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The Relation between Statistics of Input Photons and Photocounts

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The relation between the statistics of photons in input light on a detector and the measured photocounts by the detector is discussed. The averages and variances of the photocounts are compared with the averages and variances of input photons on practical detectors having quantum efficiencies of μ. This comparison was made for three kinds of inputs which include Fock state light, coherent light, and thermal light. The calculations were carried out based on the combined operator model for a detector having less-than-unit quantum efficiency.

I. INTRODUCTION

The measurement of light is one of the fundamental experiments in various areas of science, especially in optics. There are several methods for measuring light, for example, a detection of intensity with one detector, or the intensity correlation detection with two detectors [1], according to what we are trying to get.

Among them the most common measurement is the detection of light intensity with one detector. In this method there are various kinds of detectors used, such as thermal detectors for a high level of light intensity, and photodetectors for a wide range of light intensity. Photodetectors are based either on the emission of photoelectrons from photocathodes, or on changes in the conductivity of semiconductors due to incident radiation, or on photovoltaic devices where voltage is generated by the internal photoeffect.

When it comes to the measurement of very low intensity light, where nonclassical phenomena are more prominent, photodetectors such as PMT's or avalanche photodiodes are employed. Those detectors use the photoelectric effect and have finite quantum efficiencies less than 1. The quantum efficiency is the average ratio of the number of counted photoelectrons for the average photon number striking the detector surface. Therefore the value of the quantum efficiency corresponds to the probability that one photon gives rise to a photoelectron.

If we measure light with a practical detector having quantum efficiency less than one, the statistics, the average and the variance of the measured photocounts can be different from the statistics of the photons in a light beam due to the stochastic process of detection [2]. Although the relation between the statistics of incoming photons and photocounts are already known by the complicated calculation of the distribution functions [3], we'd like to show that the exact relation can be found with a different, but simple calculation of a combined annihilation operator for a practical detector [4].

We'd like to show that the combined operator model can easily be applicable to any kind of photodetection. In Sec. II we derive the general relations between the statistics of the input photons and the measured photocounts by the operator calculation. Sec. III describes the application of the general equations to some kinds of input to arrive at specific expressions. Discussion will be presented in Sec. IV.

II. RELATION BETWEEN THE STATISTICS OF INCOMING PHOTONS AND PHOTOCOUNTS

The model of a photodetector with less than unit quantum efficiency was discussed in detail by Yuen, Shapiro, and Yurke [5,6]. Less than unit quantum efficiency results from the presence of a mechanism by which photons can be lost in a photodetector without generating an observable photocount. As shown in

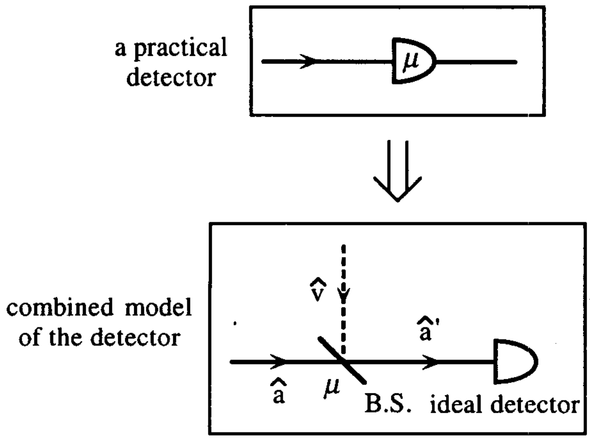

FIG. 1. The schematic diagram of a combined operator model of practical detector with the quantum efficiency μ.

Ref. 5, a detector with less than unit quantum efficiency can be described by a unit quantum efficiency detector in front of which one has placed a beam splitter that lets a fraction μ of the incoming light pass to the detector.

