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Anaysis of System Lifetime Subject to Two Classes of Random Shocks

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Abstract. We consider a system whose inherent life follows an Erlang distribution, which is subject to two heterogeneous random shocks. Minor shocks arrive according to a renewal process and each causes the system to fail independently with a certain probability. A major shock whose interarrival times follow an Erlang distribution causes the system to fail with probability one. The Laplace transform of the distribution of the time to system failure is derived in a functional form of the Laplace transform of the interarrival time distribution of minor shocks. An algorithm is given for the computation of the moments of the time to system failure.

Key Words : *system lifetime, Erlang distribution, heterogeneous random shocks, renewal process, interarrival times, Laplace transform.*

1. INTRODUCTION

Consider a system which has a random life time following an Erlang distribution and is subject to two independent and heterogeneous random shocks (minor and major shocks). Generally, the term "shock" refers to environment causes such as noise or vibration which may give rise to a system failure. An Erlang distribution provides a good description of the life length having an increasing failure rate with time; and it also has various shapes of probability density function according to its parameters. A minor shock whose interarrival times are independent and identically distributed with an arbitrary distribution may cause the system to fail with a certain probability. A major shock whose arrival time is assumed to follow an Erlang distribution makes the system out of order, i.e., upon the arrival of a major shock the system immediately fails. We are interested in the distribution as well as the moments of time until the system fails, which is determined by the minimum value among the inherent system life, the arrival time of a critical minor shock, and the time until a major shock occurs. Some traditional random shock models deal with a system subject to random shocks, where each shock deteriorates the system by a random amount of damage which is distributed with a common distribution [1 - 4]. Without consideration of the inherent system life, most studies in the literature focus on homogeneous shocks occurring in accordance to a Poisson process.

This paper presents an analytic approach to deriving the Laplace transform of the system failure time distribution, which provides an alternative method of deriving the moments in a tractable manner. Mainly two steps are involved. First we consolidate the minor and major shocks into a homogeneous shock whose arrival causes the system to fail. In this regard, the consolidated shock will be referred to as a *critical shock*. So, the time to system failure just becomes the minimum between the inherent system life and the arrival time of the critical shock. Through the renewal process involved in minor shocks, we first construct the distribution of the time to a critical shock and thereafter derive its Laplace transform which is directly used in the moments of the effective system life (time to system failure). For a preliminary step to derive the Laplace transform of the effective system life, we give a useful theorem that models the Laplace transform of the distribution of the minimum between two random variables one of which at least follows an Erlang distribution. Based on the formula of the theorem, the Laplace transform of the effective system life is obtained as a functional form of the Laplace transform of the interarrival time distribution of minor shocks. Thus a simple manipulation produces the moments of the effective system life. Basically the Laplace transform has a closed form. As a computational point of view, however, it is not trivial or straightforward to compute the moments. Algorithms are provided for computing the moments of the effective system life. The developed algorithm would be effective if it is tractable to obtain the derivatives for the Laplace transform of the interarrival time distribution of minor shocks.

The developed Laplace transform may have its applications in determining the optimal replacement or repair period [1, 5, 6] and in analyzing cellular mobile communication systems exploiting hierarchical cell-structure where a macrocell overlaying a number of microcells provides alternate rputes for dropped calls in microcells through *overflow* frpm microcell to macrocell (see [7 - 10]). In such a system, the channel holding time is a key element in deriving QoS measures such as call blocking probability. With the following interpretations the channel holding time corresponds to the effective system life: the inherent system life is interpreted as the call duration of overflow calls, the inter-arrival time of monor shocks as the microcell dwell time, and the arrival time of a major shock as the macrocell dwell time. On the other hand, note that the channel occupancy time is directly determined by the call duration and cell times which can be effectly characterized by an Erlang distribution [11]. With this context, the Laplace transform developed here gives the moments of channel holding time when exploiting a queueing policy for overflow calls.

The rest of the paper is organized as follows: Section 2 gives model description with the notations used in the paper. In Section 3, we derive the Laplace transform of the effective system life distribution and moments. An algorithm for computing the moments is given in Section 4. Finally, in Section 5 we present some numerical examples.

