자기매계변수 연성을 갖는 계의 응답의 통계적 특성

Stochastic Response of a System with Autoparametric Coupling

조 덕 상* 김 영 종**
Cho, Duk-Sang Kim, Young-Jong

요 지

본 연구에서는 광대역 불규칙가진력을 받는 자기매계변수계의 모드상호작용을 고찰하였다. 고찰대상 모델은 매우 흔한 구조물의 형태인 내부공진을 갖는 자기매계변수 동흡진기이다. Gaussian closure 방법에 의하여 계의 불규칙 응답을 나타내는 동적 모멘트방정식은 1차 및 2차 모멘트로 구성된 자율 상미분방적식으로 줄여진다. 계의 평형해와 평형해의 안정성측면에서 계의 응답이 조사되었다. 참고문헌 [18]과 [20]에서 보고된 발견한 감쇠가 안정성을 축소하기도 한다라는 이 효과는 본 연구에서 발견할 수 없었다. 또한 확정적 비선형계에 존재하는 포화현상은 발견되지 않았다.

핵심용어 : 불규칙가진력, 자기매계변수계, 동흡진기, 내부공진, Gaussian closure 방법, 동적 모멘트방정식

Abstract

The nonlinear modal interaction of an autoparametric system under a broadband random excitation is investigated. The specific system examined is an autoparametric vibration absorber with internal resonance, which is typical of many common structural configurations. By means of Gaussian closure scheme the dynamic moment equations explaining the random responses of the system are reduced to a system of autonomous ordinary differential equations of the first and second moments. In view of equilibrium solutions of this system and their stability we examine the system responses. We could not find the destabilizing effect of damping, which was reported in References (18) and (20). The saturation phenomenon, which is well known in deterministic nonlinear system, did not take place for this system subject to broadband random excitation.

Keywords : *random excitation, autoparametric system, vibration absorber, internal resonance, Gaussian closure scheme, dynamic moment equation*

1. Introduction

Modal interactions of harmonically excited nonlinear systems with internal resonance have been studied extensively[1)~11)]. Those systems have been known to exhibit complicated behaviors such as jump and saturation phenomenon, Hopf bifurcations and a sequence of period-doubling bifurcations leading to chaos[4)~11)]. In the meantime, Ibrahim and his colleagues[12)~20)] have studied influences of internal resonance on responses of randomly excited nonlinear systems. For examples, Ibrahim and Roberts[14),18)] and Roberts[19)] included cubic nonlinear terms in the analysis for autoparametric vibration absorber[4)] with 1 : 2 internal resonance, and the desta-

* 정회원 · 영남대학교 공업기술연구소, 연구원
** 정회원 · 상주대학교 기계공학부, BK21교수

• 이 논문에 대한 토론을 2001년 3월 31일까지 본 학회에 보내주시면 2001년 6월호에 그 결과를 게재하겠습니다.

bilizing effect of a damping ratio was observed[18]. This destabilizing effect is a notable observation because the stabilizing effect of damping has been known as a generally accepted idea for resonance responses. Ibrahim and Li[16] and Ibrahim[17] reported that there exists no saturation phenomenon which is common in quadratic nonlinear systems with the internal resonance and harmonic excitation.

The motive of this study is due to the following two questions relevant to the above studies. 1) Is the destabilizing effect of damping really possible? 2) How can we assure that saturation doesn't exist? In order to answer these questions we selected an autoparametric vibration absorber[4] under a broadband random excitation. Obtaining moment equations from the Fokker-Planck equation corresponding to the equation of motion, we used Gaussian closure scheme to reduce a system of 14 autonomous ordinary differential equations for the first and second moments. We examined the equilibrium solution of this system and its stability.

