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An Opportunity–based Age Replacement Policy with Warranty Analysed by Using TTT–Transforms

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Abstract. In a recent paper Iskandar & Sandoh (1999) studied an opportunity-based age replacement policy for a system which has a warranty period (0,S]. When the system fails at age $x \leq S$ a minimal repair is performed. If an opportunity occurs to the system at age x, $S \leq x \leq T$, we take the opportunity with probability p to preventively replace the system, while we conduct a corrective replacement when its fails in (S,T). Finally, if its age reaches T, we perform a preventive replacement. Under this policy the design variable is T. For the case when opportunities occur according to a homogeneous Poisson process, the long–run average cost of this policy was formulated and studied analytically by Iskandar & Sandoh (1999). The same problem is here analysed by using a graphical technique based on scaled TTT–transforms. This technique gives, among other things, excellent possibilities for different types of sensitivity analysis. We also extend the discussion to the situation when we have to estimate T based on times to failure.

Key Words : *replacement policy, warranty period, minimal repair, homogeneous Poisson process, long–run average cost, TTT–transforms.*

1. INTRODUCTION

This paper discusses an opportunity-based age replacement policy for a system which has a warranty period of (0,S]. Under the proposed policy, the system is restored from failure by a minimal repair during its warranty period. A minimal repair is a repair after which the failure rate of the system is the same as just before failure. The concept of minimal repair was first introduced by Barlow & Hunter (1960). Whenever the system fails after the warranty period (0,S] we perform a corrective replacement, i.e. the system is replaced by a new identical unit or equivalently, restored to "as-good-as-new" state. When an opportunity occurs to the system at age x, with $S \leq x \leq T$, we take the opportunity to preventively replace the system with probability p. When the age of the system reaches T we always conduct a preventive replacement.

Under such a policy the design variable is T. In this paper we discuss the long–run average cost per unit time of the proposed policy in the case when opportunities occur according to a homogeneous Poisson process and the analysis is based on a graphical approach with TTT–plots and scaled TTT–transforms as tools.

The idea associated with S and T in the above discussion is similar to that of t and T respectively, in the (t, T)–policy proposed by Ohnishi et al. (1990). Under their policy we conduct a minimal repair to the objective system if it fails when its age x satisfies $x < t$. In the case when $t \leq x \leq T$ a corrective replacement (CR) is performed to the system when it fails. When x becomes equal to T preventive replacement (PR) of the system is performed. Ohnishi et al. (1990) provided and discussed a general formulation to such a problem based on semi–Markov decision processes. However, the (t, T)–policy does not take an opportunity–based replacement into account.

Our model is of practical interest for instance when discussing behaviour related to computers. In recent years, personal computers (PC) have become an essential key component in an office as well as in manufacturing systems to preserve an update significant information. The moment a PC fails we may lose significant information stored on its hard disk. If we want to prevent such a serious loss, it is important to conduct preventive maintenance such as backup operations of files on the hard disk and to replace our PC preventively by a new one before it fails. On the other hand, technology associated with PCs has shown a remarkable development in the past two decades. New models of PCs have been released every half–year. Operating systems and their related major application software have also been released frequently. Harddisk memory sizes have become significantly larger with new versions of PCs. These facts have sometimes obliged us to replace our PC with a new one even when it has not failed. This indicates that these factors can be regarded as opportunities in the opportunistic preventive maintenance policies.

2. MODEL FORMULATION

Under the proposed replacement policy the process behaviour generates a renewal reward process (see e.g. Ross, 1970) where the renewal corresponds to a PR or CR, whichever occurs first. In this section we formulate the expression of the long-run average cost per unit time based on the renewal reward theory.

Let c_1 and $c_2(< c_1)$, respectively, denote the cost for CR and PR, respectively. Furthermore, let c_3 be the cost for a minimal repair. The reasons for the cost c_3 are for instance the following: (1) Even if a system is recovered from a failure free of charge during the warranty period we may lose significant information on the hard disk at a failure, (2) we will not be able to use our PC while it is repaired, and (3) even if an alternative PC is supplied we will waste our time on setting it up.

