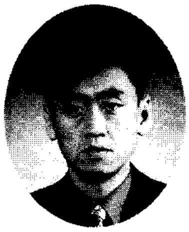
Probabilistic Analysis of Reinforced Concrete Beam and Slab Deflections Using Monte Carlo Simulation



Choi, Bong-Seob* Kwon, Young-Wung**

Abstract

It is not easy to correctly predict deflections of reinforced concrete beams and one-way slabs due to the variability of parameters involved in the calculation of deflections. Monte Carlo simulation is used to assess the variability of deflections with known statistical data and probability distributions of variables. A deterministic deflection value is obtained using the layered beam model based on the finite element approach in which a finite element is divided into a number of layers over the depth. The model takes into account nonlinear effects such as cracking, creep and shrinkage. Statistical parameters were obtained from the literature. For the assessment of variability of deflections, 12 cases of one-way slabs and T-beams are designed on the basis of ultimate moment capacity. Several results of a probabilistic study are presented to indicate general trends indicated by results and demonstrate the effect of certain design parameters on the variability of deflections. From simulation results, the variability of deflections relies primarily on the ratio of applied moment to cracking moment and the corresponding reinforcement ratio.

Keywords : deflection, variability, Monte Carlo simulation, cracking, creep, shrinkage, beam, one-way slab

* KCI member, Senior researcher, Ph.D., Hap Technological Group for Structural Safety, Co. Ltd., Korea
** KCI member, Professor, Ph.D., Dept of Architectural Engineering, University of Inchon, Korea

1. Introduction

A variety of equations and procedures have been presented in many codes of practice to predict the deflections of reinforced concrete members. However, all of these procedures provide a deterministic value for designers which is either smaller or larger than the deflection that would be likely to occur in a real structure. Even with the most sophisticated methods of analysis using experimentally determined material properties, the variability of short-time as well as long-time deflections is high. The actual behavior is a decidedly probabilistic phenomenon requiring statistical approaches for a rational analysis. Furthermore, the variability, once known, can be taken into account in the assessment of structural serviceability.

The objective of the study described in this paper is to discuss these issues and illustrate the variability of deflections and the effect of several design variables. For the probabilistic assessment of beam deflections, a deterministic layered beam model, which includes the effects of cracking, creep and shrinkage is developed using a finite element method, and it is used to perform Monte Carlo simulations which require the application of techniques to ensure that the simulation runs in the proper manner and produces output in an efficient manner.

Accordingly, the techniques used in this study and the statistical data and probability distributions of variables are outlined, and Monte Carlo simulation is conducted for each case of the designed one-way slabs and T-beams on the basis of these techniques to evaluate the variability of deflections. Three support conditions i.e. simply supported, both end fixed and equal two span continuous are considered. Two types of load-time histories are also considered for long-time deflections. Based on these design conditions, probability distributions of calculated deflections are presented in order to demonstrate how the variability is affected by the design parameters.

2. Development of Layered Beam Model

For a realistic estimate of reinforced concrete beam deflection under a given loading, nonlinear effects such as cracking, tension stiffening, creep and shrinkage must be taken into account in the analytical model. These effects result in variations in the material stiffness along the member length. The finite element method allows the variations to be taken into account by computing stiffness characteristics for localized regions of the structure and using matrix methods to assemble a global stiffness matrix for the entire structure.

Also, the beam is considered as a series of layers to allow for a variation in elastic constants with depth. The modulus of elasticity may vary from layer to layer but is considered to be constant within layers.

2.1 Analysis for Cracking Effect

In order to include the effect of cracking in the analysis, consideration is given to the tension stiffening effect of concrete which the concrete between cracks is still capable of carrying stress because of the bond between concrete and reinforcement. A tension stiffening parameter, β defined as $\beta = \varepsilon_{tu}/\varepsilon_{ti}$ is used to characterize the post peak stress-strain relationship.

In this study, it is assumed that after tensile cracking the post tensile response of concrete may be represented as a linear reduction in the tensile stress as shown in Fig. 1. It is called a linear stepped model for tension stiffening effect. In order to apply this concept to cracking analysis, it is necessary to evaluate the reduced modulus of elasticity for concrete layers due to cracking through the following iterative procedures.