2. Equations of Motion

Fig. 1 shows the autoparametric system under a broadband random excitation $F(t)$. The equations of motion of the system[4] are, for the main mass,

$$(M+m)\ddot{x}+c_1\dot{x}+k_1x-(6/5\,l)\,m(\dot{y}^2+y\ddot{y})=F(t) \tag{1}$$

and, for the cantilever,

$$m\ddot{y}+c_2\dot{y}+\{k_2-(6/5l)\,m\ddot{x}\}y + (36/25\,l^2)my(\dot{y}^2+y\ddot{y})=0 \tag{2}$$

where x and y are normal coordinates corresponding to the linearized system. Introducing

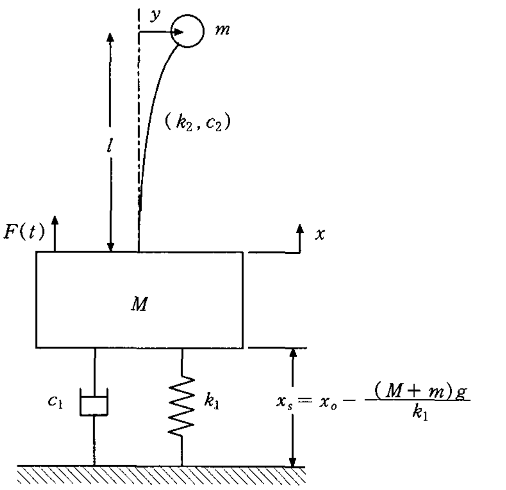

Fig. 1 Schematic diagram of an autoparametric absorber system

the notations

$$X=\frac{x}{x_s},\quad Y=\frac{y}{l},\quad \zeta_1=\frac{c_1}{2(M+m)\omega_1},$$

$$\zeta_2=\frac{c_2}{2m\omega_2},\quad \tau=\omega_1 t,\quad r=\frac{\omega_2}{\omega_1},\quad \varepsilon=\frac{l}{x_s},$$

$$R=\frac{m}{M+m},\quad \omega_1^2=\frac{k_1}{M+m},\quad \omega_2^2=\frac{k_2}{m},$$

$$W(\tau)=\frac{F(\tau/\omega_1)}{(M+m)x_s\omega_1^2},\quad \rho=\frac{6}{5},$$

we have the nondimensionalized equations as follows :

$$X''+2\zeta_1X'+X-\rho\varepsilon R(Y^2+YY'')=W(\tau) \tag{3}$$

$$Y''+2\zeta_2 rY'+(r^2-\frac{\rho}{\varepsilon}X'')Y + \rho^2 Y(Y'^2+YY'')=0 \tag{4}$$

In the above equations dot and prime denote differentiations with respect to t and τ, respectively.

For approximation we consider the system weakly nonlinear. Eliminating the nonlinear acceleration terms and neglecting the fourth or higher orders of nonlinear terms we have

$$X'' + 2\zeta_1 X' + X + \rho\varepsilon R(r^2 Y^2 + 2r\zeta_2 YY' - Y'^2) + \rho^2 R(-W(\tau)Y^2 + XY^2 + 2\zeta_1 X' Y^2) = W(\tau) \quad (5)$$

$$Y'' + 2\zeta_2 rY' + r^2 Y + \frac{\rho}{\varepsilon}(-YW(\tau) + XY + 2\zeta_1 YX') + \rho^2(1-R)(-r^2 Y^3 - 2\zeta_2 rY^2 Y' + YY'^2) = 0 \quad (6)$$

Random excitation $W(\tau)$ is assumed to be zero mean white noise having the autocorrelation function

$$R_{WW}(\varDelta\tau) = E[W(\tau)W(\tau+\varDelta\tau)] = 2D\,\delta(\varDelta\tau) \quad (7)$$

where $2D$ represents the spectral density when we express the frequency by $f(=\omega/2\pi)$, and $\delta(\varDelta\tau)$ is the Dirac delta function.