2. MODEL DESCRIPTION

First, the definitions of random variables used in this paper are given in the following:

X inherent system life with pdf $f(t)$ and cdf $F(t)$

Y_i interarrival time between $(i-1)$th and ith minor shocks with pdf $g(t)$ and cdf $G(t)$

Z arrival time of a major shock with pdf $g_Z(t)$ and cdf $G_Z(t)$

T arrival time of a minor shock causing the system to fail with pdf $g_T(t)$ and cdf $G_T(t)$

W arrival time of a critical shock with pdf $g_W(t)$, $W = \min(T, Z)$

S effective system life with pdf $g_S(t)$, $S = \min(X, W)$

q system failure probability due to a minor shock $0 < q < 1$

We assume that X has an Erlang distribution with the pdf as

$$f(t) = \frac{(\lambda t)^{m-1}}{(m-1)!}\lambda e^{-\lambda t}, \quad t > 0 \tag{1}$$

where m is a positive integer and λ is a positive constant. Also, Z is assumed to follow an Erlang distribution with parameters h and μ having the pdf as

$$g_Z(t) = \frac{(\mu t)^{h-1}}{(h-1)!}\mu e^{-\mu t}, \quad t > 0. \tag{2}$$

Note that the Erlang distribution is a preferable one in the computational point of view since its tail distribution has a simple form of summation, which facilitates the computation of complex integrals.

Now let us assume that the system starts to operate at time epoch 0. Each arriving minor shock causes the system failure independently with probability q, and a major shock immediately shuts down the system. Or, the system may fail due to its inherent life. Hence, the system experiences the failure in the following cases as illustrated in Fig. 1:

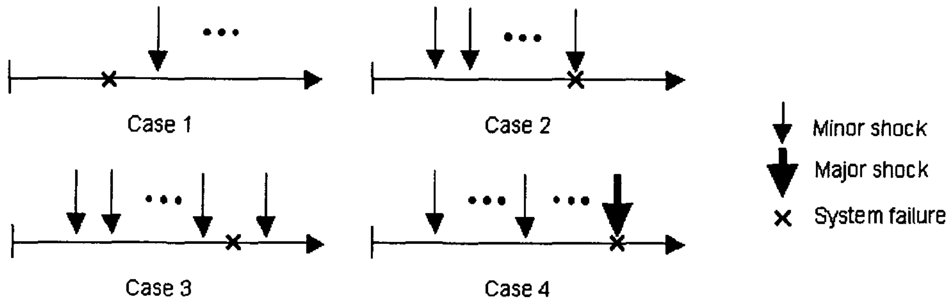

Fig. 1. Four cases of failure instances

- It fails before any minor or major shock arrives (case 1).
- It fails due to an arbitrary minor shock (case 2).
- It fails after surviving some consecutive number of minor shocks (case 3).
- It fails due to a major shock (case 4).

Note that in cases 1 and 3 the system fails due to its inherent life not due to shocks. In particular, as in case 3 a certain number of minor shocks may not affect the system. Note also that the system failure time is represented as $S = \min(X, W)$ where $W = \min(T, Z)$. The next section deals with the Laplace transformation of the distribution of S .

3. LAPLACE TRANSFORM OF SYSTEM FAILURE TIME DISTRIBUTION

This section mainly focuses on deriving the Laplace transform of effective system life distribution, through which the moments of the effective system life can be obtained. We first construct the distribution of arrival time of the critical shock. After that, we present a useful theorem as a preliminary in determining the Laplace transform.

3.1 Distribution of Critical Shock Arrival Time

The critical shock will be one of the followings: 1) a major shock arriving before any minor shock (type 1), 2) a minor shock making the system fail before a major shock after a certain number of consecutive minor shocks which have no effect on the system (type 2), or 3) a major shock arriving after a certain number of consecutive minor shocks (type 3).

Let us define a random variable T_k with pdf $g_{K_k}(t)$ as

$$T_k = \sum_{i=1}^{k} Y_i, \quad k = 1, 2, \cdots \tag{3}$$

where Y_i's are independent and identically distributed with $G(t)$. Let $g_i(t)$ denote the probability intensity of the critical shock arrival time for type $i\,(i = 1, 2, 3)$. Then, based on the definition of T_k in Eq. (3) we obtain

$$\begin{aligned}
g_1(t) &= \frac{d}{dt} P\{Z < t, Z < Y_1\} \\
&= g_Z(t)\bar{G}(t) &(4)\\
g_2(t) &= \frac{d}{dt} \sum_{k=1}^{\infty} P\{T_k < t, T_k < Z\}(1-q)^{k-1} q \\
&= \bar{G}_Z(t) \sum_{k=1}^{\infty} g_{T_k}(t)(1-q)^{k-1} q &(5)\\
g_3(t) &= \frac{d}{dt} \sum_{k=1}^{\infty} P\{Z < t, T_k < Z < T_{k+1}\}(1-q)^k \\
&= g_Z(t) \sum_{k=1}^{\infty} (1-q)^k \int_0^t \bar{G}(t-y) g_{T_k}(y) dy \\
&= g_Z(t) \sum_{k=1}^{\infty} (1-q)^k \int_0^t [g_{T_k}(y) - g_{T_{k+1}}(y)] dy &(6)
\end{aligned}$$

