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Bivariate Data Analysis for the Lifetime and the Number of Indicative Events of a System †

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Abstract. This research considers a system which has an ultimate terminal event such as death, critical failure, bankruptcy together with a certain indicative events (temporary malfunction, special treatment, kind of defaults) that frequently occurs before the terminal event comes to the system. Some investigation of a model for the corresponding bivariate data of the system have been done with an explanation of the situation in terms of two continuous variables instead of continuous-discrete variables and some other properties. Also an analysis has been carried out to evaluate the effect of intermediate observation of occurrence of indicative event so that the result can be used for a possible suggestion of an intermediate observing schedule.

Key Words : *bivariate data, lifetime, number of indicative events, terminal event.*

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1. INTRODUCTION

Common problems that arise in longitudinal studies are two kinds of modeling. The first one is to model the effects of some explanatory factors on the number of occurrences of a given event that have occured in some time interval. For example, one may wish to model the number of transfusions given to a bone marrow transplant patient in the course of their recovery, the number of admissions to the hospital of a patient with a serious illness, or the number of doses of a drug given to a patient with a heart attack in the emergency room. Also we can find many examples in computers, cars, electronic products and in business areas. The other is to model the lifetime that can be observed only one time. A lot of researches have been done in reliability and survival analysis area.

In this article, we consider the following situation consisting of the above two cases. That is about the systems which have both a critical failure event(main event) such as death, bankruptcy and an indicative event which occurs repeatedly prior to the occurrence of the critical failure event. For example one may experience number of temporary malfuctions prior to the system failure or number of doses of a drug prior to a patient s death.

The occurrences of such indicative events provide some information about the occurrence of the main event and the status information of the system gives an idea about the number of indicative events to occur in the future.

The important characteristics to be considered in the model for such situation is how to explain the dependence structure between a continuous variable representing the time to the main event and a discrete variable for the number of the indicative events prior to the occurrence of the main event. In this paper we use the joint model of time to the main event and number of indicative event using a shared frailty model to explain mutual dependency. This frailty model is introduced by Vaupel etc.(1979) who considered it to explain individual heterogeneity in a univariate survival model. There are many researchers to exploit frailty models in several fields such as reliability, survival analysis, demography, etc(Lee and Klein(1988, 1989), Lindley and Singpurwalla(1986), Lawless(1987), Nayak(1987), Bandyopadhyay and Basu(1990), Gupta and Gupta(1990), and Whitmore and Lee(1991), Park and Klein(1997), Park and Lee(1998)). Specially Lawless(1987) and Park and Klein(1997) have investigated the very similar situation with ours. Lawless studied the exact occurring times of indicative events(new tumors) during the prespecified time interval and Park and Klein applied the model to the breast cancer patients data. Following these two results we consider a more realistic situation that allows intermediate observations of the occurrence of the indicative event while Klein and Park's model used only the counting data obtained at the end of the study, that is, when the system fails or is censored.

Section 2 investigates the characteristics of the suggested model, so that a physical interpretation is attached to a key part of the model about which the two previous studies didn't make any explanation. Section 3 describes data structure and likelihood function of situation. To discuss the effect of increasing the number of intermediate observations of occurrences of indicative events on the inference in terms of the standard error. In Section 4, a conclusion is made.

2. THE MODEL

2.1 BASIC MODEL

In this section we present a model for the joint distribution of the time to the main event, we will call it death, X, and the number of indicative events which occur up to time t, $N(t)$. We let W denote a shared random effect which has a common multiplicative effect on both the rate at which death is occurring and the rate at which the indicative events are occurring. This random effect, which is allowed to vary from individual to individual, is analogous to a frailty in the usual multivariate survival modeling(See Klein(1992)) and is put into a model for unobserved heterogeneity in modeling count data (See Lawless (1987)). It represents common genetic, disease specific or environmental factors that were not measured on the individual which are affecting both the number of indicative events and the time to death. Here we assume that W has a Gamma distribution with a mean of 1 and a variance θ. That is,

$$f(w) = \frac{w^{1/\theta-1}e^{-w/\theta}}{\Gamma(1/\theta)\theta^{1/\theta}}. \tag{2.1}$$