Step 1. The modulus of elasticity in the tension region, E_t is assumed to be the same as one in the compression region until the tensile stress reaches the modulus of rupture, f_r (Case 1) as shown in Fig. 1.

Step 2. If the tensile stress exceeds the specified modulus of rupture, the modulus of elasticity, E_t may be modified to E_{t1}, which is the next lower concrete modulus (Case 2) defined by projecting from the calculated stress f_{t1} into the decreasing branch of the stress-strain diagram.

Step 3. The Analysis is repeated for other layers in the same manner.

Step 4. In the second iteration, if the tensile stress does not exceed the next lower rupture of modulus, f_{r1}, it is assumed that E_{t1} does not need to be modified (Case 3).

Step 5. If exceeding f_{r1}, E_{t1} is modified into the next lower concrete modulus, E_{t2} (Case 4).

Step 6. This iteration may continue until the stress stays inside envelope satisfying a desired convergence criteria.

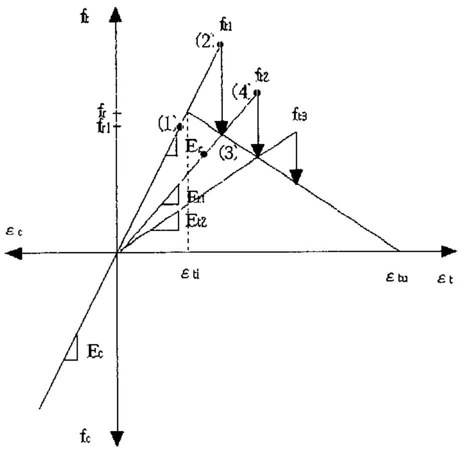


Fig. 1 Linear stepped model for tension stiffening

2.2 Analysis for Time-Dependent Effects

In this study, equations for creep and shrinkage recommended by ACI Committee Report 209R[1] are used to predict the time-dependent response of a reinforced concrete beam to given loading and environmental conditions.

For creep analysis, the age-adjusted effective modulus proposed by Trost[2] is used. This method consists of an elastic analysis with a modified elastic modulus, E_{ca} which is defined by Equation (1).

$$E_{ca} = \frac{E_{ci}}{1 + \chi\, \phi_t} \qquad (1)$$

where E_{ci} is the initial modulus of elasticity, χ is the aging coefficient and ϕ_t is the creep coefficient for standard conditions which is represented as follows:

$$\phi_t = \frac{t^{0.6}}{10 + t^{0.6}}\, \phi_u \qquad (2)$$

where t is age on the load duration and ϕ_u is ultimate creep coefficient. Therefore, in this analysis, the initial modulus of elasticity, E_c is modified to the reduced modulus of elasticity, E_{ca} as shown in Fig. 2. It is assumed that the modulus of elasticity in tension region, E_t is also changed into E_{ta}. By using the same post peak stress-strain relationship as for instantaneous and time-dependent analysis, the tensile strength under sustained stress is assumed to decrease with time.

For the case in which the reinforcement and eccentricity are constant along the span and the same in the positive and negative moment regions of continuous beam, shrinkage deflections for uniform beams are computed by Equation (3).

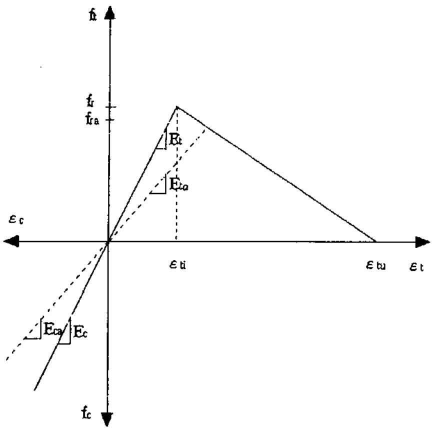


Fig. 2 Linear uniaxial stress-strain relation ship for creep effect

$$\Delta_{sh} = \xi_w \ \kappa_{sh} \ l^2 \qquad (3)$$

where ξ_w is a deflection coefficient for different boundary conditions , and κ_{sh} is the curvature due to shrinkage warping. The equivalent tensile force method, as modified by Branson[3] using $E_c/2$ and the gross section properties for better results, is used for computing shrinkage curvature in this research. It is shown in Equation (4).