where the bar (–) means the tail distribution of the corresponding distribution. The integral $\int_0^t \bar{G}(t-y) g_{T_k}(y)dy$ in Eq. (6) is the probability that the arrival time of a major shock Z is between T_k and T_{k+1} given that $Z = t$, i.e., $P\{T_k < Z < T_{k+1} | Z = t\}$. With this context, the pdf of W can be constructed by

$$g_W(t) = g_1(t) + g_2(t) + g_3(t) \tag{7}$$

because the following Eqs. (8) - (10) guarantee that $\int_0^\infty [g_1(t) + g_2(t) + g_3(t)]dt = 1$:

$$\int_0^\infty g_1(t)dt = 1 - \int_0^\infty \bar{G}_Z(t) g(t) dt \tag{8}$$

$$\begin{aligned}\int_0^\infty g_2(t)dt &= \sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^\infty \bar{G}_Z(t)g_{T_k}(t)dt\\ &= \sum_{k=1}^{\infty}(1-q)^{k-1}[1-(1-q)]\int_0^\infty \bar{G}_Z(t)g_{T_k}(t)dt\\ &= \sum_{k=1}^{\infty}(1-q)^{k-1}\int_0^\infty \bar{G}_Z(t)g_{T_k}(t)dt - \sum_{k=1}^{\infty}(1-q)^{k}\int_0^\infty \bar{G}_Z(t)g_{T_k}(t)dt\\ &= \int_0^\infty \bar{G}_Z(t)g(t)dt + \sum_{k=1}^{\infty}(1-q)^{k}\int_0^\infty \bar{G}_Z(t)[g_{T_{k+1}}(t)-g_{T_k}(t)]dt \qquad (9)\end{aligned}$$

$$\int_0^\infty g_3(t)dt = \sum_{k=1}^{\infty}(1-q)^{k}\int_0^\infty \bar{G}_Z(t)[g_{T_k}(t)-g_{T_{k+1}}(t)]dt. \qquad (10)$$

In Eqs. (9) and (10), the interchange of integral and summation is justified since all terms are nonnegative (this property is satisfied in the rest of paper).

Now we obtain the Laplace transform of $g_w(t)$. If we let $\tilde{g}_i(s) = \int_0^\infty e^{-st}g_i(t)$ $(i = 1, 2, 3)$, then we have $\tilde{g}_W(t) = \tilde{g}_1(t) + \tilde{g}_2(t) + \tilde{g}_3(t)$ (we use the tilde ($\sim$) to represent the Laplace transform of the corresponding function in the rest of paper). Since $G_Z(t)$ is Erlang distribution and its tail distribution has a form of summation, we first obtain the following through the interchange of integral and summation :

$$\begin{aligned}\tilde{g}_1(s) &= \int_0^\infty e^{-st}g_Z(t)\bar{G}(t)dt\\ &= \int_0^\infty g(y)\int_0^\infty e^{-st}g_Z(t)dtdy\\ &= \left(\frac{\mu}{s+\mu}\right)^h\int_0^\infty g(y)\left[1-\int_y^\infty \frac{(s+\mu)^h}{(h-1)!}t^{h-1}e^{-(s+\mu)t}dt\right]dy\\ &= \left(\frac{\mu}{s+\mu}\right)^h\left[1-\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}\int_0^\infty t^ie^{-(s+\mu)t}g(y)dy\right]\\ &= \left(\frac{\mu}{s+\mu}\right)^h\left[1-\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}D_Y^{(i)}(s+\mu)\right] \qquad (11)\end{aligned}$$

where

$$\begin{aligned}D_Y^{(i)}(x) &= \int_0^\infty t^ie^{-xt}g(t)dt\\ &= (-1)^i\frac{d^i}{ds^i}\tilde{g}(s)|_{s=x}. \qquad (12)\end{aligned}$$

Next, applying the similar technique in Eq. (11) gives

$$\begin{aligned}
\tilde{g}_2(t) &= \sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^{\infty} e^{-st}\bar{G}_Z(t)g_{T_k}(t)dt \\
&= \sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^{\infty}\left(\sum_{i=0}^{h-1}\frac{\mu^i}{i!}t^i e^{-(s+\mu)t}\right)g_{T_k}(t)dt \\
&= \sum_{i=0}^{h-1}\frac{\mu^i}{i!}\sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^{\infty} t^i e^{-(s+\mu)t}g_{T_k}(t)dt \\
&= \sum_{i=0}^{h-1}\frac{\mu^i}{i!}D_T^{(i)}(s+\mu) \qquad (13)
\end{aligned}$$

