Historia Mathematica Vol. 13, No. 2, 2000, 151-160

A Note on Representations for Irreducible Characters of Finite Groups

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Abstract

In this paper we prove that for all irreducible complex characters of a finite group, there exist F-representations and a finite degree field extension $F \supseteq \mathbb{Q}$, where $\mathbb{Q}$ is the rational number field.

0. Introduction

Representation theory has its origin in the study of permutation groups and algebras of matrices. It is understood as concrete realizations of axiomatic system of abstract algebras. In particular, the theory of group representations was astonishingly well established by G. Frobenius in the last two decades of nineteenth century and it was realized by both Frobenius and Burnside that the theory plays an important role in the theory of abstract finite groups. G. Frobenius initiated the study of complex representations and characters, and the representation theory of finite groups by matrices over the complex field was mainly the work of Frobenius together with significant contributions by I. Schur. In fact, there are some important results, such as Frobenius theorem which had not been proved without introduction of characters. Along with Frobenius' work, the book by Burnside in 1911 was the first one to give a systemic approach to representation theory. Also it contains many results on abstract groups which were proved using group characters. Among those, the most famous one might be Burnside's $p^a q^b$ theorem. Not long ago, purely group-theoretic proof of the theorem has been obtained by Thompson. Such a proof is of course important for the structure theory of groups. However, it is at least as complicated as the original proof by group characters.

In 1929, N. Noether has observed in [8] that representation theory can be understood by the study of modules over rings and algebras. The representation theory of rings and algebras has led to new insights in the classical theory of semisimple rings and to new investigations of rings with minimum condition centering around Nkayama's theory of Frobenius algebras and quasi-Frobenius rings.

R. Brauer's work on modular representations of finite groups is another major development in representation theory and it has many important applications to the theory of finite groups. It also draws on the representation theory of algebras and suggests new problems on modules and rings with minimum condition.

During 1950-1960, the theory of integral representations of groups and rings initiated some of problems and conjectures both in homological algebra and in the arithmetic of non-commutative rings. After the decade, another subject, algebraic K-theory exerts a strong influence on integral representation theory. In 1970's topological K-theory has been developed and applied to Atiyah-Singer Index theorem. This topological K-theory holds some nice property, so called Bott periodicity and this generalized homology theory is in some sense simple. Algebraic K-theory was guided by the topological K-theory and it can be viewed as an algebraic reformulation of the K-theory. However algebraic K-theory dose not hold the periodicity theorem. So it is more complicated than topological K-theory. Contrast to such complication algebraic K-theory suggested new problems of major importance in representation theory. These solution has led to fresh applications to topology and algebraic number theory.

As a final historical note, there is another interaction between representation theory and geometry. This has occurred in the representation of finite groups of Lie type. In the examples where character tables were known, there were certain representations that were difficult to construct by standard methods. In a dramatic breakthrough, Deligne and Lusztig found general methods for constructing these and other representations through a systematic study of actions of groups on algebraic varieties.

The purpose of this paper is to prove some properties of representations for irreducible complex characters[Propositions 2.1 and Theorems 2.2, 2.3].

Throughout this paper, every group G is a finite group and every character of a group G means a complex character. Let $Irr(G)$ be the set of all irreducible complex characters of G.

1. Preliminaries

Let F be a field and let A be an F-vector space which is also a ring with unity 1. If $(cx)y = c(xy) = x(cy)$ for all $c \in F$ and $x, y \in A$, then A is an *F-algebra*. In this paper, A means a finite dimensional F-algebra (i. e. as an F-vector space, it has a finite dimension).[7]

Let A be an F-algebra and let V be a finite dimensional F-vector space. For all $x, y \in A$, $v, w \in V$ and $c \in F$, if the followings are hold;

(a) $x(v+w) = xv + xw$,

(b) $(x+y)v = xv + yv$,

(c) $x(yv) = (xy)v$,

(d) $(cx)v = c(xv) = x(cv)$,

(e) $1v = v$,

then V is called an *A-module*.[7]

Definition 1.1. Let V be a nonzero A-module. Then V is *irreducible* if it has only two submodules $\{0\}$ and V, where A is an F-algebra.