Given the value of $W = w$, we assume that the time to death of an individual with a covariate Z_d follows a Weibull distribution with hazard rate

$$h(t|w) = w\alpha t^{\alpha-1}\exp(\phi Z_d), \ t \geq 0, \ \alpha \geq 0. \tag{2.2}$$

For the number of events we assume, given $W = w$ (and a system covariate Z_c), that $N(t)$ follows a Poisson process with a rate

$$\eta(t|w) = w\beta t^{\beta}\exp(\gamma Z_c) \ , \quad t \geq 0, \ \beta \geq 0 \tag{2.3}$$

Given $W = w$, we assume that $N(t)$ and X are independent. To study properties of this model we first need to note that $N(t)$ is only observable as long as $t \leq X$, and that $N(t) = N(X)$ when $t > X$. Thus we have, given $W = w$, with some abuse of notation,

$$\begin{aligned} &P[X = t, \ N(s) = k|w] \\ &= w\exp(\phi Z_d)\alpha t^{\alpha-1}\exp[-w\exp(\phi Z_d)t^{\alpha}] \\ &\times \frac{[w\exp(\gamma Z_c)\min(s,t)^{\beta}]^k \exp[w\exp(\gamma Z_c)\min(s,t)^{\beta}]}{k!} \end{aligned} \tag{2.4}$$

Also

$$
\begin{aligned}
&P[X > t, N(s) = k|w] \\
&= \int_t^\infty P[X = u, N(s) = k|w]\, du\ , \quad t \geq s \\
&= \int_t^s P[X = u, N(u) \ = \ k|w]\, du \\
&\quad + \ P[X > s, N(s) \ = \ k|w]\ , \quad t < s
\end{aligned} \tag{2.5}
$$

To find the unconditional distribution of X and $N(\cdot)$, we take the expectation of (2.4) with respect to W. For $\theta > 0$, this yields,

$$
\begin{aligned}
&P[X = t,\ N(s) = k] \\
&= \frac{\Gamma(\theta^{-1} + k + 1)}{k!\Gamma(\theta^{-1})}[\exp(\phi Z_d)\alpha\theta t^{\alpha-1}][\exp(\gamma Z_c)\theta \min(s,t)^\beta]^k \\
&\times\ [1 + \exp(\phi Z_d)\theta t^\alpha + \exp(\gamma Z_c)\theta \min(s,t)^\beta]^{-(\theta^{-1}+k+1)}
\end{aligned} \tag{2.6}
$$

When θ is equal to zero, W is equal to 1 almost surely and then X and $N(\cdot)$ are independent Weibull and Poisson random variables, respectively. From (2.5) we have

$$
\begin{aligned}
&P[X > t,\ N(s) = k] \\
&= \frac{\Gamma(\theta^{-1} + k)}{k!\Gamma(\theta^{-1})}[\exp(\gamma Z_c)\theta s^\beta]^k \\
&\quad \times\ [1 + \exp(\phi Z_d)\theta t^\alpha + \exp(\gamma Z_c)\theta s^\beta]^{-(\theta^{-1}+k)} \\
&\ +\ I(s-t)\frac{\Gamma(\theta^{-1} + k + 1)}{k!\Gamma(\theta^{-1})}\exp(\phi Z_d)(\exp(\gamma Z_c))^k\alpha\theta^{k+1} \\
&\quad \times \int_t^s u^{k\beta+\alpha-1}(1 + \exp(\phi Z_d)\theta u^\alpha + \exp(\gamma Z_c)\theta u^\beta)^{-(\theta^{-1}+k+1)}\, du
\end{aligned} \tag{2.7}
$$

where $I(a)$ is an indicator function with 1 when $a > 0$ and 0 when $a \leq 0$. For this model one can show that the marginal distribution of X is a univariate Burr distribution with survival function

$$
S(t) = (1 + \exp(\phi Z_d)\theta t^\alpha)^{-\frac{1}{\theta}} \tag{2.8}
$$

To find the marginal distribution of $N(s)$ we need to compute $P[X \geq, N(s) = k]$.