It is assumed that the opportunities occur according to a homogeneous Poisson process with parameter λ. As mentioned before, we take an opportunity with probability p if we have passed the warranty period. When we look at the process caused by the opportunity-based PRs, the cumulative distribution function (cdf) of the time between successive PRs is given by

$$G_p(t) = 1 - \exp(-\lambda pt), \qquad t \geq 0. \tag{2.1}$$

It should be noted that the cdf in (2.1) is that of an exponential distribution with parameter λp; see Block et al. (1985).

Let us denote by $\gamma(S)$ the residual life, or excess life, at age S. Then the survival function $R_\gamma(t)$, and the failure rate $r_\gamma(t)$ of $\gamma(S)$ are expressed by

$$\begin{aligned} R_\gamma(t) &= \frac{R(t+S)}{R(S)} \\ r_\gamma(t) &= \frac{f_\gamma(t)}{R_\gamma(t)} = r(t+s) \end{aligned}$$

where $R(t)$, $f(t)$ and $r(t)$, respectively, is the survival function, the density function and the failure rate of the system; see e.g. Ross (1970).

The long-run average cost per unit time $C(T)$ under this policy is given by (see Iskandar & Sandoh, 1999)

$$C(T) = \frac{B(T)}{A(T)}$$

where

$$A(T) = S + \int_S^T R_\gamma(t-S)\exp(-\lambda p(t-S))dt$$

and

$$\begin{aligned} B(T) =& c_1 - (c_1 - c_2)\Big[\int_S^T R_\gamma(t-S)\lambda p \exp(-\lambda p(t-S))dt \\ &+ R_\gamma(T-S)\exp(-\lambda p(T-S))\Big] + c_3 E\left[N(S)\right] \end{aligned}$$

and $N(S)$ represents the number of minimal repairs during the warranty period. It should be noted here that $E[N(S)]$ is difficult to derive in a closed form for a general failure time distribution $F(t)$.

Iskandar & Sandoh (1999) proved the following results (although the findings were not formulated as a theorem in this way).

Theorem 1 (Iskandar & Sandoh, 1999). If F is IFR then there is a unique solution T^*, with $S \leq T^* \leq \infty$, which minimises the average long–run cost $C(T)$. The solution T^* depends on the costs involved and the different possibilities are described as follows, where

$$\begin{aligned} a =& Sr(S) + 1 \\ b =& r(\infty)\left[S + \frac{1}{R(S)}\int_S^\infty \exp(-\lambda p(t-S))R(t)dt\right] \\ &+ \frac{\lambda p}{R(S)}\int_S^\infty \exp(-\lambda p(t-S))R(t)dt \\ c_0 =& c_1 + c_3 E\left[N(S)\right] \end{aligned}$$

(1) If $c_0/(c_1 - c_2) \leq a$, then we have $T^* = S$. The long–run average cost in this case is given by

$$C(T^*) = \frac{c_2 + c_3 E\left[N(S)\right]}{S}$$

(2) If $a < c_0/(c_1 - c_2) \leq b$ there exists a unique and finite optimal solution T^* with $S < T^*$. The long–run average cost is in this case given by

$$C(T^*) = (c_1 - c_2)r(T^*)$$

(3) If $c_0/(c_1 - c_2) \geq b$, then we have $T^* = \infty$, i.e. no preventive replacement should be performed.

Remark. In Iskandar & Sandoh (1999) it was said that if F is IFR then a unique finite optimal solution T^* exists. That is not quite true, as can be seen from Theorem 1. Depending on the costs c_1, c_2 and c_3 it is quite possible for T* to be infinite if e.g. $r(\infty)$ is finite, which may occur; see (3) above. However, if F is IFR with $r(\infty) = \infty$ then a finite unique solution T^* always exists, with $S \leq T^*$.

In Section 4 we will return to the problem to find the value T* which minimises the long–run average cost $C(T)$. The solution is partly graphical and based on the scaled TTT–transform which will be presented and discussed in the next Section.

3. THE TTT–PLOT AND THE SCALED TTT–TRANSFORM

Suppose that we have a complete ordered sample $0 = t_{(0)} \leq t_{(1)} \leq \cdots \leq t_{(n)}$ of times to failure from n identical and independent non–repairable units with life distribution F and survival function $R = 1 - F$. The TTT–plot (TTT = Total Time on Test) of these observations is then obtained in the following way:

- Calculate the TTT-values
  $S_j = nt_{(1)} + (n-1)(t_{(2)} - t_{(1)}) + ... + (n-j+1)(t_{(n)} - t_{(n-1)})$ for $j = 1, 2, \cdots, n$
  (for convenience we set $S_0 = 0$).
- Normalize these TTT-values by calculating $u_j = S_j/S_n$ for $j = 1, 2, \cdots, n$.
- Plot $(j/n, u_j)$ for $j = 0, 1, \cdots, n$.
- Join the plotted points by line segments.