where

$$\begin{aligned}
D_T^{(i)}(x) &= \sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^{\infty} t^n e^{-xt}g_{T_k}(t)dt \\
&= (-1)^i\sum_{k=1}^{\infty}(1-q)^{k-1}q\left(\frac{d^i}{ds^n}\tilde{g_{T_k}}(s)\Big|_{s=x}\right). \qquad (14)
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\tilde{g}_3(t) &= \sum_{k=1}^{\infty}(1-q)^k\int_0^{\infty} e^{-st}\bar{g}_Z(t)\int_0^t\left[g_{T_k}(y)-g_{T_{k+1}}(y)\right]dydt \\
&= \sum_{k=1}^{\infty}(1-q)^k\int_0^{\infty}\left[g_{T_k}(y)-g_{T_{k+1}}(y)\right]\int_y^{\infty} e^{-st}\bar{g}_Z(t)dtdy \\
&= \left(\frac{\mu}{s+\mu}\right)^h\sum_{k=1}^{\infty}(1-q)^k\int_0^{\infty}\left[g_{T_k}(y)-g_{T_{k+1}}(y)\right]\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}y^i e^{-(s+\mu)y}dy \\
&= \left(\frac{\mu}{s+\mu}\right)^h\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}\frac{1}{q}\sum_{k=1}^{\infty}(1-q)^k q\int_0^{\infty} y^i e^{-(s+\mu)y}\left[g_{T_k}(y)-g_{T_{k+1}}(y)\right]dy \\
&= \left(\frac{\mu}{s+\mu}\right)^h\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}\frac{1}{q}\left[(1-q)^k qD_T^{(i)}(s+\mu)-\left(D_T^{(i)}(s+\mu)-qD_Y^{(i)}(s+\mu)\right)\right] \\
&= \left(\frac{\mu}{s+\mu}\right)^h\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}\left[D_Y^{(i)}(s+\mu)-D_T^{(i)}(s+\mu)\right]. \qquad (15)
\end{aligned}$$

From Eqs. (11), (13), and (15) we obtain

$$\tilde{g}_W(s) = \left(\frac{\mu}{s+\mu}\right)^h\left[1-\sum_{i=0}^{h-1}\frac{(s+\mu)^i}{i!}D_T^{(i)}(s+\mu)\right]+\sum_{i=0}^{h-1}\frac{\mu^i}{i!}D_T^{(i)}(s+\mu) \qquad (16)$$

Note that $d^n/d^n s D_T^{(i)}(s+\mu) = (-1)^n D_T^{(i+n)}(s+\mu)$. Hence, the n-th derivative of $\tilde{g}_W(s)$ is given by

$$\frac{d^n}{d^n s}\tilde{g}_W(s) = (-1)^n \frac{(h+n-1)!}{(h-1)!}\frac{\mu^h}{(s+\mu)^{h+n}}\left(1 - \sum_{j=0}^{h+n-1}\frac{(s+\mu)^j}{j!}D_T^{(j)}(s+\mu)\right)$$
$$+ (-)^n \sum_{j=0}^{h-1}\frac{\mu^j}{j!}D_T^{(j+n)}(s+\mu) \tag{17}$$

Then, the n-th moment of W is

$$E\left[W^n\right] = (-1)^n \frac{d^n}{d^n s}\tilde{g}_W(s)\Big|_{s=0} . \tag{18}$$

3.2 Distribution of the Minimum of Two Random Variables

Before dealing with the system failure time directly, we digress in this section to a useful theorem, which confirms the formula of $\tilde{g}_W(s)$ developed in Section 3.1 and also gives a basic formula for the Laplace transform of the effective system life distribution that will be given in Section 3.3. Theorem 1 states that if at least one of two random variables follows an Erlang distribution, the Laplace transform of the distribution of the minimum between them can be represented in a functional form of the derivatives of another one's Laplace transform.

Theorem 1. Consider the two independent random variables B and C where B has an Eralng distribution with parameters α and β and C is arbitrary distributed with pdf $g_C(t)$. Let $A = \min(B, C)$ and $g_A(t)$ be the pdf of A . Then, the Laplace transform of $g_A(t)$ is given by

$$\tilde{g}_A(s) = \left(\frac{\beta}{s+\beta}\right)^\alpha \left(1 - \sum_{i=0}^{\alpha-1}\frac{(s+\beta)^i}{i!}D_C^{(i)}(s+\beta)\right) + \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}D_C^{(i)}(s+\beta) \tag{19}$$

and the n-th moment is given by

$$E\left[A^n\right] = \frac{(\alpha-1+n)!}{(\alpha-1)!}\frac{1}{\beta^n}\left(1 - \sum_{i=0}^{\alpha-1+n}\frac{\beta^i}{i!}D_C^{(i)}(\beta)\right)$$
$$+ \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}D_C^{(i+n)}(\beta), \quad n = 1, 2, 3, \cdots \tag{20}$$