$$\kappa_{sh} = \frac{T_s e_g}{\frac{E_c I_g}{2}} \qquad (4)$$

where $T_s = (A'_s + A_s)\varepsilon_{sh} E_s$, A'_s=area of top steel, A_s=area of bottom steel, e_g= distance between the centroid of gross concrete section and steel area and I_g= moment of inertia of the gross section.

The layered beam model was verified by comparison with available experimental results. A tension stiffening parameter, β =3.0 was found to give good correlation for both instantaneous and long-term deflections[4].

3. Techniques and Statistical Data for Monte Carlo Simulation

Monte Carlo simulation is effective and practical for problems involving random variables with known or assumed probability distributions if used with a digital computer. This simulation conducts a repeating process using a set of random variables generated in accordance with the corresponding probability distribution. A set of data from Monte Carlo simulation is similar to a set of experimental data if the sample set of data is large and probability distributions are selected properly.

For practical implementation of Monte Carlo Simulation, several steps are required. Uniformly distributed random numbers, which are between 0 and 1 may be generated using a random number generator. These are used for selecting standard random variates which are sample values for random variables with given distributions. The selected normal variate is checked whether it is in given confidence interval. If it is satisfied, this variate becomes a desired random variable on the proper scale using known mean value and standard deviation. However, it is not satisfied the cycle is repeated until a random number accepted.

In most cases, the statistical data and probability distributions were obtained from the literature[1,5,6,7]. Fourteen properties are considered as random variables, which have normal distributions and truncated normal distributions where the tails of the distribution are shortened. These include the parameters related to the concrete and steel strength, beam dimensions and time-dependent properties such as creep and shrinkage. Information contained in ACI Committee Report 209R[1] is to be used to develop statistics for the variability of creep and shrinkage.

It is assumed that these parameters are independent except for concrete compressive strength, modulus of rupture and modulus of elasticity. Modulus of rupture and modulus of elasticity are correlated with compressive strength. This is accomplished as follows. A value of compressive strength is selected randomly. This value is then used to compute a mean value for modulus of rupture and modulus of elasticity. The assumed coefficient of variation for these parameters are then used to established probability density functions. Values of modulus of rupture and modulus of elasticity are then obtained randomly from these distributions. In addition, the tension stiffening parameter, β is also considered as a random variable because of the variability of post-tensile cracking behavior. The statistical data of random variables are summarized in Table 1.

The Monte Carlo procedure may be defined as a process in which the input variables incorporated into the deterministic algorithm are randomized to be used for the calculation of deflection based on Monte Carlo techniques and their probability models. The program outline for this procedure is shown in Fig. 3. It is necessary to determine a suitable number of simulation cycles before performing the probabilistic study for the variability of deflections. It was found that results did not vary much for 500 or more runs. In this study, 1000 simulation runs were used for each case.