Definition 1.2. Let V be an A-module. If for every submodule W of V, there exists another submodule U of V such that $V = W \oplus U$ ($\oplus$ means direct sum), then V is *completely reducible*.

Lemma 1.3. Let A be an F-algebra and let $V_1, \cdots, V_n$, $W_1, \cdots, W_m$ be A-module. If $V = V_1 \oplus \cdots \oplus V_n$, $W = W_1 \oplus \cdots \oplus W_m$ be A-modules, then

$$Hom_A(V, W) \cong \begin{bmatrix} Hom_A(V_1, W_1) \cdots Hom_A(V_n, W_1) \\ \vdots \qquad\qquad \vdots \\ Hom_A(V_1, W_m) \cdots Hom_A(V_n, W_m) \end{bmatrix}$$

is an F-vector space.

In particular, if $V^{(n)} = V \oplus \cdots \oplus V$(n copies), then $End_A(V^{(n)}) \cong Mat_n(End_A(V))$ is an F-algebra.

Proof. For $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, let $\varepsilon_j : V_j \to V$ be the natural injection defined by $\varepsilon_j(v_j) = (0, \cdots, 0, v_j, 0, \cdots, 0)$ and let $\pi_i : W \to W_i$ be the natural projection defined by $\pi_i(w_1, \cdots, w_n) = w_i$. Then ε_j and π_i are A-homomorphism. If for each i and j, $\phi_{ij} \in Hom_A(V_j, W_i)$, then we can define a $\phi \in Hom_A(V, W)$ by for all $v_i \in V_i$,

$$\phi(v) = \phi(v_1 + \cdots + v_n) = \begin{bmatrix} \phi_{11} \cdots\cdots \phi_{1n} \\ \vdots \qquad \vdots \\ \phi_{m1} \cdots\cdots \phi_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= (\phi_{11}(v_1) + \cdots + \phi_{1n}(v_n) + \cdots + \phi_{m1}(v_1) + \cdots + \phi_{mn}(v_n)).$$

Conversely, if $\phi \in Hom_A(V, W)$, then $\phi_{ij} = \pi_i \circ \phi \circ \varepsilon_j \in Hom_A(V_j, W_i)$ and

$$\begin{bmatrix} \phi_{11} \cdots\cdots \phi_{1n} \\ \vdots \qquad \vdots \\ \phi_{m1} \cdots\cdots \phi_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \phi(v_1 + \cdots + v_n) = \phi(v), \text{ for all } v_i \in V_i.$$

So the proof is complete. □

Lemma 1.4. Let A be an F-algebra and let V be a completely reducible A-module. If $B = End_A(V)$, then for each $v \in V$ and $f \in End_B(V)$, there is an $a \in A$ such that $av = f(v)$.

Proof. Since V is a completely reducible and $Av = \{av \mid a \in A\}$ is a submodule of V, there is some submodule W such that $V = Av \oplus W$. Let $\pi : V \rightarrow Av$ be the projection, then $\pi \in End_A V = B$. And since $f \in End_B(V)$ and $\pi(v) = v$ thus $f(v) = f(\pi(v)) = \pi(f(v)) \in \pi(V) = Av$. □

Proposition 1.5. Let A be an F-algebra and let V be an irreducible A-module. If $B = End_A(V)$, $f \in End_B(V)$, and $v_1, \cdots, v_n \in V$, then there is an $a \in A$ such that $av_i = f(v_i)$ $(i = 1, \cdots, n)$.