where

$$D_C^{(n)}(x) = (-1)^n \frac{d^n}{d^n s}\tilde{g}_C(s)\Big|_{s=0} . \tag{21}$$

proof. The pdf of A is represented as

$$g_A(t) = g_B(t)\bar{G}_C(t) + \bar{G}_B(t)g_C(t) \tag{22}$$

where $g_B(t)$ and $\bar{G}_B(t)$ respectively represent the pdf and the tail distribution of B, and $\bar{G}_C(t)$ is the tail of C. Thus, the Laplace transform of $g_A(t)$ is obtained by

$$\tilde{g}_A(s) = \int_0^\infty e^{-st}\left[g_B(t)\bar{G}_C(t) + \bar{G}_B(t)g_C(t)\right]dt. \tag{23}$$

Since B follows the Erlang distribution, applying the same technique as in (11) we first obtain

$$\int_0^\infty e^{-st}g_B(t)\bar{G}_C(t)dt = \left(\frac{\beta}{s+\beta}\right)^\alpha \left[1 - \sum_{i=0}^{\alpha-1}\frac{(s+\beta)^i}{i!}D_C^{(n)}(s+\beta)\right] \tag{24}$$

where

$$D_C^{(n)}(x) = (-1)^n\frac{d^n}{d^n s}\tilde{g}_C(s)|_{s=x}. \tag{25}$$

With the interchange of integral and summation it follows that

$$\begin{aligned}\int_0^\infty e^{-st}g_C(t)\bar{G}_B(t)dt &= \int_0^\infty e^{-st}g_C(y)\sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}t^i e^{-\beta t}dt \\ &= \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}\int_0^\infty g_C(y)t^i e^{-(s+\beta)t}dt \\ &= \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}D_C^{(i)}(s+\beta). \end{aligned} \tag{26}$$

Hence, putting Eqs. (24) and (26) into Eq. (23), we obtain

$$\tilde{g}_A(s) = \left(\frac{\beta}{s+\beta}\right)^\alpha \left[1 - \sum_{i=0}^{\alpha-1}\frac{(s+\beta)^i}{i!}D_C^{(i)}(s+\beta)\right] + \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}D_C^{(i)}(s+\beta). \tag{27}$$

Note that $d^n/d^n sD_C^{(i)}(s+\beta) = (-1)^n D_C^{(i+n)}(s+\beta)$. Hence, by using the same method in Eqs. (17) and (18) we obtain the n-th moment of A as in Eq. (20)

Using Theorem 1 we can validate the formula of $\tilde{g}_W(s)$ presented in Section 3.1 in which a different approach is used to derive it. The following Corollary 1 gives a more systemic approach to obtaining $\tilde{g}_W(s)$.

Corollary 1. Consider a renewal process for which the interarrival times Y_i have a distribution $G(t)$. Let N denote a *stopping* time of the process that is a geometric random variable with parameter q and define $T = \sum_{i=1}^{N} Y_i$. Then, the Laplace transform of $A = \min(B, T)$ where B follows an Erlang distribution with parameters α and β is given by

$$\tilde{g}_A(s) = \left(\frac{\beta}{s+\beta}\right)^{\alpha}\left[1 - \sum_{i=0}^{\alpha-1}\frac{(s+\beta)^i}{i!}D_T^{(i)}(s+\beta)\right] + \sum_{i=0}^{\alpha-1}\frac{\beta^i}{i!}D_T^{(i)}(s+\beta) \tag{28}$$

where

$$D_T^{(i)}(x) = (-1)^i \sum_{k=1}^{\infty}(1-q)^{k-1}q\left(\frac{d^i}{ds^i}[\tilde{g}(s)]^k\,|_{s=x}\right), \quad n = 0, 1, 2, \cdots \tag{29}$$

proof. Based on Theorem 1 Eq. (28) immediately holds from Theorem 1 when

$$D_T^{(i)}(x) = (-1)^i \frac{d^i}{ds^i}\tilde{g}_T(s)|_{s=x}. \tag{30}$$

If we denote the pdf of T by $g_T(s)$, then

$$\begin{aligned}
\tilde{g}_T(s) &= \int_0^{\infty} e^{-st} g_T(t)dt \\
&= \int_0^{\infty} e^{-st}\left[\sum_{k=1}^{\infty}(1-q)^{k-1}q g_{t_k}(t)\right]dt \\
&= \sum_{k=1}^{\infty}(1-q)^{k-1}q\int_0^{\infty} e^{-st} g_{t_k}(t)dt \\
&= \sum_{k=1}^{\infty}(1-q)^{k-1}q\tilde{g}_{t_k}(t)dt.
\end{aligned} \tag{31}$$

Since $\tilde{g}_{t_k}(s) = [\tilde{g}(s)]^k$, Eq. (29) holds.