From (2.8) we see that

$$\begin{aligned}
P[N(s)=k] &= \int_0^s P[X=u, N(u)=k]du \;+\; P[X>s, N(s)=k] \\
&= \frac{\Gamma(\theta^{-1}+k+1)}{k!\Gamma(\theta^{-1})}\exp(\phi Z_d)(\exp(\gamma Z_c))^k \alpha\theta^{k+1} \\
&\times \int_0^s u^{k\beta+\alpha-1}(1+\exp(\phi Z_d)\theta u^\alpha + \exp(\gamma Z_c)\theta u^\beta)^{-(\theta^{-1}+k+1)}du \\
&+ \frac{\Gamma(\theta^{-1}+k)}{k!\Gamma(\theta^{-1})}(\exp(\gamma Z_c)\theta s^\beta)^k \\
&\times (1+\exp(\phi Z_d)\theta s^\alpha + \exp(\gamma Z_c)\theta s^\beta)^{-(\theta^{-1}+k)} \qquad (2.9)
\end{aligned}$$

2.2 CHARACTERISTICS OF THE MODEL

In order to get some physical explanations of the model, we first define the following notations.

$$\begin{aligned}
A(\theta,k) &= \frac{\Gamma(\theta^{-1}+k)}{\Gamma(\theta^{-1})k!} \\
p(t,s) &= \frac{1 \;+\; \exp(\phi Z_d)\theta t^\alpha}{1 \;+\; \exp(\phi Z_d)\theta t^\alpha \;+\; \exp(\gamma Z_c)\theta s^\beta} \\
q(t,s) &= 1-p(t,s) \\
c(t,s) &= 1 \;+\; \exp(\phi Z_d)\theta t^\alpha \;+\; \exp(\gamma Z_c)\theta s^\beta \\
S(t) &= \text{survival distribution of } X
\end{aligned}$$

(1) An explanation of $p(t,s)$

$p(t,s)$ plays an important probability role in the conditional probability distribution of $N(s)$ given $X > t$, as in the study of Lawless(1987) and Park and Klein(1997).

[1] Joint survival distribution of two components

We consider the systems with two components, A and B, say whose life times are Q and X respectively. If we assume that their conditional hazard rates, given w, are

$$\begin{aligned}
h_Q(t|w) &= w\exp(\gamma Z_c)\beta t^{\beta-1} \\
h_X(t|w) &= w\exp(\phi Z_d)\alpha t^{\alpha-1}
\end{aligned}$$

and that W has the same distribution as (2.1). Then joint survival distribution of A and B is as follows;

$$\begin{aligned} P(X>t,Q>s) &= E_W(P(X>t,Q>s|W)) \\ &= \int_0^\infty \exp(-w\exp(\phi Z_d)t^\alpha - w\exp(\gamma Z_c)s^\beta)\ f(w)\ dw \\ &= (1+\exp(\phi Z_d)\theta t^\alpha + \exp(\gamma Z_c)\theta s^\beta)^{-\frac{1}{\theta}} \end{aligned}$$

So the conditional survival distribution of A given B becomes a function of $p(t,s)$.

$$\begin{aligned} P(Q>s|X>t) &= \left[\frac{(1+\exp(\phi Z_d)\theta t^\alpha)}{(1+\exp(\phi Z_d)\theta t^\alpha + \exp(\gamma Z_c)\theta s^\beta)}\right]^{\theta^{-1}} \\ &= [p(t,s)]^{\theta^{-1}} \end{aligned} \quad (2.10)$$