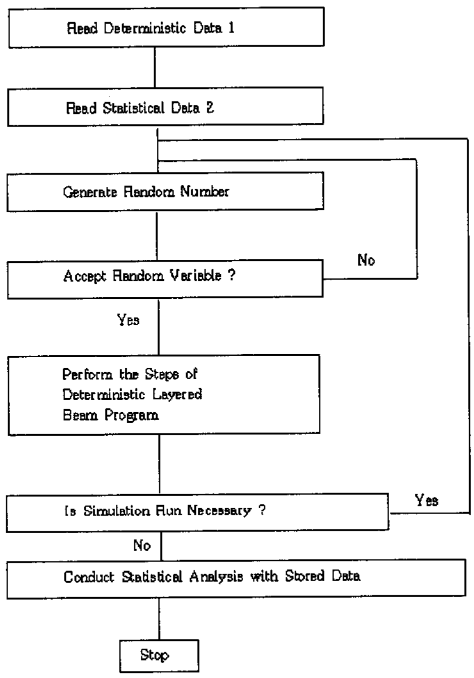


Fig. 3 Program outline for monte carlo simulation

Table 1 Probability model of random variables

	Variable	Mean (μ)	C.O.V	S.D. (σ)	Distribution
Concrete (In-situ)	f_{ck}	$0.675 f_{ck} + 1100 \leq 1.15 f_{ck}$ (psi)	0.176	-	Normal
	f_r	$8.3\sqrt{f_{ck}}$ (psi)	0.218	-	Normal
	E_c	$60{,}400\sqrt{f_{ck}}$ (psi)	0.119	-	Normal
Reinforcement	A_s	$0.99 A_{sn}$ (in^2)	0.024	-	Truncated Normal
	E_s	29,200 (ksi)	0.024	-	Normal
Beam Dimension	b	$5/32 + b_n$ (in)	0.045	-	Normal
	d_{st} , d_{sb}	d_{sn} + 1/16 (in)	$0.68/h_n$	-	Normal
Shrinkage Equation Parameters	$(\varepsilon_{sh})_u$	780×10^{-6} in/in	0.156	121.66×10^{-6} in/in	Truncated Normal
	γ	1.0	0.033	3.333×10^{-2}	Normal
	f	55 days	0.455	25 days	Truncated Normal
Creep Equation Parameters	ϕ_u	2.35	0.255	0.6	Truncated Normal
	η	0.6	0.111	6.666×10^{-2}	Normal
	d	10 days	0.667	6.666 days	Truncated Normal
Tension Stiffening Parameter	β	3.0	0.11	0.33	Normal

4. Description of Designed One-Way Slabs and T-Beams

In order to study the variability of deflections, 12 cases consisting of 6 simply supported one-way slabs, 3 both-end-fixed one-way slabs and 3 two-equal-span continuous T-beams were designed on the basis of ultimate moment capacity.

Details of nominal concrete properties and dimensions are shown in Table 2 and Fig. 4. Table 3 also shows details of the combinations of uniformly distributed dead and live loads. The live loads were selected to provide a suitable range of live to dead load ratio.

For simply supported one-way slabs, two types of compressive strength of concrete are used as

Table 2 Details of one-way slabs and t-beams

Mark	f_{ck} (psi)	l (in)	b (in)	b_w (in)	b_f (in)	h (in)	h_f (in)	l/h	d_{st} (in)	d_{sb} (in)	ρ
SS324S	3,000	240	12	-	-	10	-	24	-	8	0.0050
SS324M	3,000	240	12	-	-	12	-	20	-	10	0.0040
SS424D	4,000	240	12	-	-	14	-	17.1	-	12	0.0058
SS330S	3,000	300	12	-	-	10	-	30	-	8	0.0085
SS330M	3,000	300	12	-	-	16	-	18.8	-	14	0.0054
SS430D	4,000	300	12	-	-	18	-	16.7	-	16	0.0110
SF324S	3,000	240	12	-	-	6.0	-	40	1.0	5.0	0.0055
SF324M	3,000	240	12	-	-	8.5	-	28.2	1.0	7.5	0.0045
SF324D	3,000	240	12	-	-	11	-	21.8	1.0	10	0.0033
TC530D	5,000	300	-	12	75	18	5	16.7	2.5	15.5	0.0120
TC430M	4,000	300	-	12	75	15	5	20	2.5	12.5	0.0088
TC430S	4,000	300	-	12	75	12	5	25	2.5	9.5	0.0082

Identification Code of One-Way Slab and T-Beam : AAXBBC

AA = SS : Simply Supported One-Way Slab
= SF : Both Fixed End Slab
= TC : Two Equal Continuous T-Beam
X = f_{ck} (ksi)

BB = Span÷ 10 (in)
C = S : Shallow Depth (h)
= M : Medium Depth (h)
= D : Deepest Depth (h)