Proof. Denote $V^{(n)} = V \oplus \cdots \oplus V$ (n copies) and define $f^{(n)}: V^{(n)} \rightarrow V^{(n)}$ by $f^{(n)}(v_1 + \cdots + v_n) = f(v_1) + \cdots + f(v_n)$ for all v_i, $f(v_i) \in i$th summand V. Denote $B' = End_A(V^{(n)})$. Given any $\phi \in B'$, by Lemma 1.3 there are $\phi_{ij} \in End_A(V) = B$. Hence we see $f^{(n)}(\phi(v_1 + \cdots + v_n)) = \phi(f^{(n)}(v_1 + \cdots + v_n))$, $f^{(n)} \in End_{B'}(V^{(n)})$. By Lemma 1.4, there is an $a \in A$ such that $a(v_1 + \cdots + v_n) = f^{(n)}(v_1 + \cdots + v_n) = f(v_1) + \cdots + f(v_n)$ for all v_i, $f(v_i) \in i$th summand V. Thus $av_i = f(v_i)$ for all i $(i = 1, \cdots, n)$. □

Definition 1.6. An algebra A is *semisimple algebra* if its regular module A° is completely reducible(i. e. $A = L_1 \oplus \cdots \oplus L_m$, where L_i is irreducible left ideal).

Lemma 1.7. Let A be an finite F-algebra. Then A is semisimple algebra iff every A-module is completely reducible.

Proof. Suppose that A is semisimple. Note that left ideals of A are exactly the submodule of A°. Let $A = L_1 \oplus \cdots \oplus L_m$(L_i is irreducible left ideal) and let $V = Fv_1 \oplus \cdots \oplus Fv_n$. Then $V = AV = \sum_{i=1}^{m} \sum_{j=1}^{n} L_i v_j$. For each i, j, the map $f : L_i \rightarrow L_i v_j$ defined by $f(a) = av_j$ is a surjective A-homomorphism. Since L_i is irreducible, either $L_i v_j = \{0\}$ or $L_i v_j \cong_A L_i$. Thus V is sum of finite irreducible A-modules, so is completely reducible.

Conversely, suppose that every A-module is completely reducible. Then A is clearly semisimple. □

Proposition 1.8. Let A be a finite semisimple F-algebra. Then the followings hold.

(1) A has only finitely many nonisomorphic irreducible left ideals $L_1, \cdots, L_s$.

(2) If A_i is the sum of all left ideals of A isomorphic to L_i, then A_i is a two-sided ideal in A and a finite simple F-algebra.

(3) $A = \oplus_{i=1}^{s} A_i$

(4) If $1 = e_1 + \cdots + e_s$, then A_i has a unit element e_i, $e_i A = A_i = Ae_i$. ($i \neq j$ implies $A_i A_j = 0$).

Proof. Let $\{L_i \mid i \in I\}$ be the set of representatives of all isomorphism classes of simple left ideals, and define $A_i = \sum \{L \subseteq A \mid L \cong_A L_i\}$ for all $i \in I$. Now let V be an irreducible A-module. Then $AL_i V = L_i V$ is a submodule of V, and so either $\{0\}$ or V. If it is V, choose $v \in V$, $L_i v \neq \{0\}$. $L_i v$ is an A-submodule of V, so $L_i v = V$. Thus $L \cong_A V$ since L is irreducible as A-module. Hence $i \neq j$ implies $A_i A_j = \{0\}$. Of course $A = \sum_{i \in I} A_i$, so $A_i \subseteq A_i A = A_i A_i \subseteq AA_i = A_i$, proving that each A_i is a two-sided ideal. Write $1 = \sum_{i \in I} e_i$, $e_i \in A_i$, so all but finitely many e_i are 0, say

$1 = e_1 + \cdots + e_s$ ($e_i \neq 0$, $e_i \in A$). If $x \in A_k$, $k \in I - \{1, \cdots, s\}$, then $x = 1x = (e_1 + \cdots + e_s)x = 0$ proving (1).