Also we compute $P(Q>s|X=t)$ from these results.

$$\begin{aligned} P(Q>s|X=t) &= \left[\frac{(1+\exp(\phi Z_d)\theta t^\alpha)}{(1+\exp(\phi Z_d)\theta t^\alpha + \exp(\gamma Z_c)\theta s^\beta)}\right]^{\theta^{-1}+1} \\ &= [p(t,s)]^{\theta^{-1}+1} \end{aligned} \quad (2.11)$$

[2] The ratio of two hazard rates of A

From (2.10) and (2.11), $p(t,s)$ is written as the following ratio

$$p(t,s) = \frac{P(Q>s|X=t)}{P(Q>s|X>t)} = \frac{\left[\frac{\partial P(Q>s,X>t)}{\partial t}\right]}{P(Q>s,X>t)} \frac{P(X>t)}{f_X(t)} \quad (2.12)$$

and division of the first term in right side of (2.12) by $P(Q>s)$ gives another expression.

$$p(t,s) = \frac{h_X(t|Q>s)}{h_X(t)} \quad (2.13)$$

From this result we can give two kinds of explanation to $p(t,s)$. We assume that Q is occurring time of an indicative event and X is occurring time of the main event to explain in the view of our study. Then (2.12) is the ratio of the probability that an indicated event doesn't occur at time s when a main event occurs at time t to the probability that an indicated event doesn't occur at time s when a main event doesn't occur at time s. Here it is trivial that the ratio of two conditional probabilities, $p(t,s)$, is less than 1 when we consider the fact that there is a positive relation between Q and X because of sharing environment.

Equation (2.13) can be interpreted as a value representing how much the hazard rate the main event is decreased by the information that the indicative event does not occur up to time s.

(2) Characteristics of conditional probability of indicative events

[1] $P(N(s) = k \mid X > t)$ is calculated from equations (2.7), (2.8), (2.9).

$$P(N(s) = k \mid X > t) = A(\theta, k)[q(t,s)]^k [p(t,s)]^{\theta^{-1}} R(t, \max(t,s)) \tag{2.14}$$

where

$$\begin{aligned} R(b,a) &= \left[\frac{q(a,s)}{q(t,s)}\right]^k \left[\frac{p(a,s)}{p(t,s)}\right]^{\theta^{-1}} \left[\frac{S(a)}{S(t)}\right] \\ &- (1+k\theta)\int_b^a \left(\frac{q(u,u)}{q(t,s)}\right)^k \left(\frac{p(u,u)}{p(t,s)}\right)^{\theta^{-1}} p(u,u)\frac{S(u)}{S(t)}\, du \end{aligned}$$

For $s \leq t$ the conditional distribution of $N(s)$ given $X > t$ follows a negative binomial distribution with parameters $1/\theta$ and success probability $p(t,s)$. So the expected value and variance are as follows;

$$\begin{aligned} E(N(s)|X > t) &= \frac{q(t,s)}{[\theta p(t,s)]} \\ V(N(s)|X > t) &= \frac{q(t,s)}{\theta[p(t,s)]^2} \end{aligned}$$

[2] Another interpretation of $P(N(s) = k|X > t)$

Under the fact that the conditional distribution is a negative binomial distribution with success probability $p(t,s)$ when $s < t$, $p(t,s)$ equals to $P(Q > s|X > t)$ if W has a gamma distribution with a mean of 1 and a variance 1. So the conditional probability can be written as

$$P(N(s) = k|X > t) \;=\; (1 - P(Q > s|X > t))^k (P(Q > s|X > t)) \tag{2.15}$$

The fact that number of event $N(s)$ equals to k at time s when a main event doesn't occur at time $t(s < t)$ can be interpreted as the fact that k systems have been found to have the indicative event occurred prior to s before one which has no the indicative event up to s is observed in an experiment of investigating the occurring time of the indicative event among the systems which do not experience the main event at time t.