Table 3 Details of loads on one-wav slabs and t-beam

Mark	W_{sw} (plf)	W_{sd} (plf)	W_d (plf)	W_{lsus} (plf)	W_{lvar} (plf)	W_l (plf)	W_s (plf)	W_{co} (plf)	$\frac{W_l}{W_d}$
SS324S	120.0	2.4	122.4	37.2	50.4	87.6	159.6	210.0	0.72
SS324M	150.0	2.4	152.4	66.0	18.0	84.0	218.4	236.4	0.55
SS424D	175.2	52.8	228.0	66.0	248.4	314.4	294.0	542.4	1.38
SS330S	120.0	2.4	122.4	45.6	51.6	97.2	166.8	218.4	0.80
SS330M	200.4	18.0	218.4	72.0	146.4	218.4	290.4	436.8	1.00
SS430D	225.6	48.0	273.6	446.4	374.4	820.8	720.0	1094.4	3.00
SF324S	75.6	15.6	91.2	43.2	120.0	164.4	134.4	255.6	1.80
SF324M	106.8	51.6	158.4	96.0	220.8	316.8	254.4	475.2	2.00
SF324D	138.0	132.0	270.0	38.4	421.2	456.0	308.4	729.6	1.70
TC530D	552.0	108.0	660.0	900.0	504.0	1404.0	1560.0	2064.0	2.13
TC430M	516.0	120.0	636.0	228.0	192.0	420.0	864.0	1056.0	0.66
TC430S	480.0	1.2	481.2	6.0	114.0	120.0	487.2	612.0	0.25

- List of Symbols on Loads

W_{sw} = Self-weight of Beams W_{sd} = Superimposed Dead Load W_d = Total Dead Load (=W_{sw}+W_{sd})
W_{lsus} = Sustained Portion of Live Load W_{lvar} = Variable Portion of Live Load W_l = Total Live Load (=W_{lsus}+W_{lvar})
W_{co} = Equivalent Construction Load (=W_d + W_l) = Load Level 1
W_s = Instantaneous Load of Sustained Portion (= W_d + W_{lsus}) = Load Level 2

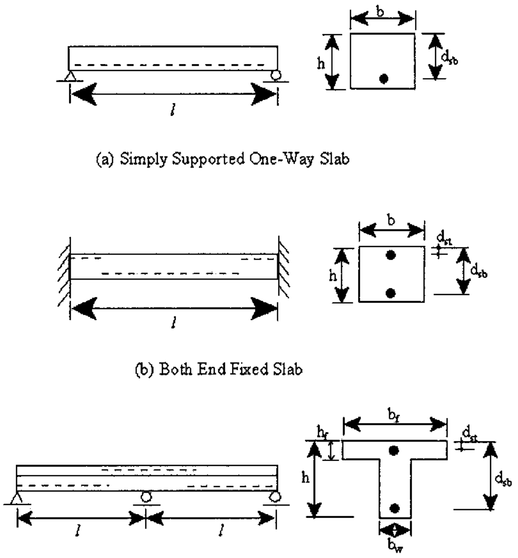


Fig. 4 Sketch for dimensions of one-way slabs and t-beams

3,000psi and 4,000 psi, and two span lengths are selected as 20 ft and 25 ft. The span to depth ratios are chosen to give depths greater than, equal to and less than the minimum thickness value, l /20 specified in Table 9.5 (a) of the ACI 318-95 Code[8]. Table 2 shows that the chosen span to depth ratios ranges from 16.7 to 30. In addition, the range of reinforcement ratios is from 0.004 to 0.011 corresponding to the combinations of loads given in Table 3.

For both-end-fixed one-way slabs, the compressive strength of concrete is 3,000 psi and span length is 20 ft. The span to depth ratios vary from 21.8 to 40 compared with a value of 28 corresponding to the minimum thickness value of Table 9.5(a) of ACI 318-95 Code. The positive moment reinforcement ratios ranged from 0.0033 to 0.0055.

For two equal-span continuous T-beams 5,000 psi and 4,000 psi are used as the compressive strength of concrete, and span length of 25 ft is used. The span to depth ratios range from 16.7 to 25 compared to 18.5 as specified in Table 9.5(a) of the ACI Code. The reinforcement ratios represented for positive moment steel range from 0.0082 to 0.012.

5. Load-Time Histories for Analysis of Variability

Long-time deflections depend on the load-time history. In this study, two load-time histories, designated LH1 and LH2, are considered as shown in Fig. 5 and Fig. 6.

