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A Family of Tests for Trend Change in Mean Residual Life using Censored Data

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Abstract. In a resent paper, Na and Kim(2000) develop a family of test statistics for testing whether or not the mean residual life changes its trend based on complete data and show that the new tests perform better than previously known tests. In this paper, we extend their tests to the randomly censored data. The asymptotic normality of the test statistics is established. Monte Carlo simulations are conducted to compare our tests with a previously known test by the power of tests.

Key Words : *Trend change in MRL, hypothesis testing, randomly censored data.*

1. INTRODUCTION

Let X denote the lifetime of an item having a continuous distribution function F such that $F(x) = 0$ for $x \leq 0$. The mean residual life(MRL) function is defined by

$$e(x) = \begin{cases} \int_x^\infty \bar{F}(t)dt/\bar{F}(x), & \text{if } \bar{F}(x) > 0 \ , \\ 0, & \text{otherwise,} \end{cases}$$

where $\bar{F}(x) = 1 - F(x)$.

Based on the behavior of MRL function, various nonparametric classes of life distributions have been defined. One such class consists of those with "increasing initially, then decreasing mean residual life (IDMRL)". The dual class is "decreasing initially, then increasing mean residual life (DIMRL)". See Guess and Proschan(1988) and the references therein for examples and applications of the IDMRL(DIMRL) class. Also it is well known that F is exponential distribution if and only if $e(x)$ is constant. We consider the problem of testing

H_0 : F is the exponential distribution

against

H_1 : F is IDMRL, but not exponential.

When the dual model is proposed, we test H_0 against

H_1' : F is DIMRL, but not exponential.

When complete data is utilized, Guess, Hollander and Proschan(1986) propose two test procedures for constant MRL against the trend change in MRL when the turing point τ is known or when the proportion $p = F(\tau)$ before the change occurs is known. Aly(1990) suggests several tests for monotonicity of MRL. These tests consider the IDMRL alternative when either the change point or the proportion is known. Hawkins, Kochar and Loader(1992) developed a test for exponentiality against IDMRL alternative when neither the change point nor the proportion is known. Lim and Park(1998) studied a family of IDMRL tests when the proportion is known. Recently, Na and Kim(2000) propose a family of tests for the trend change in MRL when the turning is known.

In the case of randomly censored data, this problem is dealt by Guess(1984) when the turning point is known or when the proportion is known.

In this paper, we develop a family of tests for testing H_0 against $H_1(H_1')$ using randomly censored data by extending Na and Kim's(2000) family of tests to accommodate censoring. We establish the asymptotic normality of the test statistics. Monte Carlo simulations are conducted to investigate the performance of the test statistics by simulating the power of tests for various turning point and sample size n.

Section 2 is devoted to derive a test statistics for testing H_0 against $H_1(H_1')$. Results of Monte Carlo simulations are presented in Section 3.

2. A FAMILY OF TESTS FOR TREND CHANGE IN MRL

In this section we generalize IDMRL(τ) tests to the randomly censored data. We assume that the turning point τ is known or has been specified by the user.

Let $v(x) = \int_x^\infty \bar{F}(u)du$ and f denote the probability density function corresponding to F. As a measure of the deviation from H_0 in favor of H_1, Na and Kim(2000) considered the following parameter

$$
\begin{aligned}
U_j(F) &= \int_0^\tau \bar{F}^j(t)(f(t)v(t) - \bar{F}^2(t))dt + \int_\tau^\infty \bar{F}^j(t)(\bar{F}^2(t) - f(t)v(t))dt \quad (0.1)\\
&= \frac{1}{j+1}\Big(\int_0^\infty \bar{F}(t)dt - (j+2)\int_0^\tau \bar{F}^{j+2}(t)dt \quad (0.2)\\
&\quad +(j+2)\int_\tau^\infty \bar{F}^{j+2}(t)dt - 2\bar{F}^{j+1}(\tau)\int_\tau^\infty \bar{F}(t)dt\Big) \quad (0.3)
\end{aligned}
$$

where j is a integer with $j \geq -1$. Na and Kim(2000) formed their test statistics by replacing F in (2.1) by the empirical distribution. In our randomly censored model, we replace F in (2.1) by the Kaplan-Meier(KM) estimator defined in (2.2) below.