[3] Another interpretation of $N(u) = k$ when $X = u$

When $s < t$ and variance of frailty W is 1, $P(N(s) = k|X = t)$ is as follows;

$$\begin{aligned} P(N(s) = k|X = t) &= k(q(s,t))^k (p(s,t))^2 \\ &= k(P(Q < s|X > t))^k \; (P(Q > s|X > t))^2 \end{aligned}$$

The number of indicative events which occurred prior to u when the system failed at time u is equivalent to the number of the systems confirmed as those with the indicative event occurred prior to time u before the two systems are found to have no indicative event at time u. Also we get

$$p(t,s) = \frac{1}{k}\frac{P(N(s)=k|X=t)}{P(N(s)=k|X>t)} \tag{2.16}$$

[4] Characteristic in situation of $s > t$

When system life time X is greater than t, $P(N(s)=k|X>t)$ and expectation $E(N(s)|X>t)$ are as follow if variance of frailty W equal to 1;

$$\begin{aligned}
&P(N(s)=k|X>t) \\
&\quad = P(N(s)=k|X \geq s)P(X>s|X \geq t) \\
&\qquad + \int_t^s P(N(u)=k|X=u)f_{X|X>t}(u)\,dt \\
&E(N(s)|X>t) = \frac{q(s,s)}{p(s,s)}P(X \geq s|X>t) \\
&\qquad + 2\int_t^s \frac{q(u,u)}{p(u,u)}f_{X|X>t}(u)\,du
\end{aligned} \tag{2.17}$$

Specially when $\alpha = \beta$, the second term of right side in the above equation is as follows;

$$\begin{aligned}
&2\int_t^s \frac{q(u,u)}{p(u,u)}f_{X|X>t}(u)\,du \\
&\quad = \frac{u}{\exp(\phi Z_d)}[2P(t<X<s|X>t) \\
&\qquad + P(X \geq s|X>t)P(X \geq s) - P(X>t)]
\end{aligned}$$

So the expectation is expressed by the cumulative distribution function.

$$\begin{aligned}
&E(N(s)|X \geq t) \\
&= \left(\exp(\gamma Z_c)s^{\alpha} + \frac{\exp(\gamma Z_c)}{\exp(\phi Z_d)}\right)P(X>s)P(X>s|X>t) \\
&- \frac{u}{\exp(\phi Z_d)}[P(X>t) - 2P(t<X<s|X>t)]
\end{aligned} \tag{2.18}$$

3. DATA ANALYSIS

3.1 DATA AND LIKELIHOOD FUNCTION

Here is the notation for the data of the individual i.

(i) Total number of observations of indicative events : $n_i + 1$

(ii) Occurring time of main event : X_i , Censoring time : L_i

(iii) Time to end of the study : $T_i = \min(X_i , L_i)$

(iv) Censoring indicator : δ_i ($\delta_i = 1$ death ; $\delta_i = 0$ censor)

(v) Time of observation of the indicative event : s_{ij} ($j = 0, 1, 2, \cdots , n_i + 1$)

(vi) Number of indicative events between s_{ij-1} and s_{ij} :k_{ij} ($j = 1, 2, \cdots , n_i + 1$)

Here s_i0 in (v) and (vi) is set to 0 and $s_{i,n+1}$ is set to T_i. The likelihood function is written as

$$\begin{aligned} LL \;=\; & \sum_{i=1}^{m} \Big[\ln\Gamma(g_i) + \sum_{l=1}^{n_i+1} k_{il} \;\log(s_{il}^{\beta} - s_{il-1}^{\beta}) + \delta_i\beta Z_{d_i} \\ & + k_i\gamma Z_{c_i} - g_i \ln(y_i) - m\log(\Gamma(1/\theta)) - \frac{m}{\theta}\log(\theta) \\ & + \delta_i log(\alpha) + (\alpha-1)\delta_i \log(t_i) - \sum_{l=1}^{n_i+1} \log(k_{il}!) \Big] \end{aligned} \tag{3.1}$$

where m is the sample size and

$$g_i = \frac{1}{\theta} \;+\; k_i \;+\; \delta_i$$

and

$$y_i = \frac{1}{\theta} \;+\; t_i{}^{\alpha}\exp(\phi Z_{d_i}) \;+\; t_i{}^{\beta}\exp(rZc_i)$$

