◈ Application Papers

A Control Scheme for a Gradual Drift in the Process Variance

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Abstract

This paper presents a study on control schemes for gradual increases (drifts) in a process variance. A new control chart, the Drifting Variance Control Chart (DVCC) is designed using Likelihood Ratio Test (LRT), and the ARL performance of the chart is evaluated for different subgroup sizes. The performance of this chart is then compared to some of the popular control schemes for the process dispersion, like the Shewhart S^2 chart, the CUSUM chart and the EWMA chart. Results are presented and discussed. Also included is a sensitivity analysis that investigates how the DVCC performs when applied to a stepped change in process variance.

1. Introduction

In order to produce a part that conforms to certain predetermined specifications, it is important that the manufacturing process that produces the part is "in-control". The term "in-control" implies that the process is producing parts with the desired target mean, and with little variation between parts. The process variance is an important parameter to control because a process operating with an on-target mean, but with a shifted variance will produce parts that do not conform closely with the desired specifications.

Many methods are used today to minimize process variability. The "upstream" practice of designed experiments is commonly used. One of the objectives of designed experiments is to select the settings of controllable variables that minimize the defects of uncontrollable factors on the process. This results in a minimized deviation from the target mean.

Despite the use of "upstream" techniques, it is important to have "downstream" checks on the variability of the process. This is most commonly done by using control charts on the variation among the parts being produced in the process under consideration. When the control chart signals that the process deviation is too large, the causes for the increased deviation must be investigated and rectified before the process is restarted.

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The control charts most often used in industrial applications today are the Shewhart $\overline{X}$ and R charts. The $\overline{X}$ chart is used to control the process mean, and the R(Range) chart is used to control the process standard deviation. In the Shewhart control charts, all the emphasis is placed on the last sample point plotted. Historical data is only important in designing control limits. The problem that arises out of this approach is that when this chart is applied to a process that has a gradually increasing shift in a process parameter, the chart will take a considerable length of time to signal[2][5].

Two other popular methodologies are the Exponentially Weighted Moving Average (EWMA) chart, and the Cumulative Sum (CUSUM) control chart. These charts are designed to use more than just the information in the last sample point. The EWMA chart uses a geometric moving average of all the previous data. This implies that the more recent data receives more importance, but that historical data is not disregarded. EWMA charts are designed and used to control the process mean and the process standard deviation[7][9]

The CUSUM chart accumulates information from all the historical data till the current sample point. CUSUM charts are designed to control the process mean and the process variance. Both the EWMA and the CUSUM charts are effective in detecting small shifts quickly, and for detecting persistent causes, i. e., causes which persist till action is taken to rectify them[6][10].

In this paper, the problem of a gradual drift in the process variance is considered. A new control chart is designed for this type of problem, and its performance is compared with those of the standard Shewhart S^2, CUSUM and EWMA schemes for detecting changes in process variance [3][12][14].

2. Problem Description

In this paper, methods to detect a gradual drift in the process variance are investigated. The problem addressed can be summarized by the following hypothesis testing statement.

$$H_0 : \sigma_1^2 = \ldots = \sigma_t^2 = 1$$
$$H_a : \sigma_1^2 = \ldots = \sigma_\phi^2 = 1$$
$$\sigma_{\phi+1}^2 = m, \ldots, \sigma_t^2 = m^{t-\phi} \qquad (2.1)$$

where m is a multiplier factor by which the process variance charges at each sampling point, ϕ is the last subgroup under the in-control process, and t is the current subgroup. In this paper, we only consider increases in the process variance, and hence m>1 indicates a drift in the variance. The chart obtained is, therefore, a one-sided control chart, and for comparison purposes, the one-sided performances of other charts need to be considered.

A Likelihood Ratio Test(LRT) is used to design a control chart for the test in Eq 2.1. The design of the chart is described in Section 3. The performance of the chart is measured in terms of its ARL (Average Run Length) values. It is then compared with the S^2 . EWMA and CUSUM charts used for detecting changes in the process variance.