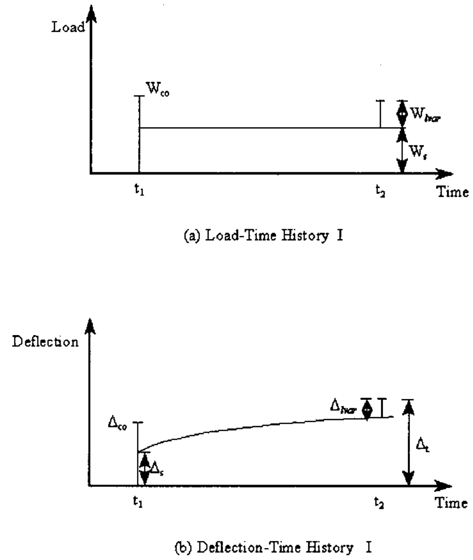


Fig. 5 Load-time history and corresponding deflection-time history, LH1

As shown in Fig. 5(a), load-time history LH1 consists of the construction load W_{co}, which is taken as equal to the service dead load plus live load applied instantaneously, followed by a sustained load W_s consisting of dead plus sustained live load. The sustained load is assumed to remain constant and the variable portion of live load is applied at the end of the assumed load duration. Consequently, two instantaneous deflections Δ_{co}, Δ_s can be induced by the corresponding to two instantaneous loads, W_{co} and W_s, and long-time total deflection Δ_t can be estimated by summing instantaneous deflection Δ_s, due to sustained load, sustained long-time deflection due to creep and shrinkage and instantaneous deflection due to variable portion of live load.

In case of load-time history LH2, the sustained load is only considered in the calculation of instantaneous deflection and long-time total deflection assuming no significant load during construction. As for the load-time history LH1, the sustained load remains constant and the variable portion of live load is applied at the end of the assumed load duration. As a result, the corresponding deflections consist of instantaneous sustained deflection Δ_s and total deflection Δ_t. This load-time history and corresponding deflection-time history are shown in Fig. 6 (a) and (b).

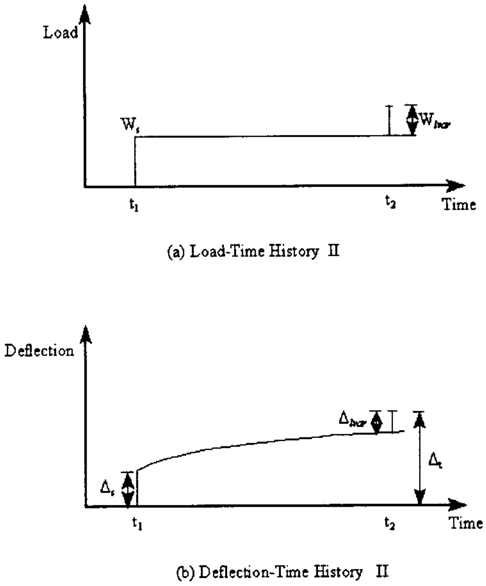


Fig. 6 Load-time history and corresponding deflection-time history, LH2

6. Probabilistic Assessment of Deflections

6.1 Probability Distributions

As discussed in the previous sections, each case was simulated 1,000 times for the two types of load-time histories to obtain mean deflection value, standard deviation and coefficient of variation. (C.O.V.) These simulated results and probability distributions for all cases considered are presented in Choi's thesis[(4)].

Several examples are presented in this section to indicate some general trends indicated by the results. Fig. 7(a) shows the probability distribution obtained for immediate deflection of slab SS324M under load history LH1. There are two distinct parts to the distribution. The majority of the results are clustered around the mean value of 0.193 in. However there are a significant number of values spread out above the mean value. The majority of cases involving lower deflection