The usual maximum likelihood estimation process for the parameters α, β, θ, ϕ, γ has been discussed by Klein and Moeschberger(1997), Park and Lee(1998).

Owing to the exploited methods for inference, we investigate the effect of the increased number of observations of occurrence of the indicative event on the inferences.

3.2 EXPERIMENT

The basic set up for the parameters is as following :

	Case								
	a	b	c	d	e	f	g	h	i
α	3.0	3.0	3.0	2.5	2.5	2.5	3.0	3.0	3.0
β	1.5	2.0	2.5	1.5	2.0	2.5	1.5	2.0	2.5
θ	2.0	2.0	2.0	2.0	2.0	2.0	0.5	0.5	0.5
common : $\varphi_0 = -4.6$ $\varphi_1 = 0.69$ $\gamma_0 = -0.69$ $\gamma_1 = 0.78$									

We assume that the time to death(main event) follows a weibull distribution and the censoring time has an exponential distribution with mean equal to the marginal mean of the system lifetime. One covariate affecting the hazard rates is assumed to be common with two values 0 and 1 indicating the two groups.

The times for the observations of occurrence of the indicative event are determined at the middle of the marginal mean of the system lifetime in case of two observations and at 1/3 and 2/3 of the mean in case of three observations and so on up to the case of ten observations. The following table shows the various standard errors for each case. Here notation $(a - \alpha)$ means the estimated standard error of the estimate of α under the setting case a.