When we denote $\hat{a}$ an annihilation operator for the input mode and $\hat{v}$ an annihilation operator for the vacuum mode, the annihilation operators for the photodetectors D with quantum efficiencies of μ (Fig. 1), can be written as

$$\hat{a}' = \sqrt{\mu}\hat{a} + \sqrt{1-\mu}\hat{v}. \quad (1)$$

Let us assume that we are measuring the photon number n by the detector D with the quantum efficiency of μ. The photocount operator $\hat{n}'$ is as follows,

$$\begin{aligned}\hat{n}' &= \hat{a}'^{\dagger}\hat{a}' \\ &= \mu\hat{a}^{\dagger}\hat{a} + \sqrt{\mu(1-\mu)}(\hat{v}^{\dagger}\hat{a} + \hat{a}^{\dagger}\hat{v}) + (1-\mu)\hat{v}^{\dagger}\hat{v}. \quad (2)\end{aligned}$$

When we perform the calculation of the operator $\hat{v}$ for the vacuum state $|0\rangle$, which generates the expectation values of all terms containing $\hat{v}$, $\hat{v}^{\dagger}$, we are left with the expectation value of photon number for the input state $|\phi\rangle$ as

$$\langle 0, \phi|\hat{n}'|0, \phi\rangle = \mu\langle\phi|\hat{n}|\phi\rangle. \quad (3)$$

Applying the commutation relation $[\hat{v}, \hat{v}^{\dagger}] = 1$, we can express the expectation value of the 2nd order of the number operator $\hat{n}'$:

$$\langle 0, \phi|\hat{n}'^2|0, \phi\rangle = \mu^2\langle\phi|\hat{n}^2|\phi\rangle + \mu(1-\mu)\langle\phi|\hat{n}|\phi\rangle. \quad (4)$$

Thus the variance of photodetection at the output is

$$(\Delta n')^2 = \mu^2(\Delta n)^2 + \mu(1-\mu)\langle\phi|\hat{n}|\phi\rangle, \quad (5)$$

where $(\Delta n)^2$ is the variance of the input photon states. This equation is nothing but the Burgess variance theorem for single light detection, which represents the relation between the variances of photocounts and input photons.

III. APPLICATION OF THE RELATIONS TO THREE KINDS OF INPUTS

Three kinds of light, Fock state light, coherent state light, and thermal light, will be considered as the input $|\phi\rangle$. According to the equations above we only have to calculate the number operator $\hat{n}'$ for the input states.

1. Fock state light

Suppose that we have Fock state light $|n\rangle$ incident on the detector D as the input state $|\phi\rangle$. We can calculate the moments of $\hat{n}$ as

$$\begin{aligned}\langle n|\hat{n}|n\rangle &= n, \\ \langle n|\hat{n}^2|n\rangle &= n^2. \quad (6)\end{aligned}$$

The average and the variance in this measurement are as follows,

$$\begin{aligned}\langle\hat{n}'\rangle &= \mu n, \\ (\Delta n')^2 &= \mu(1-\mu)n. \quad (7)\end{aligned}$$

When we calculate Mandel's Q-factor, the degree of deviation of distribution from the average, we get

$$Q = \frac{(\Delta n')^2}{\langle n'\rangle} - 1 = -\mu. \quad (8)$$

The factor always has a negative value proportional to the detector efficiency, irrespective of input photon numbers.

2. Coherent light

Coherent light $|\alpha\rangle$ is assumed to be incident on the detector D in this case as the input state $|\phi\rangle$. Considering the annihilation operators for the coherent state

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \quad (9)$$

we have the expectations of moments of $\hat{n}$ as

$$\begin{aligned}\langle\alpha|\hat{n}^2|\alpha\rangle &= |\alpha|^4 + |\alpha|^2, \\ \langle\alpha|\hat{n}|\alpha\rangle &= |\alpha|^2, \quad (10)\end{aligned}$$

in correspondence with the commutation rules, e.g. $\langle\hat{n}^2\rangle = \langle(\hat{a}^{\dagger}\hat{a})^2\rangle = \langle\hat{a}^{\dagger}\hat{a}\rangle + \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle$. The average and the variance in this measurement follow from the above equations

$$\begin{aligned}\langle\hat{n}'\rangle &= \mu|\alpha|^2, \\ (\Delta n')^2 &= \mu|\alpha|^2. \quad (11)\end{aligned}$$

The variance is exactly the same as the average, which signifies a Poisson distribution. Thus the Q factor results in zero, irrespective of the efficiency of the detector.