If $0 = x_1 + \cdots + x_s$, $x_i \in A_i$, then $0 = e_j(x_1 + \cdots + x_s) = e_j x_1 + \cdots + e_j x_s = e_j x_j = (e_1 + \cdots + e_s)x_j = 1x_j = x_j$ for all j, so (3) holds. Thus we see e_j is a unit in A_j, so (4) has been checked. Any simple left ideal of A_i is a simple left ideal of A, and so by construction is isomorphic to some L_i. Hence (2) holds. □

Let D be a division ring, D^{op} denotes the *opposite division ring* to D, that is, D^{op} has the same underlying set and addition as D. But a multiplication $\circ$ is given by $x \circ y = yx$. Let A be a finite F-algebra, and let $a \in A$. The map $\psi_a : A \to A$ defined by $\psi_a(x) = xa$ is an A-homomorphism. Thus the map $A^{op} \to End_A(A)$ defined by $a \to \psi_a$ is an algebra isomorphism. So $End_A(A) = End_A(A^o) \cong A^{op}$.[5] Now let D be an F-division algebra and let V be an n-dimensional D-vector space. Let $\{v_1, \cdots, v_n\}$ be a D-basis for V, then $V = Dv_1 \oplus \cdots \oplus Dv_n \cong D \oplus \cdots \oplus D$ (as A-module). Thus by Lemma 1.3, $End_D(V) \cong Mat_n(End_D(D)) \cong Mat_n(D^{op})$.

Proposition 1.9. Let A be a finite simple algebra, V an irreducible A-module. If $D = End_A(V)$, then $A \cong End_D(V) \cong Mat_n(D^{op})$ where $n = \dim_D V \leq \dim_F V$.

In particular, if F is an algebraically closed field, then $A \cong End_F(V) \cong Mat_n(F)$ where $n = \dim_F V$.

Proof. By Schur Lemma[7], $D = End_A(V)$ is a finite F-algebra and V is a D-vector space. If we define a map $\phi : A \to End_D(V)$ by $\phi(a)(v) = av$, then ϕ is an F-algebra homomorphism. Since A is simple algebra $Ker\,\phi = \{0\}$ and thus ϕ is one to one. Now let $f \in End_D(V)$ and let $\{v_1, \cdots, v_n\}$ be D-basis for V. Then by Proposition 1.5 there exists an $a \in A$ such that $f(v_i) = av_i$ for $i = 1, \cdots, n$. Thus for all $v \in V$, we have $f(v) = av = \phi(a)(v)$, so $f = \phi(a) \in im\phi$. Hence $End_D(V) \subseteq im\phi$, thus $End_D(V) \subseteq im\,\phi \cong A$. Therefore $A \cong End_D(V) \cong Mat_n(D^{op})$.

In particular, if F is an algebraically closed field, then $D = End_F(V) \cong F$. Thus $A \cong End_F(V) \cong Mat_n(F)$, $n = \dim_F V$. □

Theorem 1.10. Let G be a finite group and let F be an algebraically closed field of characteristic not dividing $|G|$. Then $F[G] \cong Mat_{n_1}(F) \oplus \cdots \oplus Mat_{n_s}(F)$ where $|G| = n_1^2 + \cdots + n_s^2$. $F[G]$ has exactly s-nonisomorphic irreducible modules of dimension $n_1, \cdots, n_s$, and s is the number of conjugacy classes of G.