3. Moment Equations by Gaussian Closure Scheme

Introducing the notations

$$\{X, Y, X', Y'\}^T = \{X_1, X_2, X_3, X_4\}^T = \boldsymbol{X}$$

and letting $W(\tau)$ be a formal derivative of a Brownian process, i.e., $W(\tau) = dB(\tau)/d\tau$, we can express the Eq.(5) and (6) as following Itô equation.

$$\begin{aligned} &dX_1 = X_3 d\tau, \quad dX_2 = X_4 d\tau, \\ &dX_3 = \{-2\zeta_1 X_3 - X_1 \\ &\quad + \rho\varepsilon R(X_4^2 - 2r\zeta_2 X_2 X_4 - r^2 X_2^2) \\ &\quad + \rho^2 R(-2\zeta_1 X_2^2 X_3 - X_1 X_2^2)\}d\tau \\ &\quad + (1 + \rho^2 R X_2^2) dB(\tau) \end{aligned} \quad (8)$$

$$\begin{aligned} dX_4 = &\{-2\zeta_2 rX_4 - r^2 X_2 \\ &+ \frac{\rho}{\varepsilon}(-X_1 X_2 - 2\zeta_1 X_2 X_3) \\ &+ \rho^2(1-R)(r^2 X_2^3 + 2\zeta_2 r X_2^2 X_4 - X_2 X_4^2)\} d\tau \\ &+ \frac{\rho}{\varepsilon} X_2 dB(\tau) \end{aligned}$$

The solution process of this equation is a Markov process and the Fokker-Planck equation may be applied for the Markov vector $\boldsymbol{X}$ in the form

$$\frac{\partial}{\partial\tau} p(\mathbf{x}, \tau) = -\sum_{i=1}^{4} \frac{\partial}{\partial x_i}[a_i(\mathbf{x}, \tau) p(\mathbf{x}, \tau)] + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial^2}{\partial x_i \partial x_j}[b_{ij}(\mathbf{x}, \tau) p(\mathbf{x}, \tau)] \quad (9)$$

where $p(\mathbf{x}, \tau)$ is the joint probability density function, and $a_i(\mathbf{x}, \tau)$ and $b_{ij}(\mathbf{x}, \tau)$ are the first and second incremental moments of the Markov process $\boldsymbol{X}(\tau)$. These are defined as follows :

$$a_i(\mathbf{x}, \tau) = \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} E\{X_i(\tau + \delta\tau) - X_i(\tau) \mid \boldsymbol{X}(\tau) = \mathbf{x}\} \quad (10)$$

$$b_{ij}(\mathbf{x}, \tau) = \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} E\{[X_i(\tau + \delta\tau) - X_i(\tau)][X_j(\tau + \delta\tau) - X_j(\tau)] \mid \boldsymbol{X}(\tau) = \mathbf{x}\} \quad (11)$$

From Eq.(8) a_i and b_{ij} are evaluated as follows :

$$\begin{aligned} &a_1 = x_3, \quad a_2 = x_4 \\ &a_3 = -2\zeta_1 x_3 - x_1 + \rho\varepsilon R(x_4^2 - 2r\zeta_2 x_2 x_4 - r^2 x_2^2) \\ &\qquad + \rho^2 R(-2\zeta_1 x_2^2 x_3 - x_1 x_2^2) \\ &a_4 = -2\zeta_2 r x_4 - r^2 x_2 + \frac{\rho}{\varepsilon}(-x_1 x_2 - 2\zeta_1 x_2 x_3) \\ &\qquad + \rho^2(1-R)(r^2 x_2^3 + 2\zeta_2 r x_2^2 x_4 - x_2 x_4^2) \end{aligned} \quad (12)$$

$$b_{33} = 2D + 4\rho^2 R x_2^2 D, \quad b_{34} = b_{43} = 2\frac{\rho}{\varepsilon} x_2 D,$$
$$b_{44} = 2\frac{\rho^2}{\varepsilon^2} x_2^2 D, \quad \text{all other } b_{ij} = 0$$

