mathbb{Z}^n/S_\omega$  is not torsion-free, then  $M_k(\mathbb{C})$  is non-trivial. So  $A_\omega \otimes M_l(\mathbb{C})$  is isomorphic to  $A \otimes M_{kl}(\mathbb{C})$  where  $A$  is isomorphic to  $C^*(F_\rho, \omega_1|_{F_\rho}) \otimes C(\widehat{S_\omega})$ . This contradicts Theorem 3.4 if  $\mathbb{Z}^n/S_\omega$  is not torsion-free.

Therefore,  $A_\omega \otimes M_l(\mathbb{C})$  has a non-trivial bundle structure unless  $\mathbb{Z}^n/S_\omega$  is torsion-free.  $\square$

We have obtained that  $A_\omega \otimes M_l(\mathbb{C})$  has the trivial bundle structure if and only if  $\mathbb{Z}^n/S_\omega$  is torsion-free.

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