Let $X_1, X_2, \cdots, X_n$ be independent identically distributed according to a continuous life distribution function F and let $C_1, C_2, \cdots, C_n$ be independent identically distributed according to a continuous life distribution G. C_i is the censoring time associated with X_i, $i = 1, 2, \cdots, n$. In random censoring case we can only observe $(Y_1, \delta_1), \cdots, (Y_n, \delta_n)$ where $Y_i = \min(X_i, C_i)$, and $\delta_i = I(X_i \leq C_i)$, for $i = 1, \cdots, n$. It is assumed that X_i and C_i are independent. The random variable Y_i is said to be uncensored or censored according as $\delta_i = 1$ or $\delta_i = 0$. Therefore $Y_1, \cdots, Y_n$ are observations from a life distribution H with reliability function $\bar{H} = \bar{F}\bar{G} = (1-F)(1-G)$. The KM estimator of $\bar{F}(x)$ is defined as

$$
\hat{\bar{F}}_{KM}(x) = 1 - \hat{F}_{KM}(x) = \prod_{\{i: X_{(i)} \leq x\}} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} \quad (0.4)
$$

where $Y_{(1)} \leq \cdots \leq Y_{(n)}$ are the ordered Y's and $\delta_{(i)}$ is the censoring status corresponding to $Y_{(i)}$. We treat $Y_{(n)}$ as uncensored observation whether it is uncensored or not. When censored observations are tied with uncensored we treat the uncensored as preceeding the censored.

As to the problem of testing H_0 against H_1, we propose a family of test statistics $U_j(\hat{F}_{KM})$ by replacing F in (2.1) by $\hat{F}_{KM}$. The computational simpler expression of $U_j(\hat{F}_{KM})$ is

$$
\begin{aligned}
U_j(\hat{F}_{KM}) &= \sum_{i=1}^{i^*} B_{1j}\Big(\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big)(Y_{(i)} - Y_{(i-1)}) + B_{1j}\Big(\prod_{v=1}^{i^*} c_v^{\delta_{(v)}}\Big)(\tau - Y_{(i^*)})\\
&+ B_{2j}\Big(\prod_{v=1}^{i^*} c_v^{\delta_{(v)}}\Big)(Y_{(i^*+1)} - \tau) + \sum_{i=i^*+2}^{n} B_{2j}\Big(\prod_{v=1}^{i-1} c_v^{\delta_{(v)}}\Big)(Y_{(i)} - Y_{(i-1)}),
\end{aligned}
$$

where $0 = Y_{(0)} < Y_{(1)} < \cdots < Y_{(i^*)} \leq \tau < Y_{(i^*+1)} < \cdots < Y_{(n)}$, $c_v = (n-v)/(n-v+1)$,

$$
B_{1j}(u) = \frac{1}{j+1}\{u - (j+2)u^{j+2}\} \text{ and}
$$

$$B_{2j}(u) = \frac{1}{j+1}\{(1-2\hat{\bar{F}}_{KM}^{j+1}(\tau))u + (j+2)u^{j+2}\}.$$

Since $U_j(\hat{F}_{KM})$ is not scale invariant, we use the following scale invariant test statistic

$$U_j^c = \frac{U_j(\hat{F}_{KM})}{\hat{\mu}_F}$$

where

$$\hat{\mu}_F = \sum_{i=1}^{n}\left\{\prod_{v=1}^{i-1}\left(\frac{n-v}{n-v+1}\right)^{\delta_{(v)}}\right\}(Y_{(i)} - Y_{(i-1)}).$$

When there is no censoring this test statistic reduces to the one which is obtained by replacing F in (2.1) with empirical distribution.

To establish asymptotic normality of U_j^c, we assume the following conditions on the distributions F and G.

$$\text{(i)} \quad \int_0^\infty \bar{F}^\beta(x)dx < \infty \quad \text{and} \quad \int_0^\infty \{\bar{F}^{2\beta}(x)\bar{G}(x)\}^{-1}dF(x) < \infty,$$

for some $\beta \in (0, 1/2)$, and

$$\text{(ii)} \quad \sqrt{n}\int_{Y_{(n)}}^\infty \bar{F}(x)dx \xrightarrow{p} 0.$$

The derivation of the asymptotic normality of U_j^c is similar to that of Guess(1984), using the techniques of Joe and Proschan(1982) and Gill(1983). The asymptotic distribution of U_j^c is summarized in Theorem 2.1.