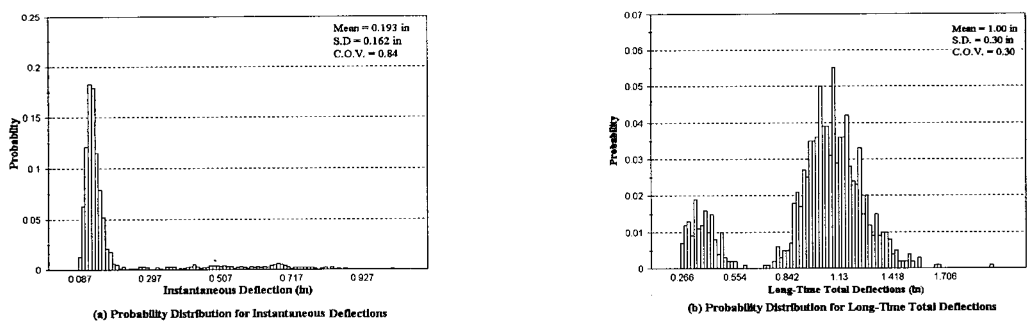


Fig. 7 Probability distribution of SS324M for deflections in LH1

values are associated with an uncracked member. If cracking is detected however, this section has a low reinforcement ratio, causing the stiffness to drop significantly after cracking resulting in higher deflections.

Fig. 7(b) shows the distribution of total long-time deflections. In this case the shape of the distribution is significantly different than for immediate deflection. A bi-modal distribution is indicated. The group of higher deflections is associated with a cracked section indicating that many cases that were uncracked under immediate loading have developed cracking under long time loading.

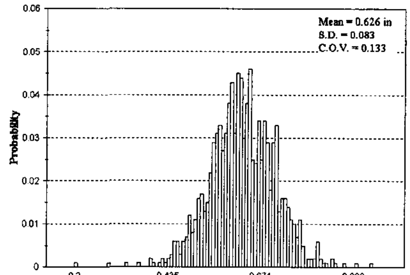
Fig. 8(a) shows the probability distribution for immediate deflection under load history LH1 for slab SS430D. This slab has the highest reinforcement ratio and highest applied moment to cracking moment ratio of the simply supported slabs. The probability distribution is approximately normal and the behavior of the slab is dominated by cracking. The coefficient of variation is only 13.3% compared with 84% for slab SS324M. For total deflection under long time loading, the same general shape is indicated in Fig. 8(b). While the standard deviation increases from 0.083 in. to 0.194 in., the C.O.V. decreases from 13.3 to 12.1% due to the increase in the mean deflection under long time loading.

For these cases, the behavior appears to be dominated by the cracked transformed section and the variability is affected by primarily by the modulus of elasticity.

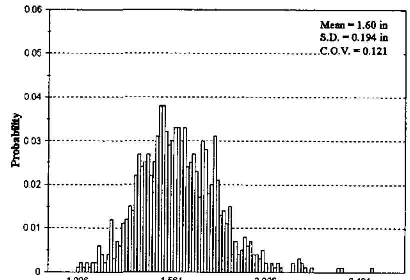
Results for immediate deflection under load history LH1 are presented in Fig. 9(a) for T-beam example TC430M. In this case the distribution divides into two groups. The lower deflections are associated with an uncracked beam while the higher deflections are associated with cracking. In Fig. 9(b), the probability distribution for total long time deflection is seen to be approximately normal indicating that in many cases cracking has developed under sustained loading. The trend is similar to that indicated for slab SS324M.

6.2 Discussion of Results

The results indicate that for members with low reinforcement ratios and, as a result, applied moments that are close to the cracking moment, the stiffness is very sensitive to the development of cracking. While the most likely condition is for the slab to be uncracked, there is still a significant probability of cracking occurring and hence there is increased uncertainty and variability in deflection of members with low reinforcement ratios.