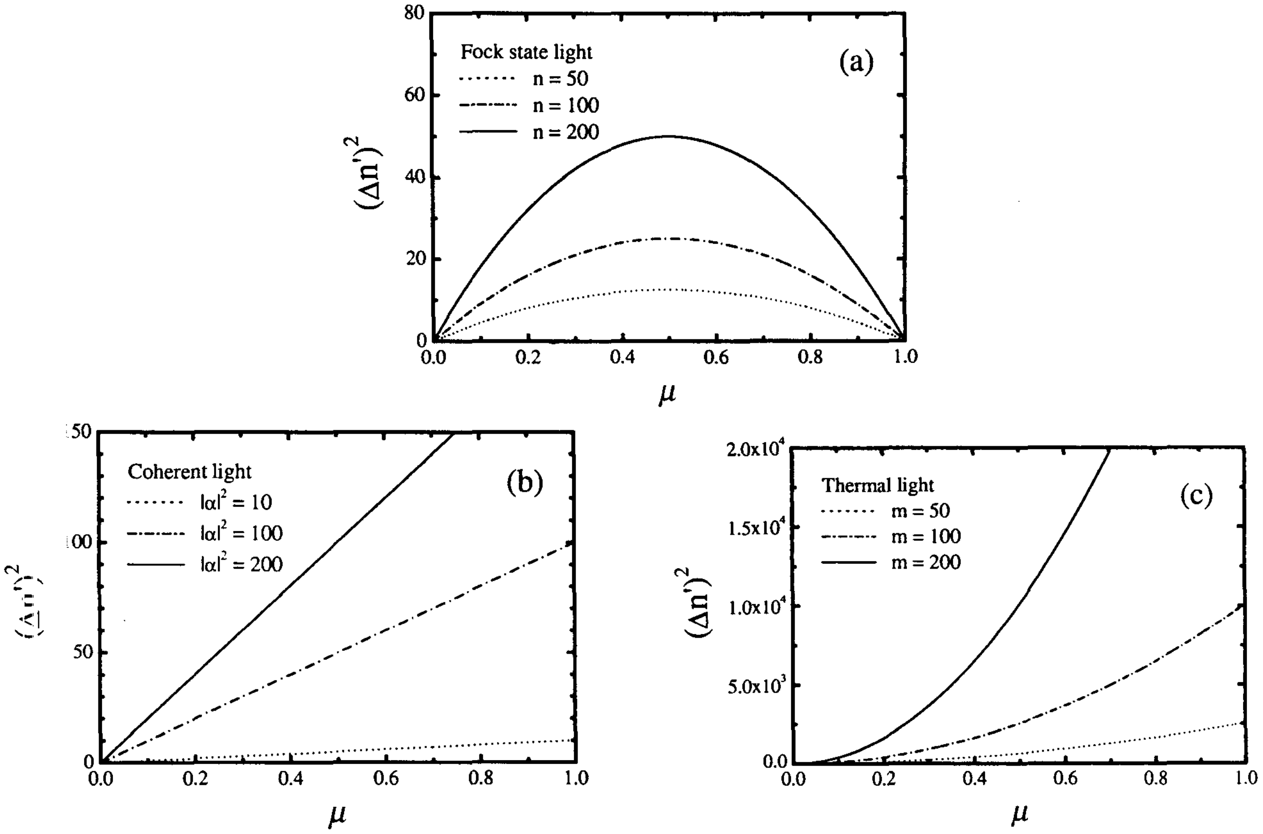

FIG. 2. The variance of the single detection as a function of quantum efficiency μ for the 3 kinds of lights. (a) the Fock state, (b) the coherent state, (c) the thermal light