Theorem 2.1 Suppose F and G are continuous distributions. Assume that F' exists at τ and $F'(\tau)$ is positive. If conditions (i) and (ii) above are satisfied, then

$$\sqrt{n}(U_j^c - U_j(F))/\mu_F \xrightarrow{d} N(0, \sigma^2(U_j, G)/\mu_F^2)$$

where

$$\begin{aligned}\sigma^2(U_j, F) &= \int_0^\infty\int_0^\infty U_j(F(x))U_j(F(y))\bar{F}(x)\bar{F}(y)\int_0^{x\wedge y}\frac{dF}{F^2G}dxdy \\ &- 4\bar{F}^j(\tau)\int_\tau^\infty \bar{F}(u)du\int_0^\infty U_j(F(x))\bar{F}(x)\bar{F}(\tau)\int_0^{x\wedge\tau}\frac{dF}{F^2G} \\ &+ 4\bar{F}^{2j}(\tau)\left(\int_\tau^\infty \bar{F}(u)du\right)^2\bar{F}(x)\bar{F}(y)\int_0^{x\wedge y}\frac{dF}{F^2G}dxdy.\end{aligned}$$

where $x \wedge y = \min\{x, y\}$.

Under H_0, i.e. $F(x) = F_0(x) = 1 - \exp(-x/\mu)$, we find

$$\varsigma_j^2 \equiv \sigma^2(U_j, G)/\mu_{F_0}^2 = \frac{1}{(j+1)^2}\left\{ \int_0^1 \frac{g_{1j}(z)}{\bar{H}(-\mu \log z)} dz \right. \tag{0.5}$$

$$\left. + \; 4\{1 - \bar{F}^{j+1}(\tau)\} \int_0^{\bar{p}} \frac{g_{2j}(z)}{\bar{H}(-\mu \log z)} dz \right\}, \tag{0.6}$$

where $g_{1j}(z) = (j+2)^2 z^{2j+3} - 2(j+2)z^{j+2} + z$ and $g_{2j}(z) = (j+2)z^{j+2} - \bar{F}^{j+1}(\tau)z$.

Since the null asymptotic variance ς_j^2 depends on H, we need consistent estimator of ς_j^2. We can obtain a consistent estimator of ς_j^2, $\hat{\varsigma}_j^2$, by replacing $\bar{H}$ in (2.3) with $\bar{H}_n$, the empirical reliability function of $Y_1, \cdots, Y_n$. For computational purpose, we have

$$\begin{aligned}
\hat{\varsigma}_j^2 &= \frac{1}{(j+1)^2}\left[\frac{(j+1)^2}{2(j+3)}\right. \\
&+ \sum_{i=1}^{n-1} \frac{n}{(n-i+1)(n-i)}\left(\frac{j+2}{2}B_i(2j+4) - \frac{2j+4}{j+3}B_i(j+3) + \frac{1}{2}B_i(2)\right) \\
&- n\left(\frac{j+2}{2}B_n(2j+4) - \frac{2j+4}{j+3}B_n(j+3) + \frac{1}{2}B_n(2)\right) \\
&+ 4(1 - \hat{\bar{F}}_{KM}^{j+1}(\tau))\left\{\frac{n}{n-k}\frac{j+1}{2(j+3)}\hat{\bar{F}}_{KM}^{j+3}(\tau)\right. \\
&+ \sum_{i=k+1}^{n-1} \frac{n}{(n-i+1)(n-i)}\left(\frac{j+2}{j+3}B_i(j+3) - \frac{1}{2}\hat{\bar{F}}_{KM}^{j+1}(\tau)B_i(2)\right) \\
&- \left.\left. n\left(\frac{j+2}{j+3}B_n(j+3) - \frac{1}{2}\hat{\bar{F}}_{KM}^{j+1}(\tau)B_n(2)\right)\right\}\right],
\end{aligned}$$

where $B_i(a) = \exp(-aY_{(i)}/\hat{\mu}_F)$ and $Y_{(k)} \le -\hat{\mu}_F \log \hat{\bar{F}}_{KM}(\tau) < Y_{(k+1)}$.

The DMRL(τ) test procedure rejects H_0 in favor of the alternative $H_{1\tau}$ at the approximate significant level α if $\sqrt{n}U_j^c/\hat{\varsigma}_j \ge z_\alpha$. Analogously, the approximate significant level α test of H_0 versus $H_{1\tau}'$ reject H_0 if $\sqrt{n}U_j^c/\hat{\varsigma}_j \le -z_\alpha$.