Proof. By Maschke's Theorem[4, 5, 7]. $F[G]$ is a semisimple algebra. Let V_i be the irreducible $F[G]$-module. Then by Proposition 1.8, $F[G]$ has s-nonisomorphic irreducible $F[G]$-modules $V_1, \cdots, V_s$ and thus by Proposition 1.9, $F[G] \cong Mat_{n_1}(F) \oplus \cdots \oplus Mat_{n_s}(F)$, $n_i = \dim_F V_i$. Thus $\dim F[G] = \dim Mat_{n_1}(F) + \cdots + \dim Mat_{n_s}(F)$ and so we have $|G| = n_1^2 + \cdots + n_s^2$. Let Z be the center of $F[G]$. Since $Z(Mat_{n_i}(F)) = \{aI \mid a \in F\} \cong F$, therefore $\dim_F Z(F[G]) = s$. For each conjugacy class $\mathscr{C}_i$ of G, let $C_i = \sum_{x \in \mathscr{C}_i} x \in F[G]$. If $g \in G$, then $g^{-1} C_i g = C_i$ so all $C_i \in Z$. The C_i are obviously F-linearly independent elements of $F[G]$. If $\sum_{g \in G} a_g g \in Z$ for some $a_g \in F$, then for any $h \in G$, $\sum a_g g = h^{-1}(\sum a_g g) h = \sum a_g h^{-1} g h$. This means $a_g = a_{h^{-1}gh}$ and implies that $\sum a_g g$ is an F-linear combination of the C_i. $\{C_i\}$ is an F-basis of Z, so $s = \dim_F Z =$ the number of C_i's $=$ number of conjugacy classes of G. □

Let F be a field, and let G be a finite group. A representation $T : G \to GL_n(F)$ is a group homomorphism, where $GL_n(F)$ is the general linear group of $n \times n$ matrices on F. In this case n is called the degree of the representation T. It is easy to construct modules from representations and representations from modules.[7]

Let M be an $F[G]$-module with a finite dimension n as an F-vector space where $F[G]$ is a group algebra. Let a representation $T : G \to GL_n(F)$ correspond to M, i. e. for each $v \in M$ and $g \in G$, $vg = vT(g)$. If M is an irreducible $F[G]$-module, then the representation T is said to be irreducible. We put $\theta(g) = tr T(g)$, the trace of $T(g)$, then θ is called a *character* afforded by T. If T is irreducible then θ is also said to be irreducible. Let $Irr(G)$ be the set of all irreducible complex characters.

2. Main Results

Let $F \subseteq E$ be a field extension and let T be an F-representation of a group G into a group of nonsingular matrices over F that, of course, are also nonsingular over E. We may, therefore, view T as an E-representation of G. As such we denote it by T^E. If T_1 and T_2 are similar F-representations of G, then $T_1{}^E$ and $T_2{}^E$ are similar and it follows that if T corresponds to the $F[G]$-module V, then there is a uniquely defined $E[G]$-module V^E (Note $V^E = V \otimes_F E$). We shall not, however, need to refer to V^E again, since it is usually easier to work with the representation T^E. The F-representation T of G may be extended by linearity to obtain a representation of $F[G]$, which we shall continue to call T. Under this convention, the $E[G]$-representation T^E is an extension of the $F[G]$-representation T. If T^E is irreducible, then clearly so is T. However, T^E may well reducible, even if T is irreducible.

Proposition 2.1. Let F be a subfield of the complex number field C, and let $\theta \in Irr(G)$. If there exists an F-representation T_θ that affords θ, then these representations T_θ are pairwise nonsimilar and every irreducible F-representation of G is similar to one of the T_θ .

Proof. Let $\theta, \psi \in Irr(G)$ and let T_θ and T_ψ be the F-representations of G that afford θ and ψ, respectively. Suppose T_θ and T_ψ are similar. Then there is a non-singular P such that $T_\theta = P^{-1}T_\psi P$. Thus for all $g \in G$, we have $\theta(g) = trT_\theta(g) = tr(P^{-1}T_\psi(g)P) = trT_\psi(g) = \psi(g)$ so $\theta = \psi$. Therefore the different characters are afforded by nonsimilar representations.

Now let T be an irreducible F-representation of G that affords θ. Since $F \subseteq$ C, T is also a C-representation of G and hence θ is a C-character of G. Thus we may put $\theta = \theta_1 + \cdots + \theta_s$ ($\theta_i \in Irr(G)$). If T_{θ_i} is the corresponding F-representation that afford θ_i for each $i = 1, \cdots, s$, then for $g \in G$, $T(g)$ is similar to a matrix transformation of the form

$$\begin{bmatrix} T_{\theta_1}(g) & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdot \cdots & T_{\theta_s}(g) \end{bmatrix}.$$

But T is an irreducible F-representation. Thus T is similar to one of the T_{θ_i}. □

Theorem 2.2. Let F be the field of all algebraic elements of $\mathbb{C}$. Then there exist F-representations T_θ for all $\theta \in Irr(G)$.