Now let us return to Section 3.1. The arrival time of a minor shock which makes the system fail can be represented as $\sum_{i=1}^{N} Y_i$ where Y_i's are independent and identically distributed with $G(t)$ and N is a geometric random variables with parameter q . So, the arrival time of a critical shock, W, is simply the minimum between the quantities Z and $\sum_{i=1}^{N} Y_i$. Note that in Corollary 1 T can be interpreted as the arrival time of the minor shock which causes the system to fail. If we interpret C as the arrival time of a major shock, then A corresponds to the arrival time of the critical shock W. Hence we see that Corollary 1 gives the same result with Eqs. (14) and (16) in Section 3.1.

Corollary 1 can be extended to the case where N is a negative binomial random variable with parameters r and p. Such case arises if we suppose that each minor shock will partially degrade the system with probability p and the system will fail when a total of r such minor shocks are accumulated. In this case, Eq. (31) is simply modified as

$$\tilde{g}_T(s) = \sum_{k=1}^{\infty} \binom{k-1}{r-1} p^r (1-p)^{k-r} \tilde{g}_{T_k}(s). \tag{32}$$

3.3 Laplace Transform of System Failure Time Distribution

Now we derive the Laplace transform of the system failure time distribution, $\tilde{g}_S(s)$. From Theorem 1,

$$\tilde{g}_S(s) = \left(\frac{\lambda}{s+\lambda}\right)^m \left[1 - \sum_{i=0}^{m-1} \frac{(s+\lambda)^i}{i!} D_W^{(i)}(s+\lambda)\right] + \sum_{i=0}^{m-1} \frac{\lambda^i}{i!} D_W^{(i)}(s+\lambda) \tag{33}$$

where

$$D_W^{(n)}(x) = (-1)^n \frac{d^n}{d^n s} \tilde{g}_W(s)|_{s=x} \tag{30}$$

and $\tilde{g}_W(s)$ is given in Eq. (16). Hence, we obtain the nth moment of S as follows:

$$\begin{aligned} E\left[S^n\right] = & \frac{(m-1+n)!}{(m-1)!} \frac{1}{\lambda^n} \left(1 - \sum_{i=0}^{m-1+n} \frac{\lambda^i}{i!} D_W^{(i)}(\lambda)\right) \\ & + \sum_{i=0}^{m-1} \frac{\lambda^i}{i!} D_W^{(i+n)}(\lambda), \quad n = 1, 2, 3, \cdots \end{aligned} \tag{35}$$

where

$$\begin{aligned} D_W^{(i)}(\lambda) = & \frac{(h+i-1)!}{(h-1)!} \frac{\mu^h}{(\lambda+\mu)^{h+i}} \left(1 - \sum_{j=0}^{h+i-1} \frac{(\lambda+\mu)^j}{j!} D_T^{(j)}(\lambda+\mu)\right) \\ & + \sum_{j=0}^{h-1} \frac{\mu^j}{j!} D_T^{(j+i)}(\lambda+\mu). \end{aligned} \tag{36}$$

Recall that m and λ are shape and scale parameters in pdf of X, respectively, while h and μ are shape and scale parameters in pdf of Z, respectively. To compute $E[S^n]$ in Eq. (35), we first need $D_W^{(i)}(\lambda)(i = 0, \cdots, m+n-1)$. In Eq. (36) we know that $D_T^{(j)}(\lambda+\mu)(j = 0, \cdots, h+i-1)$ is needed to compute $D_W^{(i)}(\lambda)$. From the definition in Eq. (14), $D_T^{(j)}(x)$ has a complex form and thus cannot be solved directly. So an algorithmic approach is needed. In Section 4, a computation algorithm for $D_T^{(j)}(\cdot)$ and thus $E\left[S^n\right]$ will be given through a recursive method.

Note that our system is simply interpreted as a series system having three components whose life time distributions are $F(t)$, $G_Z(t)$, and $G_T(t)$, respectively. Thus, the system reliability is given by $\bar{G}_S(t) = \bar{F}(t)\bar{G}_Z(t)\bar{G}_T(t)$. In case that Y_i follows an Erlang distribution with shape parameter a and scale parameter b, the system reliability is given as follows:

$$\bar{G}_S(t) = \left(\sum_{i=0}^{m-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}\right) \left(\sum_{i=0}^{h-1} \frac{(\mu t)^i}{i!} e^{-\mu t}\right) \left(\sum_{k=1}^{\infty} (1-q)^{k-1} q \sum_{i=0}^{ka-1} \frac{(bt)^i}{i!} e^{-bt}\right). \quad (37)$$