Since it is impossible to obtain the exact solution $p(\mathbf{x}, \tau)$ to the Fokker-Planck equation, we are trying to examine the system responses by means of moment equations. First of all, introducing the following notations for the nth-order moments of the system responses :

$$m_{\alpha,\beta,\gamma,\eta}(\tau)=E[X_1^{\alpha}X_2^{\beta}X_3^{\gamma}X_4^{\eta}]$$
$$=\int\int\int\int_{-\infty}^{\infty}x_1^{\alpha}x_2^{\beta}x_3^{\gamma}x_4^{\eta}\,p(\mathbf{x},\tau)\,dx_1dx_2dx_3dx_4$$

with $n=\alpha+\beta+\gamma+\eta$, we can derive a set of dynamic moment equations of any order by multiplying Eq.(9) by $x_1^{\alpha}x_2^{\beta}x_3^{\gamma}x_4^{\eta}$ and integrating by parts over the entire state space $-\infty < x_i < \infty$. This procedure results in the following general dynamic moment equation.

$$\begin{aligned}
m'_{\alpha,\beta,\gamma,\eta} &= \alpha m_{\alpha-1,\beta,\gamma+1,\eta}+\beta m_{\alpha,\beta-1,\gamma,\eta+1}\\
&-\gamma m_{\alpha+1,\beta,\gamma-1,\eta}-\varepsilon\rho R r^2\gamma m_{\alpha,\beta+2,\gamma-1,\eta}\\
&-\rho^2 R\gamma m_{\alpha+1,\beta+2,\gamma-1,\eta}+\varepsilon\rho R\gamma m_{\alpha,\beta,\gamma-1,\eta+2}\\
&+(\gamma-1)\gamma D m_{\alpha,\beta,\gamma-2,\eta}\\
&+2\rho^2R(\gamma-1)\gamma D m_{\alpha,\beta+2,\gamma-2,\eta}-2\zeta_1\gamma m_{\alpha,\beta,\gamma,\eta}\\
&-2\rho^2R\zeta_1\gamma m_{\alpha,\beta+2,\gamma,\eta}\\
&-2\varepsilon\rho R r\zeta_2\gamma m_{\alpha,\beta+1,\gamma-1,\eta+1}-r^2\eta m_{\alpha,\beta+1,\gamma,\eta-1}\\
&-\frac{\rho}{\varepsilon}\eta m_{\alpha+1,\beta+1,\gamma,\eta-1}\\
&+\rho^2(1-R)r^2\eta m_{\alpha,\beta+3,\gamma,\eta-1}\\
&-\rho^2(1-R)\eta m_{\alpha,\beta+1,\gamma,\eta+1}\\
&+\frac{\rho^2}{\varepsilon^2}\eta(\eta-1)D m_{\alpha,\beta+2,\gamma,\eta-2}\\
&-2\frac{\rho}{\varepsilon}\zeta_1\eta m_{\alpha,\beta+1,\gamma+1,\eta-1}-2r\zeta_2\eta m_{\alpha,\beta,\gamma,\eta}\\
&+2\rho^2(1-R)r\eta m_{\alpha,\beta+2,\gamma,\eta}+2\frac{\rho}{\varepsilon}\gamma\eta D m_{\alpha,\beta+1,\gamma-1,\eta-1}
\end{aligned}\tag{13}$$

Eq.(13) constitutes a set of infinite coupled equations. In other words, the differential equation of order n contains moment terms of order $n+1$ and $n+2$. The Gaussian closure is based on the assumption that the response process is nearly Gaussian and is carried out by setting third -and fourth-order cumulants to zero. In this case we can generate 14 coupled differential equations for first- and second-order moments depend on the first through fourth moments. The third- and fourth-order moments can be expressed in terms of lower -order moments as follows[20),21)]:

$$\begin{aligned}
E[X_\alpha X_\beta X_\gamma] &= E[X_\alpha]E[X_\beta X_\gamma]\\
&+E[X_\beta]E[X_\alpha X_\gamma]+E[X_\gamma]E[X_\alpha X_\beta]\\
&-2E[X_\alpha]E[X_\beta]E[X_\gamma]
\end{aligned}\tag{14}$$

$$\begin{aligned}
E[X_\alpha X_\beta X_\gamma X_\eta] &= E[X_\alpha X_\beta]E[X_\gamma X_\eta]\\
&+E[X_\alpha X_\gamma]E[X_\beta X_\eta]+E[X_\alpha X_\eta]E[X_\beta X_\gamma]\\
&-2E[X_\alpha]E[X_\beta]E[X_\gamma]E[X_\eta]
\end{aligned}\tag{15}$$

