▩ 응용논문

-최우비(RML)검정에 의한 Weibull분포의 공정능력 평가-

Evaluation of Process Capability for Weibull Distribution by the RML Test

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요 약

대수정규분포와 와이블분포는 비정규분포를 하는 공정특성에 반영된다. Hahn 과 Shapir(1967)의 수치 예를 통해 이러한 대수정규분포와 와이블분포를 식별하는 검정방법으로 최우비 검정(RML)을 사용하였다. 검정결과 와이블분포를 하는 공정으로 판단되어, 와이블분포의 모수 추정은 실무에서 그래프와 수식으로도 쉽게 추정할 수 있는 수정적률추정량(MME)를 이용하여 공정 능력을 평가하였다. 공정능력의 평가 척도로 공정능력지수와 불량률(PPM)의 척도와 병행하여 나타내었다. 공정능력 평가의 결과로 와이블분포를 하는 공정을 대수정규분포로 잘못 판단함으로써 발생하는 오류를 2가지 척도로 표현함으로써 올바르게 공정능력을 판단할 수 있도록 종합적인 정보를 제시하였다.

1. Introduction

Let the probability density function of the log-normal be written as

$$f_{\ln}(x;\mu,\sigma)=\exp\left\{-0.5\left(\frac{\ln x-\mu}{\sigma}\right)^2\right\}\frac{1}{\sqrt{2\pi}\,\sigma x}\quad \text{for}\quad x>0, \tag{1. 1}$$

and the two parameter Weibull distribution can be written as

$$f_w(x;\delta,\beta)=\frac{\delta}{\beta}(x/\beta)^{\delta-1}\ \exp\{-(x/\beta)^{\delta}\}\qquad \text{for}\quad x>0, \tag{1. 2}$$

It is noted in the paper by Dumonceaux, Antle and Haas(1973) that whenever the two models are location and scale parameter distributions the distribution of the ratio of maximized likelihoods(hereafter called the RML)does not depend upon the value of the unknown parameters and it is therefore reasonable to obtain critical values for this test by simulation whenever they cannot be calculated analytically. In statistical process control(SPC), some quality characteristics may not follow the widely used normal

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distribution and can be better represented in terms of Weibull distribution. Such situations are encountered while dealing with the failure of electrical components like bulbs, electronic devices like diodes, transistors, integrated circuits, mechanical components such as bearings, and structural elements in aircraft and automobiles(Montgomery, 1991). The estimation of process capability of any manufacturing process that affects time to failure of the output is a very important step in SPC, and without controling the process, a demonstration or an analysis of reliability of the output in terms of observed time to failure may not mean much. Not much attention has been paid so far to evaluating process capability for Weibull distribution quality characteristics.

In this paper, We estimated in terms of a modification of the moment estimator(MME) of Weibull parameters, evaluated based on process capability index suggested by Clements(1989).

Parameter estimation in the two parameter Weibull distribution can sometimes be troublesome. Moment estimators(ME), though easy to evaluate, do not make full use of available sample information and as a result estimate variance are often unduly large. Although maximum likelihood estimators(MLE) are optimal in many respects, their calculation in time consuming. In this study we present a modification of the moment estimator(MME) which enjoys some of the advantages of both the ME and the MLE, without their more serious disadvantages.

Attention was first directed to the MME in a simulation study of various estimators of Weibull parameters by two of the authors, cohen and whitten(1982). Estimators for Weibull parameter based on selected percentiles have previously been considered by such authods as Kao(1959), Dubey(1966), Wyckoff, Bain, and Engelhardt(1980), Zanakis(1977, 1979a, 1979b), and Zanakis and Mann(1981).

(1) Some fundamentals

Left X denote a random variable that is Weibull cumulative distribution function is

$$F_w(\chi\,;\,\delta,\beta)=1-\exp\left\{-\left(\frac{\chi}{\beta}\right)^{\delta}\right\} \tag{1. 3}$$

In the notation, δ is the shape parameter and β is the scale parameter. The mean, median, mode, variance for Weibull distribution are

$$\mu_x=E(X)=\beta\,\Gamma_1 \tag{1. 4}$$

$$M_e=\beta\,(\ln 2)^{\frac{1}{\delta}} \tag{1. 5}$$

$$M_o=\beta\left(1-\frac{1}{\delta}\right)^{\frac{1}{\delta}} \tag{1. 6}$$

$$V(X)=E\,(X-\mu_x)^2=\beta^2[\,\Gamma_2-\Gamma_1^2] \tag{1. 7}$$

(2) The modified estimator

As estimating equations, Cohen, whitten and Ding (1985) employ $E(X)=\bar{x}$, $V(X)=s^2$, and $E(X_1)=x_1$; where $\bar{x}=\sum_{n=1}^{n}\frac{x_i}{n}$, $s^2=\sum_{n=1}^{n}\frac{(x_i-\bar{x})^2}{n-1}$, n is the sample size, and x_1, is the smallest sample value.