Accordingly, the TTT-plot consists of line segments starting in the lower left corner and ending in the upper right corner of the unit square. We also want to emphasise that the TTT-plot is by definition independent of scale. When the sample size n increases to infinity the TTT-plot converges to the scaled TTT-transform of the life distribution $F(t)$ from which the sample has come, see e.g. Langberg et al. (1980). This is illustrated in Figure 1.

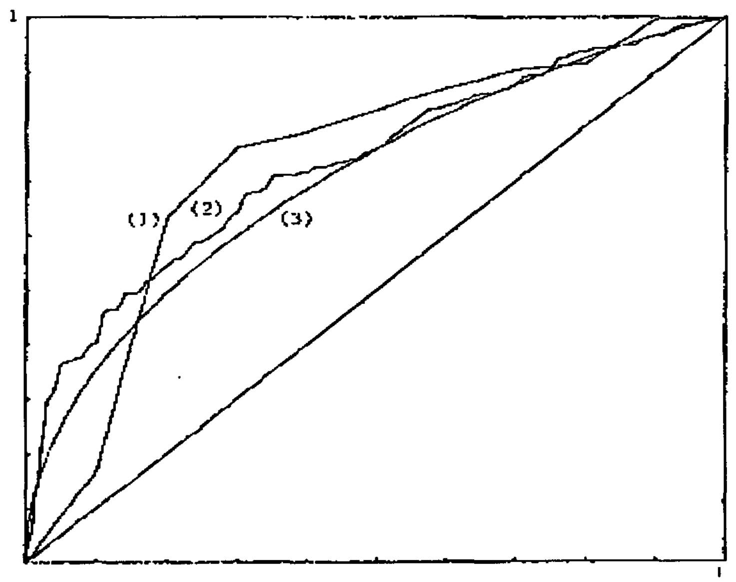

Figure 1. TTT-plots based on simulated data from a Weibull distribution with $\beta = 2.0$, $n = 10i$ (1) and $\beta = 2.0$, $n = 100i$ (2) and the scaled TTT-transform of a Weibull distribution with shape parameter $\beta = 2.0i$ (3). (From Bergman & Klefsjö, 1984.)

The scaled TTT-transform of a life distribution $F(t)$ is defined as

$$\varphi(u) = \frac{1}{\mu} \int_o^{F(u)} R(t)dt \qquad \text{for} \quad 0 \leq u \leq 1$$

where $R(t) = 1 - F(t)$ is the survival function and μ is the finite mean.

The scaled TTT–transform is, as the name indicates, independent of scale and transforms every life distribution (i.e., a distribution function with $F(0) = 0$) to a curve within the unit square. For instance, for a Weibull distribution with survival function $R(t) = \exp(-(t/\alpha)^{\beta})$, $t \geq 0$, the transform depends only on β and is independent of the value of α. Furthermore, every exponential distribution $F(t) = 1-\exp(-\lambda t)$, $t \geq 0$, is transformed to the diagonal in the unit square, independently of the value of λ; see Figure 2. Different deviations from the diagonal in the unit square accordingly means different deviations from the exponential distribution. For instance it is well–known that $F(t)$ has increasing failure rate (IFR) if and only if the scaled TTT–transform is concave; see Barlow & Campo (1975). This is illustrated by, for instance, curve number (2) in Figure 2.