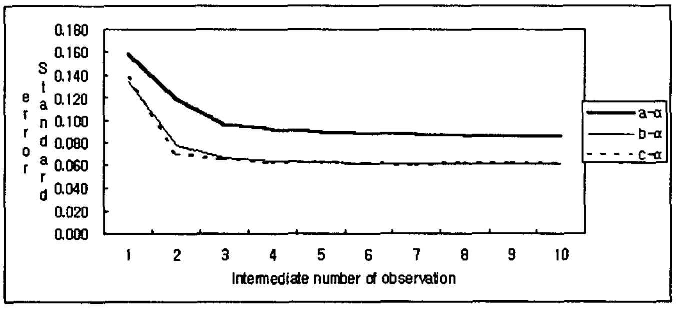

Figure 3.1

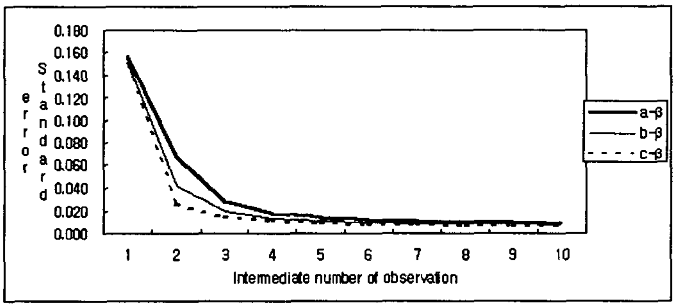

Figure 3.2

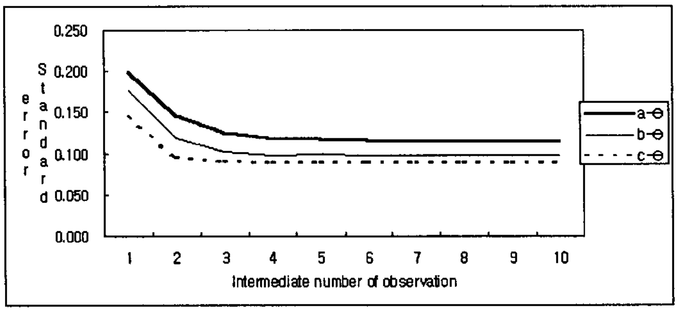

Figure 3.3

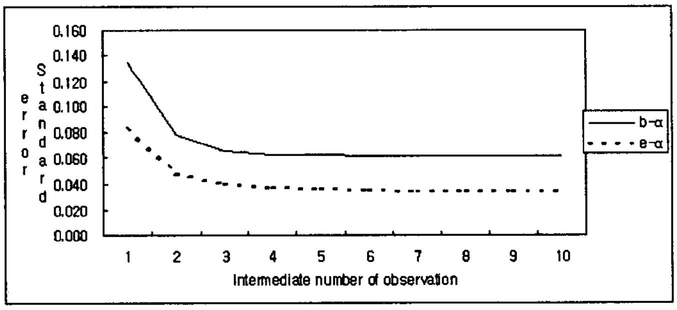

Figure 3.4

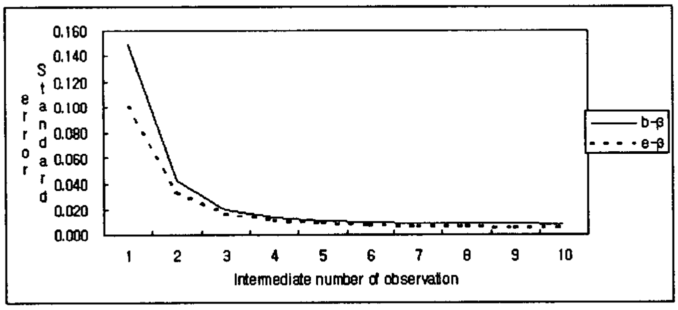

Figure 3.5

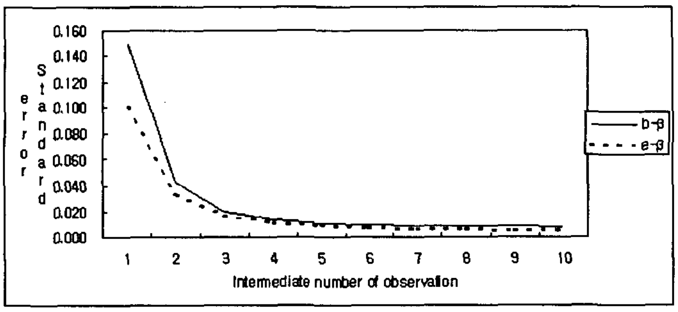

Figure 3.6

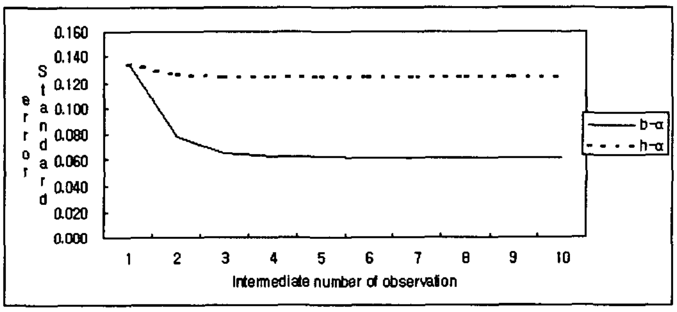

Figure 3.7

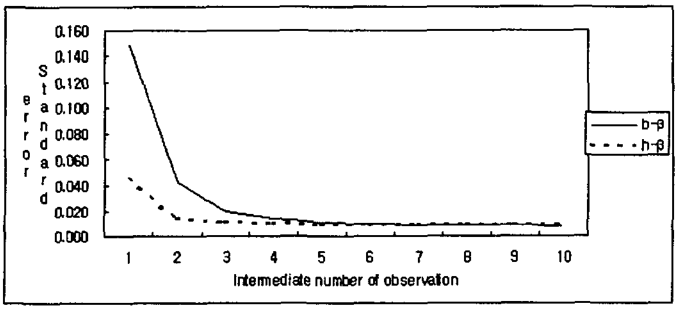

Figure 3.8

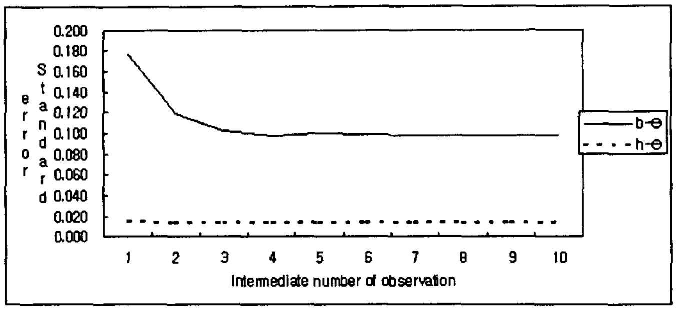

Figure 3.9

[1] Influence of shape parameter β

Figure (3.1) to Figure (3.3) shows the decreasing rates of the standard error of the estimates of parameters α, β, θ when $\alpha = 3$ and $\theta = 2$ and $\beta = 1.5,\ 2.0,\ 2.5$. All three figures confirm that the standard errors are decreasing very fast when the number of observations is increased from 1 to 2 in similar patterns. As is expected the standard errors of $\hat{\beta}$ are decreasing much more than those of $\hat{\alpha}$ and $\hat{\theta}$.

[2] Influence of shape parameter α

In Figure (3.4) to Figure (3.6), comparing the cases of ($\alpha = 3,\ \beta = 2,\ \theta = 2$) and ($\alpha = 2.5,\ \beta = 2,\ \theta = 2$) more decrease is found when $\alpha = 3$, which suggests to increase the number of observations when the hazard rate of the main event gets large.

[3] Influence of environment parameter θ

In Figure (3.7) to Figure (3.9), showing the decreasing patterns for cases of ($\alpha = 3,\ \beta = 2,\ \theta = 2$) and ($\alpha = 3,\ \beta = 2,\ \theta = 0.5$) , all three estimates $\hat{\alpha}$, $\hat{\beta}$, $\hat{\theta}$ tend to have large effect when $\theta = 2$ so that the more the heterogeneity unexplained by covariates the more intermediate observations are recommended.

4. CONCLUSION