3. SIMULATION STUDY

To compare the power of our tests based on U_j^c, $j = 0, 1, 2, 3$ with that of Guess(1984), the random numbers are generated from

$$\begin{aligned}
\bar{F}_{\alpha,\beta,\gamma}(x) &= \left\{\frac{\beta}{\beta + \gamma\exp(-\alpha x)(1 - \exp(-\alpha x))}\right\}\left\{\frac{[1+d]^2 - c^2}{[\exp(\alpha x) + d]^2 - c^2}\right\}^{1/2\alpha\beta} \\
&\times \left\{\frac{[\exp(\alpha x) + d - c]}{[\exp(\alpha x) + d + c]}\frac{[1 + d + c]}{[1 + d - c]}\right\}^{\gamma/4\alpha\beta^2 c}, \quad x \ge 0
\end{aligned}$$

where $d = \gamma/2\beta$, $c^2 = (4\beta\gamma + \gamma^2)/(4\beta^2)$. This distribution has MRL function $e_{\alpha,\beta,\gamma}(x) = \beta + \gamma\exp(-\alpha x)(1 - \exp(-\alpha x))$, $x \geq 0$. The motivation(see Hawkins, Kochar and Loader, 1992) for choosing $\bar{F}_{\alpha,\beta,\gamma}$ is that $\bar{F}_{\alpha,\beta,\gamma}$ has IDMRL structure with the turning point $\tau = \ln 2/\alpha$ for any choice of (α, β, γ) and $\bar{F}_{\alpha,\beta,\gamma}$ is exponential distribution if $\gamma = 0$.

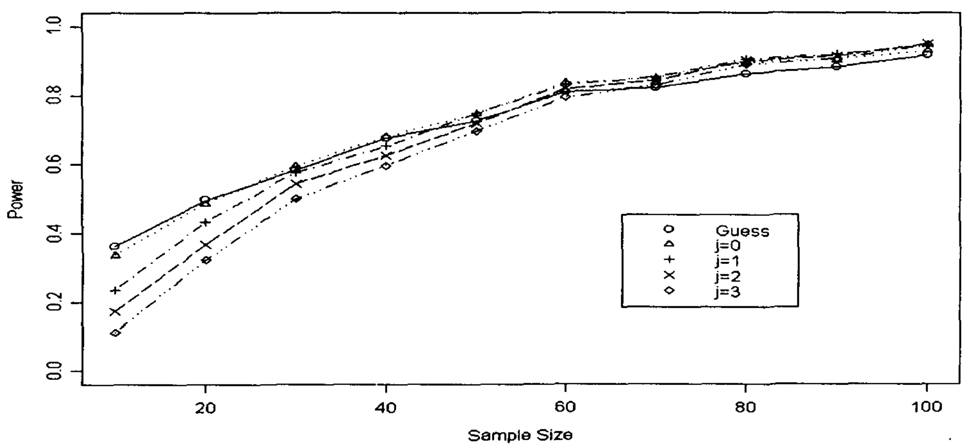

Figure 1: Monte Carlo power comparison from 1000 replications with $\alpha = 1$, $\beta = 1$ and $\gamma = 1$.

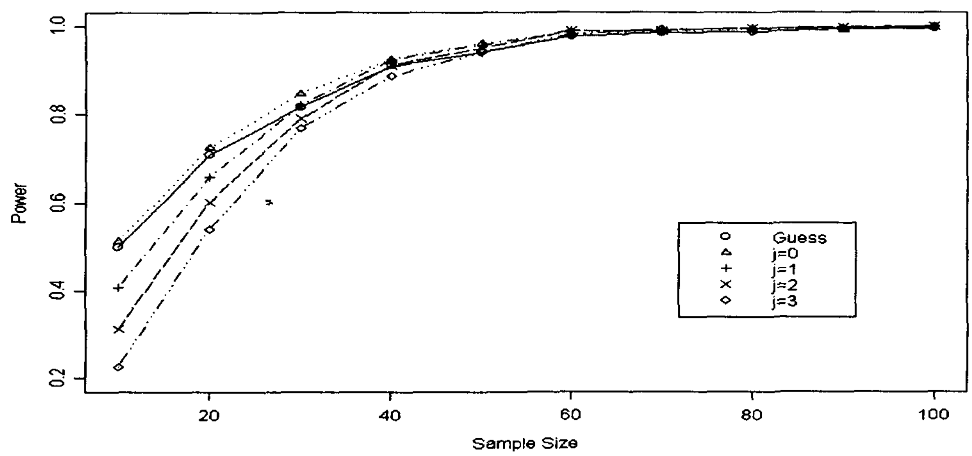

Figure 2: Monte Carlo power comparison from 1000 replications with $\alpha = 1$, $\beta = 1$ and $\gamma = 2$.