The estimating equation become accordingly

$$\frac{s^2}{(\bar{x}-x_1)^2}=\frac{\hat{\Gamma}_2-\hat{\Gamma}_1^2}{\left[\left(1-n^{-\frac{1}{\hat{\delta}}}\right)\hat{\Gamma}_1\right]^2} \tag{1. 8}$$

$$\hat{\beta}=\frac{n^{\frac{1}{\hat{\delta}}}(\bar{x}-x_1)}{\left(n^{\frac{1}{\hat{\delta}}}-1\right)\hat{\Gamma}_1} \tag{1. 9}$$

Equivalent expressions for calculating $\hat{\beta}$ which is sometimes more convenient for routine calculation is

$$\hat{\beta}=s\cdot C(\hat{\delta}) \tag{1.10}$$

where

$$C(\hat{\delta})=[\Gamma_2-\Gamma_1^2]^{-\frac{1}{2}} \tag{1.11}$$

The frequency curve of the Weibull distribution is J-shaped and become highly skewed when $\delta \leq 1$. When $\delta=1$, We obtain the simple well known exponential distribution as a special case of Weibull distribution. For $\delta > 1$ the Weibull distribution become bell-shaped.

As shown in table 1, the skewness α_3 is 6.61876 when $\delta=0.5$.

As $\delta \to 0$, the skewness in increases quite rapidly.

Table 1. Values of $C(\delta)$ and α_3

δ	$C(\delta)$	$\alpha_3=\frac{\Gamma_3-3\Gamma_2\Gamma_1+2\Gamma_1^3}{[\Gamma_2-\Gamma_1^2]^{\frac{3}{2}}}$
0.5	0.22361	6.61876
0.6	0.37805	4.59341
0.7	0.54020	3.49837
0.8	0.70020	2.81465
1.0	1.00000	2.00000
1.3	1.39580	1.34593
1.5	1.63149	1.07199
2.0	2.15866	0.63111
2.5	2.63389	0.35863
3.0	3.08119	0.16810
3.5	3.51206	0.02511
4.0	3.93258	-0.08724
5.0	4.75490	-0.25411
6.0	5.56274	-0.37326
7.0	6.36237	-0.46319

2. Process capability indices for non-normal data

There are several publication in current literature that suggest technique for determining process capability for general non-normal processes.

Clements(1989) has developed a technique for adjusting C_p or C_{pk} non-normal situations based on Pearson curves.

Clements replaced 6σ by $U_p - L_p$ where U_p is the 99.865 percentile and L_p is the 0.135 percentile determined from Gruska et al. table(1989) for the particular values of skewness and kurtosis that are estimated from the data

$$\widehat{C_p} = \frac{USL - LSL}{\widehat{U_p} - \widehat{L_p}} \tag{2. 1}$$

where $USL$and LSL denote the upper and lower specification limits, respectively

$$\widehat{C_{pk}} = \min\left[\frac{USL - \widehat{M_e}}{\widehat{U_p} - \widehat{Me}}, \frac{\widehat{M_e} - LSL}{\widehat{M_e} - \widehat{L_p}}\right] \tag{2. 2}$$

where M_e denote median

Farnum(1996~97) is used Johnson curves to describe non-normal process data.

3. Illustrative example

To illustrate the use of the RML test for discriminating between the log-normal and Weibull distributions, suppose the following observations, as given by Hahn and Shapiro(1967) might lead to the Weibull distribution.

For the sample

1	4	6	7	8
9	10	11	12	12
13	13	14	14	15
16	17	20	25	31

are used to test H_o = Log-normal vs H_1 : Weibull

3.1 The RML test for discrimination between the Log-normal and the Weibull distribution

Let $f_{\ln}(x;\mu,\sigma)$be the log-normal density by(1.1) and let $f_w(x;\delta,\beta)$be the Weibull density given by(1.2). The modification of the moment estimator(MME) of δ and β were obtained by the four equations of (1.8)~(1.11). Table 2 givers critical value for the test statistic, $(RML)^{\frac{1}{n}}$, and the power of the test for this problem. For this problem it is easily seen that

$$(RML)^{\frac{1}{n}} = (2\pi e\hat{\sigma}^2)^{\frac{1}{2}}\left[\prod_i^n x_i f_w(x_i;\hat{\delta},\hat{\beta})\right]^{\frac{1}{n}} \tag{3. 1}$$

where

$$\hat{\sigma}^2 = \sum\frac{(\ln x_i - \hat{\mu})^2}{n} \quad \text{and} \quad \hat{\mu} = \frac{1}{n}\sum \ln x_i$$

and we reject the log-normal in favor of the Weibull distribution whenever

$$(RML)^{\frac{1}{n}} \geq (RML)_c^{\frac{1}{n}} \tag{3. 2}$$

Table 2. Critical values of $(RML)^{1/n}$ and power of test for H_0 : Log-normal and H_1 : Weibull

n	$\alpha=0.20$		$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.01$	
	$(RML)_c^{\frac{1}{n}}$	power	$(RML)_c^{\frac{1}{n}}$	power	$(RML)_c^{\frac{1}{n}}$	power	$(RML)_c^{\frac{1}{n}}$	power
20	1.015	0.75	1.038	0.61	1.082	0.48	1.144	0.22
30	0.993	0.86	1.020	0.75	1.044	0.63	1.095	0.39
40	0.984	0.93	1.007	0.85	1.028	0.76	1.070	0.53
50	0.976	0.96	0.998	0.91	1.014	0.83	1.054	0.63

The values in table 2 were the result of simulation in which 15,000 samples were used for each sample size by Dumonceaux and Antle(1973).