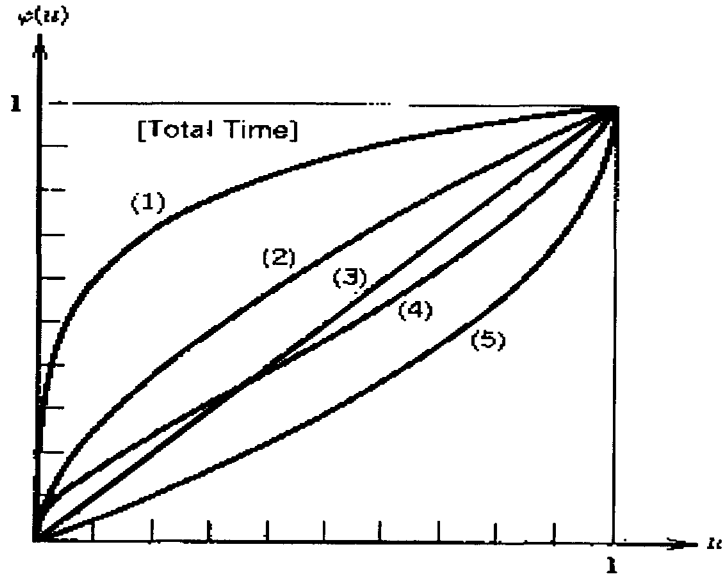

Figure 2. Scaled TTT–transforms from five different life distributions: (1) normal with $\mu = 1$, $\sigma = 0.3$; (2) gamma with shape parameter 2.0; (3) exponential distribution; (4) lognormal with $\mu = 0$, $\sigma = 1$; (5) Pareto distribution with survival function $R(t) = (1+t)^{-2}$, $t \geq 0$. (From Bergman & Klefsjö, 1988a.)

The scaled TTT–transform and the empirical counterpart, the TTT–plot, were introduced by Barlow & Campo (1975). These tools were first used for model identification purposes. Since then several other applications have appeared, both theoretical and practical. Among these are the analysis of different aging properties (see e.g. Klefsjö, 1982) and for optimization when studying different replacement problems of non–repairable units (see e.g. Bergman & Klefsjö, 1982, 1983, 1988b). Recently also applications for repairable units have been presented (see e.g. Klefsjö & Kumar, 1992). The TTT–plot and the TTT–transform have also proven to be useful tools in more theoretical discussions, e.g. when studying different test statistics;

see e.g. Bergman & Klefsjö (1989), Klefsjö (1983, 1989), Klefsjö & Westberg (1996). It should also be noticed that the TTT–transform is closely related to the Lorenz transform, which is widely used in economics (see e.g. Chandra & Singpurwalla, 1981, Klefsjö, 1984).

4. ANALYSIS BY USING TTT–TRANSFORMS

If we study the expression in Section 2 for the average long–run cost $C(T)$ it is easily seen that we can rewrite

$$B(T) = (c_1 - c_2)\left(\frac{c_0}{c_1 - c_2} - \lambda p \int_S^T [1 - H_\gamma(t - S)dt]\right) - 1 + H_\gamma(T - S)$$

where

$$c_0 = c_1 + c_3 E\left[N(S)\right]$$

and

$$1 - H_\gamma(t) = \exp(-\lambda p t) R_\gamma(t)$$

i.e. $1 - H_\gamma(t)$ is the survival function of a series system consisting of two independent components, one of which has an exponential time to failure with parameter λp, and the other has the survival function $R_\gamma(t)$, $t \geq 0$. We note here that the failure rate $r_H(t)$ of $H_\gamma(t)$ is equal to $\lambda p + r_\gamma(t)$, since the failure rate of a series system is equal to the sum of the failure rates (see e.g. Barlow & Proschan, 1981). This means for instance, that $H_\gamma(t)$ is IFR if $F(t)$ is IFR, a fact we soon will use.

In the same way we can write

$$A(T) = S + \int_S^T [1 - H_\gamma(t - S)]\, dt$$

From this we get that

$$\begin{aligned} C(T) &= \text{const} + \text{const}\frac{c_5 + H_\gamma(T - S)}{S + \int_S^T [1 - H_\gamma(t - S)]\, dt} \\ &= \text{const} + \text{const}\frac{c_5 + H_\gamma(T - S)}{\frac{S}{\mu} + \varphi(H_\gamma(T - S))} \end{aligned} \quad (4.1)$$

where

$$c_5 = \frac{c_0}{c_1 - c_2} - \lambda p S - 1$$

and $\varphi(t)$ is the scaled TTT–transform of $H_\gamma(t)$, i.e the life distribution of the series system. Since the second constant in the expression (4.1) above is positive we get that $C(T)$ is minimal at the same value of T as

$$D(T) = \frac{c_5 + H_\gamma(T - S)}{c_6 + \varphi(H_\gamma(T - S))} \quad (4.2)$$

where $c_6 = S/\mu$. Since

$$c_5 = \frac{c_1 + c_3 E\left[N(S)\right]}{c_1 - c_2} + \lambda p S - 1 > 0$$

we can analyse $D(T)$ by using a technique based on the scaled TTT–transform similar to the one described in Bergman & Klefsjö (1983).

