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Strong Consistent Estimator for the Expectation of Fuzzy Stochastic Model

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Abstract. This paper concerns with the consistent estimator for the fuzzy expectation of a random variable taking values in the space $F(R^p)$ of upper semicontinuous convex fuzzy subsets of R^p with compact support. We introduce the concept of a fuzzy sample mean and show that the fuzzy sample mean is a strong consistent estimator for the fuzzy expectation. Some examples are given to illustrate the main result.

Key Words : *Fuzzy random variables, Fuzzy sample means, Consistent estimators, Strong law of large numbers.*

1 . INTRODUCTION

In recent years, there has been increasing interest in statistical inference for fuzzy stochastic model since Puri and Ralescu (1986) introduced the concept of fuzzy random variables. Schnatter (1992) introduced the concept of fuzzy sample mean and fuzzy sample variance in order to discuss the generalization of statistical methods to fuzzy data. Yao and Hwang (1996) studied point estimation for random sample with one vague data. Recently, Grzegorzewski (2000) proposed a definition of fuzzy test for testing statistical hypotheses with vague data and Korner (2000) also suggested a method to test hypotheses about expectation of fuzzy random variable.

This paper concerns with the strong consistent estimator for the fuzzy expectation of a random variable taking values in the space $F(R^p)$ of upper semicontinuous convex fuzzy subsets of R^p with compact support. To this end, strong laws of large numbers (for short, SLLN) for fuzzy random variables should be considered. The SLLN for fuzzy random variables was obtained by Klement et al. (1986), Inoue (1991), Molchanov (1999), Joo and Kim (preprint) and etc. Our result generalizes the results of earlier works to the case of a more general setting.

2 . PRELIMINARIES

Let $K(R^p)$ denote the family of non-empty compact convex subsets of the Euclidean space R^p. For $A, B \in K(R^p)$, let us denote

$$\delta(A,B) = \sup_{a \in A} \inf_{b \in B} |a - b|,$$

where $|.|$ denotes the Euclidean norm. Then the space $K(R^p)$ is metrizable by the Hausdorff metric defined by

$$h(A,B) = \max\{\delta(A,B), \delta(B,A)\}.$$

A norm of $A \in K(R^p)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a \in A} |a|.$$

It is well-known that $K(R^p)$ is complete and separable with respect to the Hausdorff metric h (See Debreu (1966)). The addition and scalar multiplication on $K(R^p)$ are defined as usual:

$$\begin{aligned} A \oplus B &= \{a + b : a \in A, b \in B\} \\ \lambda A &= \{\lambda a : a \in A\} \end{aligned}$$

for $A, B \in K(R^p)$ and $\lambda \in R$.

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space. A set-valued function $X : \Omega \to K(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B) = \{\omega : X(\omega) \cap B \neq \emptyset\}$$

is a measurable set. It is well-known that the measurablility of X is equivalent to the measurability of X considered as a map from Ω to the metric space $K(R^p)$ endowed with the Hausdorff metric h. A set-valued function $X : \Omega \to K(R^p)$ is called a random set if it is measurable.

A random set X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random set X is defined by

$$E(X) = \{Ef : f \in L(\Omega, R^p) \,\text{and}\, f(\omega) \in X(\omega) \,a.s.\}.$$

It is well-known that if $E\|X\| < \infty$, then $E(X) \in K(R^p)$, and that

$$\begin{aligned} E(X_1 \oplus X_2) &= E(X_1) \oplus E(X_2), \\ E(\lambda X) &= \lambda E(X). \end{aligned}$$

The following SLLN for random sets was proved by Artstein and Vitale (1975) and generalized by Artstein and Hansen (1985).

Theorem 2.1. *Let $\{X_n\}$ be a sequence of independent and identically distributed random sets. If $E\|X_1\| < \infty$, then*

$$\lim_{n\to\infty} h(\frac{1}{n} \bigoplus_{i=1}^{n} X_i, EX_1) = 0 \ a.s.$$

3 . MAIN RESULTS

In what follows, $\overline{A}$ denotes the closure of a set $A \subset R^p$. Let $F(R^p)$ denote the family of all fuzzy sets $u : R^p \to [0,1]$ with the following properties;

(1) u is normal, i.e., there exists $x \in R^p$ suchthat $u(x) = 1$;

(2) u is upper semicontinuous;

(3) u is a convex fuzzy set,i.e., $u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y))$ for $x, y \in R^p$ and $\lambda \in [0,1]$;

(4) $supp\, u = \overline{\{x \in R^p : u(x) > 0\}}$ is compact.