Most of systems have various kinds of indicative events such as repairable failures, temporary malfunctions, doses of drugs, transfusions followed by main events such as unrepairable failure, death, bankruptcy which may or may not be dependent of the indicative ones. In this research, we consider a mixed type bivariate distribution with dependence induced by an environmental factor explained through a frailty model. Data considered consists of discrete type one for the numbers of occurrence of the indicative event observed before the main event occurs and continuous type one for the time to the main event. Our consideration can be regarded an extension of the studies done by Lawless(1987) and Park and Klein(1997) up to the multi-observation of occurrences of the indicative event(Park and Klein) during the random interval(Lawless). The two aspects discussed mainly in this paper are about physical interpretation of the model and about the effect of the intermediate observations of indicative events. This interpretations given to the probability involved in the conditional distribution function of the number of indicative events is driven based on the joint distribution of two continuous lifetime random variables whose dependence is also induced by a frailty model so that one can get a deep insight about the situation described in the model. The effect analysis suggests that more intermediate observations are needed if the degree of heterogeneity unexplained by covariates gets bigger and the hazard rate of the main event becomes larger confirming the estimate of β has the largest benefit with a decreased standard error as is expected. Further research of connecting the increased number of intermediate observations and the increased cost for the observations is aiming at an optimal experimental design. Another effort is being given to an application of the model to the real world financial data and motor lifetime data which have lots of noises.

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