The technique consists of two phases.

- Firstly, we substitute $u = H_\gamma(T - S)$ in (4.2) and look for the value u^* of u, $0 \le u \le 1$, for which

$$\frac{c_5 + u}{c_6 + \varphi(u)} \tag{4.3}$$

  is minimised.

- Secondly, we get T^*, the optimal value of T, by solving the equation $u^* = H_\gamma(T^* - S)$.

The first problem here is easily illustrated and solved by using the scaled TTT–transform of $H_\gamma(t)$. This procedure is illustrated in Figure 3.

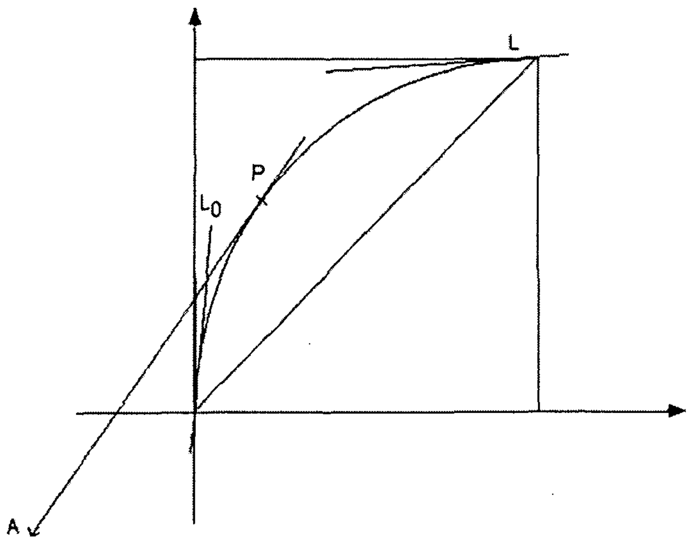

Figure 3. To minimise the expression in (4.3) we draw the line through $A = (-c_5, -c_6)$ which touches the scaled TTT–transform of $H_\gamma(t)$ and has the largest slope. If the life distribution $F(t)$ has increasing failure rate (IFR) and accordingly also $H_\gamma(t)$ is IFR, i.e. the scaled TTT–transform is concave, that line touches the transform at $P = (u, \varphi(u))$, $0 \le u \le 1$. If P coincides with $(0, 0)$ we get that $u^* = 0$, which gives $T^* = S$. If, instead P coincides with (1,1), then $u^* = 1$, and $T^* = \infty$.

From Figure 3 we realise that a unique value u^* which minimises the expression in (4.3) exists if the scaled TTT–transform is concave. This is the case if H_γ has increasing failure rate (IFR) which in turn happens if F has IFR. If the line through A with the largest slope touches the scaled TTT–transform in a value of u, with $0 < u < 1$, the unique and finite value T^* is the solution of $u^* = H_\gamma(T^* - S)$. If the line through A with the largest slope touches the transform in (0,0) the optimal value T^*, which is the solution then to $0 = H_\gamma(T^* - S)$ gives $T^* = S$. If, on the other hand, the line with the largest slope touches the transform at (1,1) we get the optimal T–value T^* from the equation $1 = H_\gamma(T^* - S)$, i.e. $T^* = \infty$.

The slope $l_{\tan}$ of the tangency to the scaled TTT–transform at $P = (u, \varphi(u))$ is equal to

$$l_{\tan} = \frac{1}{\mu r_H(H_\gamma(u))}$$

where $H_\gamma^{-1}(u) = \inf(x : H_\gamma(x) \geq u)$; see Barlow & Campo (1975). Accordingly, we get that $T^* = S$ if the slope l_{A0} of the line L_{A0} through A and $(0,0)$ satisfies

$$l_{A0} \geq l_0 \tag{4.4}$$

where l_0 is the slope of the tangency L_0 at $(0,0)$. The condition in (4.4) is equal to $c_6/c_5 \geq l_0$. In the same way $T^* = \infty$ if

$$l_{A1} \leq l_\infty \tag{4.5}$$

where l_∞ is the slope of the tangency L_∞ at (1,1) and l_{A1} is the slope of the line L_{A1} through A and (1,1). The inequality in (4.5) is equivalent to