For a fuzzy set u in R^p, the α-level set of u is defined by

$$L_\alpha u = \begin{cases} \{x : u(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ supp\ u, & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that $u \in F(R^p)$ if and only if $L_\alpha u \in K(R^p)$ for each $\alpha \in [0,1]$. The linear structure on $F(R^p)$ is defined as usual;

$$(u \oplus v)(z) = \sup_{x+y=z} \min(u(x), v(y)),$$

$$(\lambda u)(z) = \begin{cases} u(z/\lambda), & \lambda \neq 0 \\ I_{\{0\}}, & \lambda = 0, \end{cases}$$

for $u, v \in F(R^p)$ and $\lambda \in R$, where $I_{\{0\}}$ is the indicator function of $\{0\}$. Then it is known that $L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$ and $L_\alpha(\lambda u) = \lambda L_\alpha u$ for each α.

Lemma 3.1. *For $u \in F(R^p)$, we define*

$$f_u : [0,1] \longrightarrow (K(R^p), h), f_u(\alpha) = L_\alpha u.$$

Then the followings hold;

(1) f_u *is left continuous on* $(0,1]$,

(2) f_u *has right-limits on* $[0,1)$*and* f_u *is right-continuous at* 0.

Proof. See Lemma 2.2 of Joo and Kim (2000). □

We denote $\overline{\bigcup_{\beta>\alpha} L_\beta u}$ by $L_{\alpha^+}u$. Then the right limit of f_u at α is $L_{\alpha^+}u$. Now we define, for $J \subset [0,1]$,

$$w_u(J) \;=\; \sup_{\alpha_1,\alpha_2\in J} h(L_{\alpha_1}u, L_{\alpha_2}u) \tag{3.1}$$

then it follows that for $0 \le \alpha < \beta \le 1$,

$$w_u(\alpha,\beta) \;=\; w_u(\alpha,\beta] \;=\; h(L_{\alpha^+}u, L_\beta u),$$

and

$$w_u[\alpha,\beta) \;=\; w_u[\alpha,\beta] \;=\; h(L_\alpha u, L_\beta u).$$

Lemma 3.2. *For each* $u \in F(R^p)$ *and* $\epsilon > 0$, *there exist a partition* $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ *of* $[0,1]$ *such that*

$$w_u(\alpha_{i-1},\alpha_i) < \epsilon,\; i = 1,2,\ldots,r. \tag{3.2}$$

Proof. See Lemma 2.3 of Joo and Kim (2000). □

Now, in order to generalize the Hausdorff metric on $K(R^p)$ to $F(R^p)$, we define the two metrics d_1, d_∞ on $F(R^p)$ by

$$d_1(u,v) = \int_0^1 h(L_\alpha u, L_\alpha v)\, d\alpha$$

$$d_\infty(u,v) = \underset{0\le\alpha\le1}{\rightarrow} \sup h(L_\alpha u, L_\alpha v).$$

Also, the norm of u is defined as

$$\|u\| = d_\infty(u,0) = \sup_{x\in L_0 u} |x|,$$

Then it is well-known that $F(R^p)$ is complete with respect to two metrics d_1 and d_∞ , and that $F(R^p)$ is separable with respect to d_1 but not with respect to d_∞(see Klement et al. (1986)).

A fuzzy set valued function $X : \Omega \to F(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B)(\omega) = \sup_{x\in B} X(\omega)(x)$$

is measurable when considered as a function from Ω to $[0,1]$. It is well-known that X is measurable if and only if for each $\alpha \in [0,1], L_\alpha X$ is measurable as a set-valued function. A fuzzy set valued function $X : \Omega \to F(R^p)$ is called a fuzzy random variable if it is measurable .