3. Thermal light

Now we are taking thermal light as the input, which is a single mode with the average photon number m. The density operator for the mixed state can be described as

$$\hat{\rho} = \sum_{n=0}^{\infty} \frac{m^n}{(1+m)^{1+n}} |n\rangle\langle n|. \tag{12}$$

If we measure the photons in the mode with the detector D, we find after some algebra the moments of the photon number operator as below

$$\begin{aligned} \langle \hat{n} \rangle &= \mathrm{Tr}(\hat{\rho}\hat{n}) = m, \\ \langle \hat{n}^2 \rangle &= \mathrm{Tr}(\hat{\rho}\hat{n}^2) = 2m^2 + m. \end{aligned} \tag{13}$$

Here we used the relation in the summation,

$$\sum_{n=0}^{\infty} n(n+1)C^n = \frac{2C}{(1-C)^3}, \tag{14}$$

where C denotes $\frac{m}{1+m}$. Therefore the average and the variance in this measurement are given as

$$\begin{aligned} \langle \hat{n}' \rangle &= \mu m, \\ (\Delta n')^2 &= \mu m + \mu^2 m^2. \end{aligned} \tag{15}$$

Let's check Mandel's Q factor again,

$$Q = \frac{\mu m(1+\mu m)}{\mu m} - 1 = \mu m. \tag{16}$$

As we can see here is this equation, the value of Q is related to the efficiency of the detector employed as well as the input states.

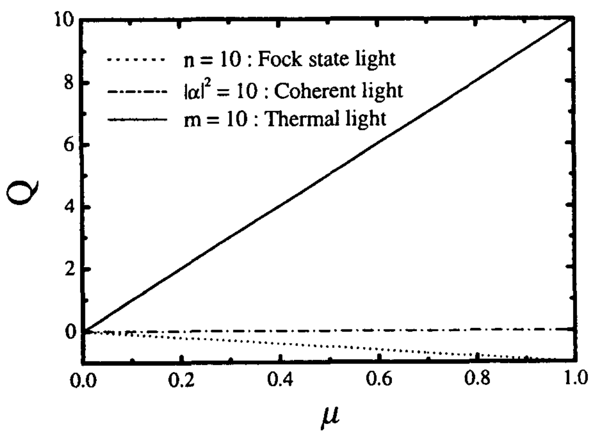

FIG. 3. The Q parameter as a function of the quantum efficiency μ for the 3 kinds of lights.

IV. DISCUSSION

In this paper we have shown that the relation between statistics of measured photocounts and incoming photons can be derived by a combined operator model for a detector for three different kinds of light input. The variances of photocounts by the detectors are depicted in Figs. 2(a)-(c) for the three different inputs; Fock state light, coherent state light, and thermal light. The curves in Fig. 2(a) show the variances versus the quantum efficiency of the detector for the three cases of photon number state of $n = 1, 50, 100$. They have maxima for quantum efficiency of 0.5, showing symmetry for lower and higher values of quantum efficiency.

For coherent light the variances are just proportional to the efficiency of the detector employed, as shown in Fig. 2(b). The straight lines are for the three inputs ($|\alpha|^2 = 10, 100, 200$), which are exactly the same as the averages in the photocounts. Therefore Mandel's Q factor is zero for the coherent state, irrespective of the efficiency of the detectors and the average photon numbers. That's why coherent state light is considered to be a reference for any other light.

In the case of thermal light, the variances depend on the square of the efficiency [Fig. 2(c)]. As in Eq. (15), the average photocount is always greater than the variance, which means a photon bunching effect takes place in the thermal inputs. Fig. 3 represents the Q values for the three different inputs. As we expected, the Q has a negative value in nonclassical Fock state light, while it is positive for chaotic light, i.e. thermal light.

As we mentioned, there are other ways to get the statistical relations between the photocount and input photons such as using Bernoulli's binomial distribution. However, we have shown that those relations can be most easily obtained by the combined operator model for practical detectors in this paper.

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