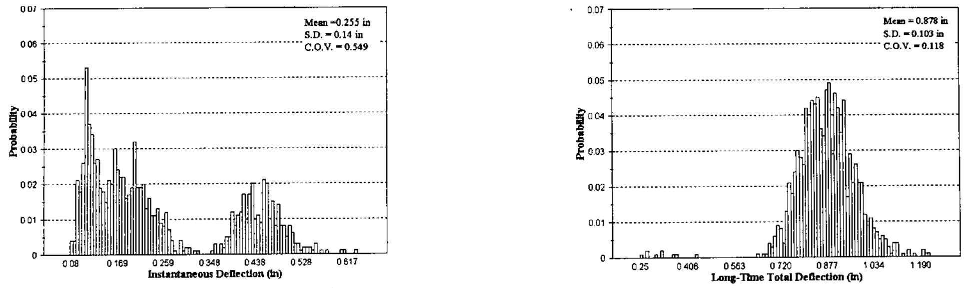


(a) Probability Distribution for Instantaneous Deflections



(b) Probability Distribution for Long-Time Total Deflections

Fig. 8 Probability distribution of SS430D for deflections in LH1



(a) Probability Distribution for Instantaneous Deflections

(b) Probability Distribution for Long-Time Total deflections

Fig. 9 Probability distribution of SS430M for deflections in LH1

For members with higher reinforcement ratios there is a high likelihood of cracking occurring and as a result less uncertainty about the stiffness.

Variability of deflections is therefore lower for higher reinforcement ratios. These results confirm the conclusions of Ramsay et al(9) in their study of short-term deflections.

7. Conclusions and Ecommen Dations

From simulation results, it is illustrated that variability depends primarily on the ratio of applied moment to cracking moment. It is caused by two factors. First of all, if the ratio of applied moment to cracking moment is close to unity, it is uncertain whether the member will be cracked or uncracked. Secondly, if the ratio of applied moment to cracking moment is close to unity it is likely that the member will have a low reinforcement ratio with a resulting significant difference between cracked and uncracked stiffness.

The results confirm that the variability of deflections can be high due to the random nature of the many variables that influence deflection. Calculations based on mean values or close to mean values of material parameters may produce deflection values that are siginficantly higher or lower than the deflections that will occur in the field. This uncertainty can not be removed by more complicated deterministic calculation procedures.

For design situations in which deflection control is a significant design consideration, the sensitivity of calculated deflections to various input parameters can be considered to determine the likely range of values that could be expected in the field. In particular, members with low reinforcement ratios may be sensitive to cracking and variations in cracking strength may have a significant effect. It may also be appropriate for design codes to use a lower concrete tensile strength value for deflection calculations involving members with low reinforcement ratios to account for the increased uncertainty indicated in this and other studies.

References

1. ACI Committee 209, "Prediction of Creep, Shrinkage, and Temperature Effects in Concrete Structures," ACI Manual of Concrete Practice, Part 1, 1992, pp.209R1-209R47.
2. Trost, H., "Implications of the Superposition principle in Creep and Relaxation Problems for Concrete and Prestressed Concrete," Beton and Stahlbetonbau (West Berlin), V. 62, 1967, pp. 230-238, 261-269.
3. Branson, B.E., "Instantaneous and Time-Dependent Deflections of Simple and Continuous Reinforced Concrete Beams," HPR Publication 7, Part 1, 1-78, Albama Highway De-

partment, Bureau of Public Roads, August 1963.
4. Choi, B.S., "Variability of Reinforced Concrete Beam Deflection," Ph.D. Thesis, Dept. of Civil Engineering, Pennsylvania State University, University Park, U.S.A, 1998.
5. Mirza, S.A., Hatzinikolas, M., and MacGregor, J.G., "Statistical Descriptions of the Strength of Concrete," ASCE Structural Division, V. 105, No. ST 6, June 1979, pp.1021-1037.
6. Mirza, S.A., and MacGregor, J.G., "Variability of the Mechanical Properties of Reinforcing Bars," ASCE Structural Division, V. 105, No. ST 5, May 1979, pp.921-937.
7. Naaman, A.E., and Amnuayporn Siriaksorn, "Reliability of Partially Prestressed Beams at Serviceability Limit States," PCI Journal, November-December 1982, pp. 66-85.
8. ACI Committee 318, "Building Code Requirements for Reinforced Concrete and commentary," American Concrete Institute, Farmington Hills, Michigan, 1995.
9. Ramsay, R. J., Mirza, S., and MacGregor, J.G., "Monte Carlo Study of Short Time Deflections of Reinforced Concrete Beams," ACI Journal, Proceedings V.76, No. 8, 1979, pp.879-918.