A fuzzy random variable X is called integrably bounded if $E\|X\| < \infty$. The expectation of integrably bounded random fuzzy set X is a fuzzy subset of R^p defined by

$$E(X)(x) = \sup\{\alpha \in [0,1] : x \in E(L_\alpha X)\}.$$

It is well-known that if $E\|X\| < \infty$, then $E(X) \in F(R^p)$, and $L_\alpha E(X) = E(L_\alpha X)$ for all $\alpha \in [0,1]$, and that

$$E(X_1 \oplus X_2) = E(X_1) \oplus E(X_2),$$
$$E(\lambda X) = \lambda E(X).$$

The fuzzy random variables $X_1, X_2, \ldots, X_n$ are called independent if for every closed subsets $B_1, B_2, \ldots, B_n$ of R^p, the random variables

$$X_1^{-1}(B_1), X_2^{-1}(B_2), \ldots, X_n^{-1}(B_n)$$

are independent in the usual sense. Then it follows that $X_1, X_2, \ldots, X_n$ are independent if and only if the Borel σ-fields

$$\sigma\{L_\alpha X_1 : \alpha \in [0,1]\},\ \sigma\{L_\alpha X_2 : \alpha \in [0,1]\}, \ldots, \sigma\{L_\alpha X_n : \alpha \in [0,1]\}$$

are independent in the usual sense. Also, the fuzzy random variables $X_1, X_2, \ldots, X_n$ are said to have the same fuzzy distribution as X if for every closed subsets B of R^p, the random variables

$$X_1^{-1}(B), X_2^{-1}(B), \ldots, X_n^{-1}(B)$$

have the same distribution as $X^{-1}(B)$ in the usual sense. It follows that if the fuzzy random variables $X_1, X_2, \ldots, X_n$ have the same fuzzy distribution as X, then for each α, the random sets $L_\alpha X_1, L_\alpha X_2, \ldots, L_\alpha X_n$ have the same distribution as $L_\alpha X$. The fuzzy random variables $X_1, X_2, \ldots, X_n$ are called a fuzzy random sample from the population with fuzzy distribution as a fuzzy random variable X if they are independent and have the same fuzzy distribution as X. For a fuzzy random sample $X_1, X_2, \ldots, X_n$, the fuzzy sample mean is defined by

$$\bar{X}_n = \frac{1}{n} \bigoplus_{i=1}^{n} X_i.$$

It follows that the fuzzy sample mean is an unbiased estimator for the fuzzy expectation $E(X)$, i.e., $E(\bar{X}_n) = E(X)$. A strong law of large numbers by Klement et al.(1986) implies that the fuzzy sample mean $\bar{X}_n$ is a strong consistent estimator for the fuzzy expectation $E(X)$ with respect to the metric d_1. The next theorem shows that the fuzzy sample mean $\bar{X}_n$ is a strong consistent estimator for the fuzzy expectation $E(X)$ with respect to the metric d_∞. This result is a generalization of the result obtained by Joo and Kim (preprint).

Theorem 3.3. *Let $\{X_n\}$ be a fuzzy random sample from the population with fuzzy distribution as a fuzzy random variable X. If $E\|X\| < \infty$, then*

$$\lim_{n\to\infty} d_\infty(\bar{X}_n, EX) = 0 \ a.s..$$

Proof. Let $\epsilon > 0$ be given. Then applying Lemma 3.2 to $u = E(X)$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r$ of $[0,1]$ such that

$$h(L_{\alpha_{i-1}^+}E(X), L_{\alpha_i}E(X)) < \epsilon,\ i = 1, 2, \ldots, r. \tag{3.3}$$

If $0 < \alpha \le 1$, then $\alpha_{k-1} < \alpha \le \alpha_k$ for some k. Since $L_\alpha \bar{X}_n \subset L_{\alpha_{k-1}^+}\bar{X}_n$ and $L_\alpha E(X) \supset L_{\alpha_k}E(X)$, we have, by (3.3),

$$\begin{aligned}\delta(L_\alpha \bar{X}_n, L_\alpha E(X)) &\le \delta(L_{\alpha_{k-1}^+}\bar{X}_n, L_{\alpha_k}E(X)) \\ &\le h(L_{\alpha_{k-1}^+}\bar{X}_n, L_{\alpha_k}E(X)) \\ &\le h(L_{\alpha_{k-1}^+}\bar{X}_n, L_{\alpha_{k-1}^+}E(X)) + \epsilon.\end{aligned}$$