Substituting Eq.(14) and (15) into Eq.(13) we can obtain a system of 14 differential equations for 14 first- and second- order moments. For convenience the system is expressed as follows :

$$\boldsymbol{m}'=\boldsymbol{f}(\boldsymbol{m}),\quad \boldsymbol{m}\in R^{14}\tag{16}$$

where $\boldsymbol{m}=\{m_{1,0,0,0}, m_{0,1,0,0},\cdots, m_{0,0,1,1}\}^T$ is the moment vector and $\boldsymbol{f}(\boldsymbol{m})=\{f_1(\boldsymbol{m}), f_2(\boldsymbol{m}),\cdots, f_{14}(\boldsymbol{m})\}^T$ is the vector field of the system. We can obtain the equilibrium solution $\boldsymbol{m}_0$ from

$$\boldsymbol{f}(\boldsymbol{m}_0)=\boldsymbol{0}\tag{17}$$

In order to investigate the stability of the equilibrium solution, we let

$$\boldsymbol{m}=\boldsymbol{m}_0+\delta\boldsymbol{m}$$

where $\delta\boldsymbol{m}$ is a small disturbance. The disturbance $\delta\boldsymbol{m}$ satisfies, to the first order

$$\delta\boldsymbol{m}'=\left.\frac{\partial f}{\partial\boldsymbol{m}}\right|_{\boldsymbol{m}=\boldsymbol{m}_0}\delta\boldsymbol{m}\tag{18}$$

If real parts of all eigenvalues of the Jacobian matrix are negative, the solution $\boldsymbol{m}_0$ is considered asymptotically stable.

From Eq.(16) we can see that the system has the following equilibrium solution

$$m_{2,0,0,0} \equiv E[X_1^2] = D/2\zeta_1,$$
$$m_{0,0,2,0} \equiv E[X'^2_1] = D/2\zeta_1 \quad (19)$$

and all other moments are zero. This equilibrium solution tells that the autoparametric vibration absorber undergoes the main system motion (X) with no cantilever motion ($Y=0$), in other words, the motion is unimodal.

4. Numerical Results

First of all, we slove the Jacobian matrix in Eq.(18) to examine the stability of the system. When the solution becomes unstable, we investigate the long-term behavior of the moments by integrating numerically the ordinary differential Eq.(16). Fig. 2 shows how the mean square values of the steady-state motion depend on the frequency ratio $r=\omega_2/\omega_1$ when mass ratio, R is equal to 0.2 and 0.15. In Fig. 2(a) and 2(b), two horizontal lines far from $r=0.5$ imply that the corresponding response is a stationary process, because the mean square values are independent of τ as well as r. The results showing the facts that the main system motion excited directly doesn't encourage the cantilever motion and the responses by a stationary excitation are stationary, coincide with the response characteristics of linear systems. According to the stability analysis, the equilibrium solution loses the stability at r_{b1} and r_{b2} by Hopf bifurcations. Therefore, in the region of $r_{b1} < r < r_{b2}$, the moments can have the long-term behaviors such as periodic, quasi-periodic, and chaotic. In the figures the upper and lower limits of two moments are shown. These results show that the energy has been transferred from the main system motion excited directly to the cantilever motion not excited directly. Since the mean square values of this motion vary between both limit, the response is nonstationary. Due to the internal resonance condition ($r=0.5$) strengthening the couplings between the nonlinear terms, the system response shows the response characteristics of nonlinear systems. Decreasing the mass ratio, R leads to a increase in the cantilever motion. However bifurcation points, r_{b1} and r_{b2} are never changed according to varying mass ratio, R.