Proof. Let T be the $\mathbb{C}$-representation of G that affords θ. Let M and N be the corresponding irreducible $\mathbb{C}[G]$-module and $F[G]$-module, respectively. Write $\mathbb{C}[G] = J_1 \oplus \cdots \oplus J_r$ and $F[G] = I_1 \oplus \cdots \oplus I_r$, where J_i and I_j are minimal two-sided ideals and r is the number of conjugacy classes in G. Note that $\mathbb{C}[G] \cong F[G] \otimes_F \mathbb{C}$ which is the tensor product of algebras,[9] By Wedderburn-Artin theorem[8], we get $J_i = \oplus_{j=1}^{f_i} M_{ij}$, and $\dim_{\mathbb{C}} J_i = f_i^2$ where $f_i = \dim_{\mathbb{C}} M_{ij}$ and $M_{i1} \cong M_{i2} \cong \cdots \cong M_{if_i}$ are irreducible one-sided ideals. Similarly, we have $I_i = \oplus_{j=1}^{f_i'} N_{ij}$, and $\dim_{\mathbb{C}} I_i = f_i'^2$ where $f_i' = \dim_{\mathbb{C}} N_{ij}$ and $N_{i1} \cong N_{i2} \cong \cdots \cong N_{if_i'}$. Note that $N_{ij} \otimes_F \mathbb{C}$ are one-sided ideals in $F[G] \otimes_F \mathbb{C}$. By uniqueness, up to an isomorphism, of the representation of $\mathbb{C}[G]$ as a direct sum of minimal one-sided ideals it is easy to see that $f_i = f_i'$ and $M_{ij} \cong N_{ij} \otimes_F \mathbb{C}$. In particular, $M \cong N \otimes_F \mathbb{C}$ for some irreducible(minimal) one-sided ideals of $F[G]$. Let $f = \dim_{\mathbb{C}} M = \dim_F N$. Consider $S : G \to GL_f(F)$ be the irreducible F-representation corresponding to N, then

$$S(g) = \begin{bmatrix} s_{11}(g) \cdots\cdots s_{1f}(g) \\ \vdots \qquad\qquad \vdots \\ s_{f1}(g) \cdots\cdots s_{ff}(g) \end{bmatrix}.$$

If $\{e_1, \cdots, e_f\}$ is a basis for N, then $e_i g = \sum_{j=1}^{f} e_j s_{ji}(g)$. Since $\{e_1 \otimes 1, \cdots, e_f \otimes 1\}$ is a basis for $N \otimes_F \mathbb{C}$, S can be viewed as a $\mathbb{C}$-representation of G corresponding to $N \otimes_F \mathbb{C} \cong M$. Thus T and S are similar $\mathbb{C}$-representations of G since they induce isomorphic $\mathbb{C}[G]$-module structures. It follows that they afford the same character θ. □

Theorem 2.3. There exists a finite degree field extension $F \supseteq \mathbb{Q}$ such that F-representations T_θ exist for all $\theta \in Irr(G)$, where $\mathbb{Q}$ is the rational number field.

Proof. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. By Theorem 2.2, for each $\theta \in Irr(G)$, there is a $\overline{\mathbb{Q}}$-representation T_θ that affords θ. Let A be the set of all entries in $\overline{\mathbb{Q}}$, then A is a finite set of algebraic numbers. Let $F = \mathbb{Q}[A]$. Then F is the field generated by finitely many elements of $\mathbb{C}$ that are algebraic over $\mathbb{Q}$. It follows that $[F:\mathbb{Q}] < \infty$. Moreover T_θ is, in fact, F-representation that affords θ. □

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