It is interesting to compare our system with a shock free system. If we focus on the quantity $E[S]$, the expected time to a system failure in a shock free system is just $E[X]$. But, the $E[S]$ in our system is given by from Eq. (35),

$$E[S] = \frac{m}{\lambda} - \frac{1}{\lambda} \sum_{i=0}^{m-1} (m-1) \frac{\lambda^i}{i!} D_W^{(i)}(\lambda). \quad (38)$$

We also know that

$$\begin{aligned} E[S] &= \int_0^\infty \bar{F}(t)\bar{G}_W(t)dt \\ &= E[x] - \int_0^\infty \bar{F}(t)G_W(t)dt. \end{aligned} \quad (39)$$

If we denote by X^r the *excess* of X having its distribution $F^e(t)$ as the equilibrium distribution of $F(t)$ and its pdf $f^e(t)$, then the following is satisfied:

$$\begin{aligned} \int_0^\infty \bar{F}(t)G_W(t)dt &= \int_0^\infty g_W(u) \int_u^\infty \bar{F}(t)dudt \\ &= E[X] \int_0^\infty g_W(u) \int_u^\infty \frac{\bar{F}(t)}{E[X]} dudt \\ &= E[X] \int_0^\infty g_W(u) \int_u^\infty f^e(t)dudt \\ &= E[X] P\{X^r > W\} \int_0^\infty \frac{g_W(u)\bar{F}^e(u)}{P\{X^r > W\}} \\ &= E[X] P\{X^r > W\}. \end{aligned} \quad (40)$$

That is, $E[S] = E[X](1 - P\{X^r > W\})$. Hence, the system failure time in the random shock environment is reduced by the proportion of $P\{X^r > W\}$ as compared to that of a shock free system.

Now, let us illustrate a special case where X and Z are exponentially distributed ($m = h = 1$) and $G(t)$ is an exponential distribution with parameter γ. Then, $E[S]$ reduces to $E[S] = (\lambda + \mu + q\gamma)^{-1}$, which corresponds to our intuition because T follows an exponential distribution with parameter $q\gamma$.

Finally, if there is no information about how long the system has been in operation, the remaining effective system life at an arbitrary epoch when the system is still alive, denoted by R, is the excess of the effective system life and its moment is obtained directly from that of the effective system life as follows:

$$E[R^n] = \frac{E[S^{n+1}]}{(n+1)E[S]}, \quad n = 1, 2, 3, \cdots \tag{41}$$

4. AN ALGORITHM FOR COMPUTATION OF MOMENTS

Note that for any arbitrary differentiable nonnegative functions $a(s)$ and $b(s)$, it is satisfied that

$$[a(s)b(s)]^{(j)} = \sum_{l=0}^{j} \binom{j}{l} a(s)^{(l)} b(s)^{(j-l)}, \quad n = 0, 1, 2, \cdots \tag{42}$$

where the subscript (j) denotes the j-th derivative of the corresponding function. Note that the Laplace transform of $g_{T_k}(t)$ is

$$\tilde{g}_{T_k}(s) = [\tilde{g}(s)]^k, \quad n = 1, 2, 3, \cdots \tag{43}$$

where $\tilde{g}(s)$ is the Laplace transform of $g(t)$. Then, $[g_{T_k}(s)]^{(j)}$ can be computed in a recursive manner as follows:

$$[\tilde{g}_{T_k}(s)]^{(j)} = \begin{cases} [\tilde{g}(s)]^{(j)} & k = 1 \\ \sum_{l=0}^{j} \binom{j}{l} [\tilde{g}(s)]^{(l)} [\tilde{g}_{T_k}(s)]^{(j-l)} & k = 2, 3, 4, \cdots \end{cases} \tag{44}$$

Hence, for a sufficiently large number K we can compute $D_T^{(j)}(s)$ in Eq. (14) as follows:

$$D_T^{(j)}(s) \approx (-1)^n \sum_{k=1}^{K} (1-q)^{k-1} q [\tilde{g}_{T_k}(s)]^{(j)}. \tag{45}$$

Note that the quantity $(1-q)^{k-1} q [\tilde{g}_{T_k}(s)]^{(j)}$ converges to 0 for a given j as k increases.

Now, $E[S^n]$ in Eq. (35) can be computed in recursive manner with a proper terminating condition using Eqs. (44) and (45).