Fig. 3 represents time histories of mean square values of the system response at the steady

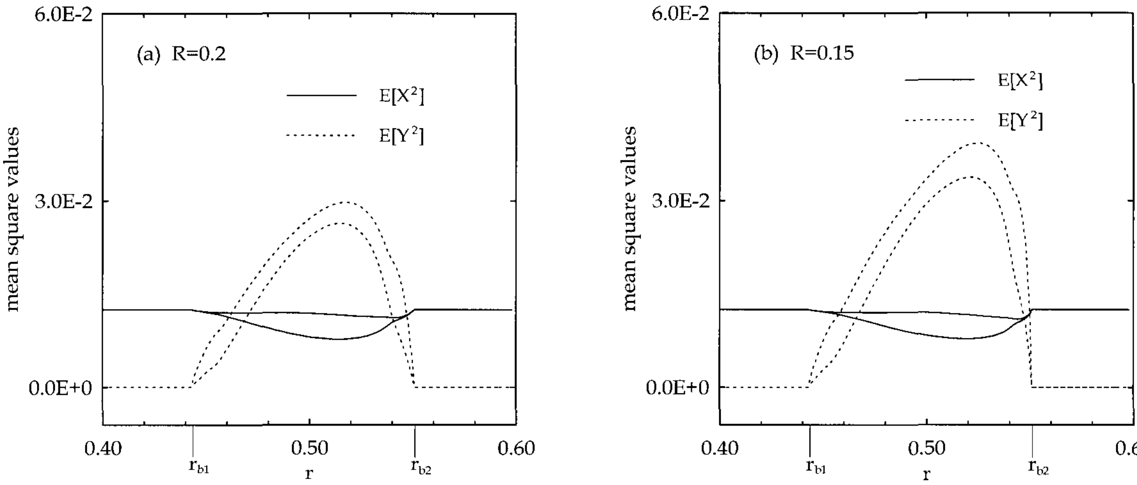

Fig. 2 Limits of mean square responses $E[X^2]$ and $E[Y^2]$ as functions of the frequency ratio r ($\zeta_1=0.01$, $\zeta_2=0.01$, $\varepsilon=2$, $2D=0.0005$).

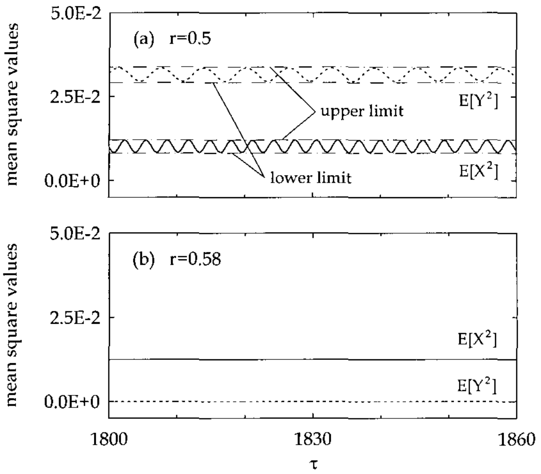

Fig. 3 Time histories of mean square responses $E[X^2]$ and $E[Y^2]$ ($\zeta_1=0.01, \zeta_2=0.01, \varepsilon=2, R=0.15, 2D=0.0005$).

state. Fig. 3(a) and (b) show the nonstationary ($r=0.5$) and stationary($r=0.58$) processes, respectively. Each of the mean square values $E[X^2]$ and $E[Y^2]$ is oscillating with twice the corresponding mode natural frequency. This coincides with the fact that square of a harmonic function is oscillating with twice its frequency.

Fig. 4 and 5 show how the system dampings affect Hopf bifurcation points in $2D-r$ planes, for $\zeta_2=0.01$ and $\zeta_1=0.005$, respectively. The figures show that stable regions expand as the system damping ratios increase. The result of Fig. 4 contradicts References (18) and (20) saying 'increase of the primary system damping ratio appears to have a destabilizing effect.' Normalizing the mean square displacement of the primary system, they couldn't avoid the constraint $D=2\zeta_1$, which might cause them to misunderstand the influence of the damping ratio ζ_1 on the stability.