$$\frac{1 + c_5}{1 + c_6} \leq l_\infty$$

Since

$$l_\infty = \frac{1}{\mu r_H(\infty)} = \frac{1}{\mu[\lambda p + r_\gamma(\infty)]} = \frac{1}{\mu[\lambda p + r(\infty)]}$$

and

$$l_0 = \frac{1}{\mu r_H(0)} = \frac{1}{\mu\,[\lambda p + r_\gamma(0)]} = \frac{1}{\mu\,[\lambda p + r(S)]}$$

we now, by straight–forward calculations, get the results by Iskandar & Sandoh (1999) presented in Theorem 1 above.

It is easily seen here that it is quite possible that $T^* = S$ or $T^* = \infty$ depending on the shape of the TTT–transform and on where the point A is situated. Different sensitivity analyses regarding the costs are also easily performed by using this graphical technique by moving the point A.

5. Estimation of T^* when F is unknown

In most cases in practice, we do not know the life distribution. If we do not know $F(t)$ how can we then proceed to estimate T^*, with T_n^* say ? One way is to estimate T^* by using an ordered sample of n times to failure, $0 = t_{(0)} \leq t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}$ from our system with the life distribution $F(t)$. It is then natural to get an estimator of u^* by studying the empirical counterpart to the expression in (4.3) and let T_n^* be the value of $t_{(k)}$ which corresponds to u^*, including $T_n^* = 0$ if $u^* = 0$ and $T_n^* = \infty$ if $u^* = 1$.

This can be solved in a similar way as discussed in Section 4 by using a generalised TTT–plot which is obtained by first plot $(H_n(t_j), A_j)$ for $j = 0, 1, \cdots, n$, where $1 - H_n(t) = \exp(-\lambda pt)[1 - j/n]$ for $t_{(j)} \leq t < t_{(j+1)}$ and $j = 0, 1, \cdots, n-1$.

is an estimator of $H_\gamma(t)$ based on the Kaplan–Meier estimator of $F(t)$, see Kaplan & Meier (1958), and

$$A_j = \frac{\int_S^{t_{(j)}} [1 - H_n(t)]dt}{\int_S^{t_{(n)}} [1 - H_n(t)]dt}$$

and then connect the plotted points by line segments. For more information about generalised TTT–plots, see e.g. Bergman & Klefsjö(1983), (1988).

Then we draw the line from $A = (-c_6, -c_5)$ which touches the generalised TTT–plot and has the largest slope (cf. Figure 4). If this line touches the generalised TTT–plot at $(H_n(t_k), A_k)$, where k is one of $1, 2, \cdots, n-1$ then $T_n^* = t_{(k)}$. If $k = n$ then $T_n^* = \infty$ and if $k = 0$ we get $T_n^* = S$. This process is illustrated in more detail in Bergman & Klefsjö (1983) in connection with the age replacement problem with discounted costs which leads to a similar function to study.

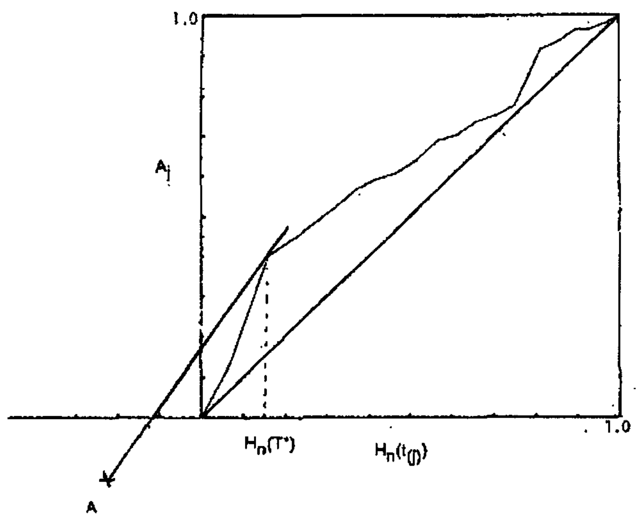

Figure 4. Illustration of the graphical procedure to determine T_n^* based on the generalised TTT–plot. The sample is simulated from $R(t) = (1 + t)\exp(-t)$, $t \geq 0$. (From Bergman & Klefsjö, 1983.)

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