Similarly, since $L_\alpha E(X) \subset L_{\alpha_{k-1}^+}E(X)$ and $L_\alpha \bar{X}_n \supset L_{\alpha_k}\bar{X}_n$, we obtain

$$\begin{aligned}\delta(L_\alpha E(X), L_\alpha \bar{X}_n) &\le \delta(L_{\alpha_{k-1}^+}E(X), L_{\alpha_k}\bar{X}_n) \\ &\le h(L_{\alpha_{k-1}^+}E(X), L_{\alpha_k}\bar{X}_n) \\ &\le h(L_{\alpha_k}E(X), L_{\alpha_k}\bar{X}_n) + \epsilon.\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}d_\infty(\bar{X}_n, E(X)) \le & \underset{1\le k\le r}{\rightarrow} \max h(L_{\alpha_{k-1}^+}\bar{X}_n, E(L_{\alpha_{k-1}^+}X)) \\ &+ \underset{1\le k\le r}{\rightarrow} \max h(L_{\alpha_k}\bar{X}_n, E(L_{\alpha_k}X)) + \epsilon.\end{aligned}$$

By Theorem 2.1, we obtain

$$\overline{\lim}_{n\to\infty} d_\infty(\bar{X}_n, E(X)) \le \epsilon \ a.s..$$

Since ϵ is arbitrary, this completes the proof. □

Example 1. Let $u \in F(R^p)$ be fixed and $X(\omega) = u(x - Y(\omega))$ be a fuzzy random variable with the same fuzzy distribution as the population, where Y is a random vector taking values in R^p with $E|Y| < \infty$. Since

$$L_\alpha X(\omega) = Y(\omega) + L_\alpha u,$$

we have

$$E(L_\alpha X) = EY + L_\alpha u.$$

Hence, $E(X)(x) = u(x - EY)$. Now if $\{Y_n\}$ is a random sample from the population with distribution of Y, and $X_n(\omega) = u(x - Y_n(\omega))$, then $\{X_n\}$ is a fuzzy random sample from the population with fuzzy distribution of X, and the fuzzy sample mean is

$$\bar{X}_n = u(x - \bar{Y}_n),$$

where $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$ is the usual sample mean of $Y_1, Y_2, \ldots, Y_n$. Hence, by the above theorem,

$$\lim_{n\to\infty} d_\infty(u(x - \bar{Y}_n), u(x - EY)) = 0 \ a.s.$$

Example 2. The triangular fuzzy number u in R is a fuzzy set $u : R \to [0,1]$ defined by

$$u(x) = \begin{cases} \frac{x-l}{m-l}, & \text{if } l < x < m \\ 1, & \text{if } x = m \\ \frac{r-x}{r-m}, & \text{if } m < x < r \\ 0, & \text{otherwise,} \end{cases}$$

where $l < m < r$. The above triangular fuzzy number u is denoted by $< l, m, r >$. Let $F_t(R)$ be the family of all triangular fuzzy numbers in R. If $X : \Omega \to F_t(R)$ and $X(\omega) =< l(\omega), m(\omega), r(\omega) >$, then it follows that X is a fuzzy random variable if and only if l, m, r are random variables in the usual sense. Furthermore, X is integrably bounded if and only if l, r are integrable, in this case, $E(X) =< E(l), E(m), E(r) >$. If $X_1, X_2, \ldots, X_n$ constitute a fuzzy random sample from the population with fuzzy distribution as $X(\omega) =< l(\omega), m(\omega), r(\omega) >$, then we can write

$$X_i =< l_i(\omega), m_i(\omega), r_i(\omega) >, i = 1, 2, \ldots, n,$$

where $l_i, m_i, r_i, i = 1, 2, \ldots, n$ are random samples of l, m, r, respectively. Then the fuzzy sample mean is given by

$$\bar{X}_n =< \bar{l}_n, \bar{m}_n, \bar{r}_n >,$$

where $\bar{l}_n = \frac{1}{n}\sum_{i=1}^n l_i, \bar{m}_n = \frac{1}{n}\sum_{i=1}^n m_i, \bar{r}_n = \frac{1}{n}\sum_{i=1}^n r_i$.

Since

$$d_\infty(\bar{X}_n, E(X)) = \max(|\bar{l}_n - E(l)|, |\bar{m}_n - E(m)|, |\bar{r}_n - E(r)|),$$

it follows that by the classical strong law of large numbers,

$$d_\infty(\bar{X}_n, E(X)) \longrightarrow 0 a.s..$$

This coincides with the Theorem 3.3.

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