Fig. 6 represent limits of mean square displacements as functions of the spectral density $2D$ proportional to mean square excitation. For the region of spectral density $2D$ below the Hopf bifurcation point $2D_b$, the mean square value of the main system motion excited directly increases linearly as $2D$, while the mean square value of the cantilever motion remains zero. In other words, the system response shows the response characteristics of linear system when $0 \le 2D \le 2D_b$. For $2D > 2D_b$, limits of mean square values of the main system motion as well as the cantilever motion increase as $2D$. Thus there exists no saturation, which implies

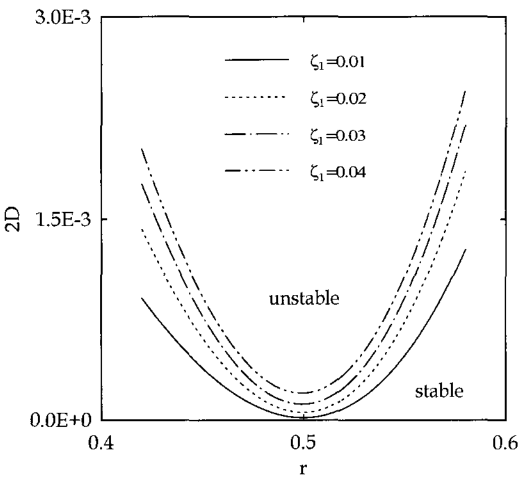

Fig. 4 Influence of damping ζ_1 on the stability boundary ($\zeta_2=0.01, \varepsilon=2$)

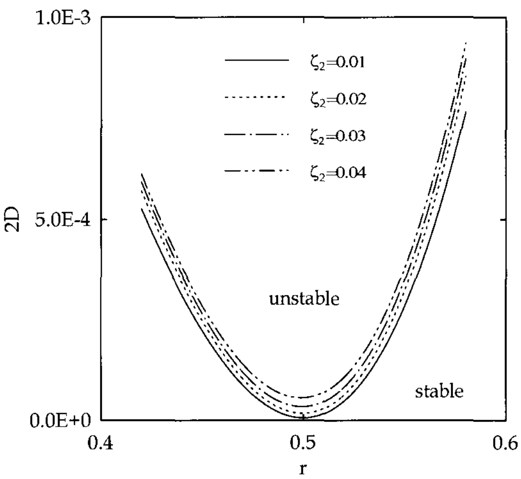

Fig. 5 Influence of damping ζ_2 on the stability boundary ($\zeta_1=0.005, \varepsilon=2$).

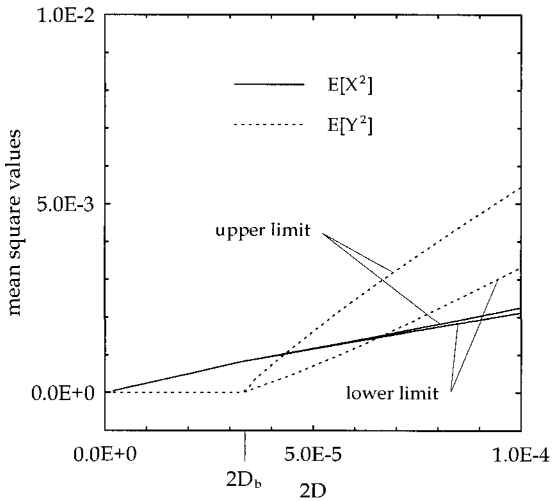

Fig. 6 Limits of mean square responses $E[X^2]$ and $E[Y^2]$ as functions of the spectral density 2D ($r=0.49$, $\zeta_1=0.01$, $\zeta_2=0.01$, $\varepsilon=2$, $R=0.15$)

a phenomenon that the motion excited directly stops increasing when the excitation level reaches a critical value.

5. Conclusions

In order to investigate the influences of the internal resonance on the system responses of a two-degree-of-freedom system with a random excitation, we examined an autoparametric vibration absorber with a broadband random excitation to the main mass. The results exhibit the following features.

(1) The stability regions in the parameter space are expanded as the system dampings increase. This is a remarkable contrast with statements in References (18) and (20). We believe that their misunderstanding is due to the normalization procedure of the analysis.

(2) By contrast to the cases of harmonic excitation, there exists no saturation of response level of the mode excited directly.

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(접수일자 : 1999. 7. 13)