5. NUMERICAL EXAMPLES

In our numerical examples, we perform two main tests when evaluating the system failure time : 1) the comparison of non–exponential distribution with exponential distribution, 2) the effect of the variance of Y. In each case the mean values of X, Y, and Z are adjusted to have the same value. We assume that Y_i follows a gamma distribution with parameters a and b ($E[Y] = a/b$). Note that when $G(t)$ is the gamma distribution, it is satisfied that $[\tilde{g}(s)]^{(j)} = (-1)^j b^a a(a+1)\cdots(a+j-1)(s+b)^{-a-j})$, which makes our algorithm more tractable. Table 1 and 2 represent the results of test 1 and 2, respectively. Each result is given for the quantities $E[S]$, $E[S^2]$, and $E[S^3]$, respectively, in increasing values of $q (0 \leq q \leq 1)$. In particular, the case that $q = 0$ corresponds the situation where there exists only major shock without no minor shock, while the case that $q = 1$ represents such shock model that the minor shock behaves same as major shock with a different arrival time distribution.

From Table 1, we see that $E[S]$ has a larger value in non-exponential model than in exponential model at each given level of q. This shows that modeling the related random variable as exponential distribution may underestimate the expected length of the effective system life. Also, the difference of $E[S]$ between them is not thought to be negligible under the same mean values of X, Y, and Z, respectively. Table 2 shows that $E[S]$ has a greater value in the model of large variance of Y than in that of relatively small variance. Note that when $q = 0$, $E[S]$ has the same value (0.9967) in each case. If we compare Case A in Table 1 with Case C in Table 2, then the mean level of Y is greater in Case C than Case A. From the two tables we see that Case C has a greater value of $E[S]$ than Case C. This result also corresponds to our intuition.

Table 1. $E[S^n]$ for non-exponential and exponential models

q	Non-exponential model (Case A) $m=4, \lambda=4$, $h=8, \mu=2$, $a=4, b=8$			Exponential model (Case B) $m=1, \lambda=1$, $h=1, \mu=\frac{1}{4}$, $a=1, b=2$		
	$E[S]$	$E[S^2]$	$E[S^3]$	$E[S]$	$E[S^2]$	$E[S^3]$
0.0	0.9967	1.2363	1.8292	0.8000	1.2800	3.0720
0.1	0.9117	1.0540	1.4679	0.6897	0.9512	1.9681
0.2	0.8363	0.8995	1.1762	0.6061	0.7346	1.3357
0.3	0.7692	0.7684	0.9403	0.5405	0.5844	0.9476
0.4	0.7093	0.6568	0.7496	0.4878	0.4759	0.6964
0.5	0.6557	0.5617	0.5953	0.4444	0.3951	0.5267
0.6	0.6077	0.4804	0.4703	0.4082	0.3332	0.4080
0.7	0.5645	0.4109	0.3692	0.3774	0.2848	0.3224
0.8	0.5256	0.3513	0.2874	0.3509	0.2462	0.2592
0.9	0.4904	0.3002	0.2213	0.3279	0.2150	0.2115
1.0	0.4586	0.2562	0.1679	0.3077	0.1893	0.1748

Table 2. Effect of variance of Y on $E[S^n]$

q	Large variance of Y (Case C)			Small variance of Y (Case D)		
	$m=4,\ \lambda=4$ $h=8,\ \mu=2$ $a=3,\ b=3$			$m=4,\ \lambda=4$ $h=4,\ \mu=4$ $a=3,\ b=\frac{3}{2}$		
	$E[S]$	$E[S^2]$	$E[S^3]$	$E[S]$	$E[S^2]$	$E[S^3]$
0.0	0.9967	1.2363	1.8292	0.9967	1.2363	1.8292
0.1	0.9908	1.2186	1.7844	0.9858	1.2088	1.7674
0.2	0.9849	1.2010	1.7400	0.9750	1.1818	1.7070
0.3	0.9791	1.1836	1.6960	0.9644	1.1551	1.6479
0.4	0.9733	1.1662	1.6525	0.9539	1.1290	1.5902
0.5	0.9675	1.1489	1.6093	0.9435	1.1032	1.5337
0.6	0.9617	1.1318	1.5666	0.9333	1.0779	1.4786
0.7	0.9560	1.1147	1.5242	0.9231	1.0530	1.4246
0.8	0.9503	1.0978	1.4823	0.9131	1.0286	1.3719
0.9	0.9446	1.0810	1.4407	0.9032	1.0045	1.3204
1.0	0.9389	1.0643	1.3995	0.8934	0.9808	1.2701

6. CONCLUSIONS

A non–exponential random shock model with two heterogeneous types of shocks was considered. Based on the characterization of the inherent system life by an Erlang distribution, the Laplace transform of the effective system life was derived in a functional form of the Laplace transform of the interarrival time distribution of minor shocks. The developed methodology can be easily extended to the hyper-Erlang distribution of related time variables, which makes it possible to handle more general cases. Also, the derived Laplace transform can facilitate its extension to multi–shock models. Finally, it has application to a telecommunication area such as cellular mobile systems exploiting queueing policy.

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