For these data we find for the modification of the moment estimators(MME) in the Weibull model, $\hat{\delta}=2.90$ and $\hat{\beta}=20.805$. We also find $\hat{\sigma}^2=0.4988$, $\hat{\mu}=2.37$ which results in a value of 1.957 for $(RML)^{1/n}$. From table 2 we see that critical value for this test at the $\alpha=0.05$ level is 1.082, at the $\alpha=0.01$ level is 1.144. Consequently we may be rejected the log-normal in favor of the Weibull.

3.2 Evaluation of process capability

To illustrate the calculation of process capability, We will use specification limit of USL=30, LSL=2 for these data.

(1) Normal distribution

If a characteristic is normally distributed, normal-based process capability indices will be used

$$\widehat{C_P}=\frac{USL-LSL}{6\hat{\sigma}} \tag{3. 3}$$

$$\widehat{C_{PK}}=\min\left[\frac{\hat{\mu}-LSL}{3\hat{\sigma}}, \frac{USL-\hat{\mu}}{3\hat{\sigma}}\right] \tag{3. 4}$$

For these data we find for estimates of σ, μ, namely $\hat{\sigma}=6.9350$, $\hat{\mu}=12.9$.

Accordingly, $\widehat{C_P}=0.67$, $\widehat{C_{PK}}=0.52$

(2) Log-normal distribution

With these data, a supposed case, we will be accept the log-normal in favor of the *Weibull distribution and then the evaluation of process capability for* log-normal distribution are estimated from the two equations of (2. 1), (2. 2).

For these data we find for estimates of median(Me), μ, σ, U_p, L_p, namely, $\widehat{M_e}=10.6974$, $\hat{\mu}=13.7275$, $\hat{\sigma}=11.0397$, $\widehat{U_P}=89.0235$, $\widehat{L_P}=1.2854$. The quantile function of the long-normal distribution is

$$x_P = \exp[\mu + \Phi^{-1}(p)\sigma] \tag{3. 5}$$

where $\Phi(\chi)$ is the cumulative distribution function of the standard normal distribution evaluated at χ. Accordingly, $\widehat{C_P} = 0.32$, $\widehat{C_{PK}} = 0.25$

(3) Weibull distribution

For these data we find estimates of median(Me), μ, σ, L_p, U_p, namely, $\widehat{M_e} = 18.335$, $\hat{\mu} = 18.563$, $\hat{\sigma} = 6.9228$, $\widehat{U_P} = 39.8972$, $\widehat{L_P} = 2.1317$.

The Weibull p quantities is

$$x_P = \beta\,[-\log(1-p)]^{\frac{1}{\delta}} \tag{3. 6}$$

Accordingly, $\widehat{C_P} = 0.74$, $\widehat{C_{PK}} = 0.54$

Therefore, for example, the evaluation of process capability are tabulated in the table 3.

Table 3. The Evaluation of Process Capability

Population type \ Measure of process capability		Process capability indices: $\widehat{C_P}$	$\widehat{C_{PK}}$: $\widehat{C_{PL}}$	$\widehat{C_{PK}}$: $\widehat{C_{PU}}$	Nonconforming(PPM): P(X<LSL)	Nonconforming(PPM): P(X>USL)	Remark $\left(\begin{matrix} PPM_{total} \\ Eq\cdot C_{PK} \end{matrix}\right)$
Normal		0.67	(0.52)	0.82	58,210	6,756	64,956 PPM $Eq\cdot C_{PK} = 0.51$
Non-Normal	Log-normal	0.32	(0.25)	0.89	8,656	72,140	80,796 PPM $Eq\cdot C_{PK} = 0.47$
	Weibull	0.74	(0.54)	1.01	1,124	5,5593	56,717 PPM $Eq\cdot C_{PK} = 0.53$

4. Summary and conclusions

The main objective of this study to propose a comprehensive methods for a measure of evaluation of process capability from discriminating between the long- normal and Weibull distributions. For this problem, it is used critical value for the test statistic, $(\mathrm{RML})^{1/n}$, using a modification of the moment estimator(MME). But, from the power in table 2 that the ability select between the Weibull with sample size 20 is not good.

The use of process capability indices and the percentage nonconforming is illustrated in reference to a distribution with normal, long-normal, Weibull. From calculated results in the table 3, particularly, If Weibull assumed to be log-normal, significant error in estimation PPM nonconforming would result.

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