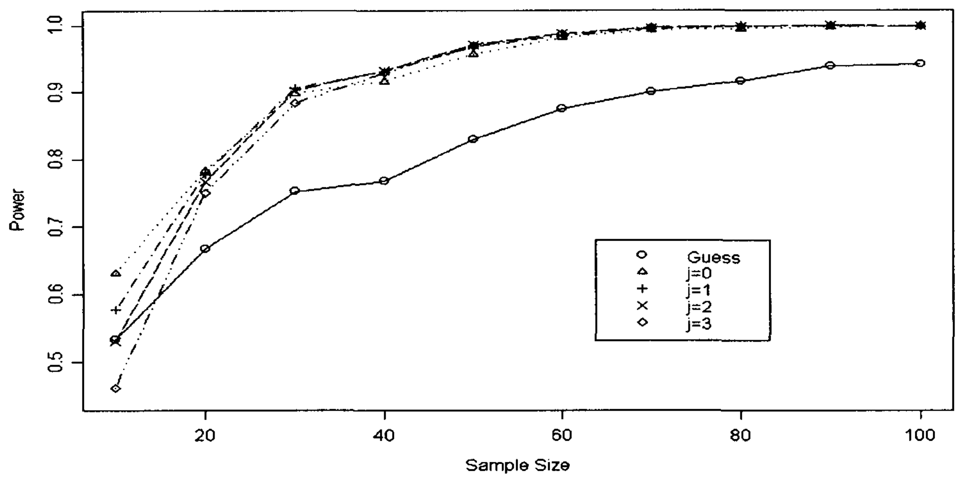

Figure 3: Monte Carlo power comparison from 1000 replications with $\alpha = 3$, $\beta = 1$ and $\gamma = 2$.

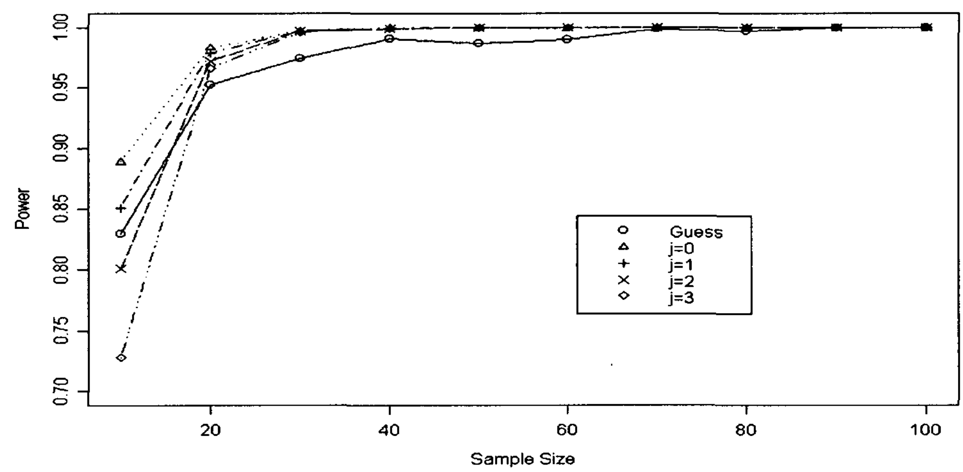

Figure 4: Monte Carlo power comparison from 1000 replications with $\alpha = 1$, $\beta = 1$ and $\gamma = 2$.

The censoring random numbers are generated from $\bar{G}(x) = \bar{F}^{\rho}_{\alpha,\beta,\gamma}(x)$, here ρ is viewed as a censoring parameter since the probability that an observation will be censored is $\Pr(\delta = 0) = \rho/(1+\rho)$.

Figures 3.1–3.4 contain Monte Carlo estimated powers based on 1000 replications of sample size $n = 10, 20, \cdots, 100$ from $\bar{F}_{\alpha,\beta,\gamma}$ for various choice of α and γ with $\beta = 1$, and the amount of censoring $\rho = 1/3$ when the level of significance is 0.10. From the figures, we notice that the powers of all tests increase rapidly as γ increases when α is fixed and also as α increases (i.e., the turning point τ decreases) when γ

is fixed. It is further better to increase γ than α. This is generally to be expected since the width of $e(x)$ increases as γ increases. Figures show that our tests generally dominate the Guess(1984) test except when $\alpha = 1$, $\gamma = 1$ and $n \leq 20$. The power of our tests increase more rapidly than that of the Guess(1984) test as n increases for any α and γ.

The results for the other sample size, the other values of α and the other amount of censoring are not given here, but can be obtained with FORTRAN program from the first author.

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