3. Design of the DVCC (Drifting Variance Control Chart)

In this section, the design of the DVCC is presented. The pair of hypotheses we wish to test are specified in Eq 2.1 as the following.

$$H_0 : \sigma_1^2 = \ldots = \sigma_t^2 = 1$$
$$H_a : \sigma_1^2 = \ldots = \sigma_\psi^2 = 1, \quad \sigma_{\psi+1}^2 = m, \ldots, \sigma_t^2 = m^{t-\psi}$$

A LRT is used to design a control chart to test this pair of hypotheses. Since there are two unknowns, namely m and ψ, m is estimated first, and then the change point ψ is estimated.

In order to estimate m, the change point ψ is fixed and the following pair of hypotheses are tested when $\psi < i \leq t$.

$$H_0 : \sigma_i^2 = 1, \qquad H_a : \sigma_i^2 = m^{i-\psi} \tag{3.1}$$

$$\frac{L_a}{L_0} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{m^{i-\psi}}}\right)^n \exp\left\{-\frac{1}{2m^{i-\psi}} \sum_{j=1}^{n} Z_{ij}^2\right\}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} Z_{ij}^2\right\}} \tag{3.2}$$

$$\ln \frac{L_a}{L_0} = -\frac{n}{2}(i-\psi)\ln(m) - \frac{1}{2} \sum_{j=1}^{n} Z_j^2 (m^{-i+\psi} - 1) \tag{3.3}$$

Differentiating Eq. 3.3 with respect to m, and equating to zero in order to solve for m, we get

$$\frac{\partial}{\partial m} \ln \frac{L_a}{L_0} = -\frac{n}{m} + \sum_{j=1}^{n} Z_{ij}^2 (m^{-i+\psi-1}) = 0 \quad \Rightarrow \quad \hat{m} = \left(\frac{\sum Z_{ij}^2}{n}\right)^{\frac{1}{i-\psi}} \tag{3.4}$$

We now revert to our original hypotheses in Eq. 2.1 in order to use the LRT. The likelihoods under H_0 and H_a are

L_0 = likelihood under null hypothesis

$$= \prod_{i=1}^{t} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} Z_{ij}^2\right\} = \left(\frac{1}{\sqrt{2\pi}}\right)^{nt} \exp\left[-\frac{1}{2} \sum_{i=1}^{t} \sum_{j=1}^{n} Z_{ij}^2\right] \tag{3.5}$$

where j indexes the subgroup size and varies between 1 and n.

L_a = likelihood under alternate hypothesis, given ψ

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{n\psi} \exp\left[-\frac{1}{2} \sum_{i=1}^{\psi} \sum_{j=1}^{n} Z_{ij}^2\right] * \left\{\left(\frac{1}{\sqrt{2\pi}}\right)^{n(t-\psi)} \prod_{i=\psi+1}^{t} \left(\frac{1}{m^{i-\psi}}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2} \sum_{i=\psi+1}^{t} \sum_{j=1}^{n} Z_{ij}^2\right]\right\} \tag{3.6}$$

$$\frac{L_a}{L_0} = \frac{\prod_{i=\psi+1}^{t} \left(\frac{1}{m^{i-\psi}}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2} \sum_{i=\psi+1}^{t} \sum_{j=1}^{n} \frac{1}{m^{i-\psi}} Z_{ij}^2\right]}{\exp\left[-\frac{1}{2} \sum_{i=\psi+1}^{t} \sum_{j=1}^{n} Z_{ij}^2\right]} \tag{3.7}$$

If ψ were known, a test could be constructed based on L_a/L_0. Since ψ is unknown, the test is based on

$$\max_{0 \leq \psi \leq t} \frac{L_a}{L_0} \tag{3.8}$$

Considering increases in the variance (m > 1), we conclude in favor of H_a when

$$\frac{L_a}{L_0} > A, \quad \text{or when} \quad \ln\frac{L_a}{L_0} > \ln A.$$

$$\ln\frac{L_a}{L_0} = \frac{n}{2}\sum_{i=\phi+1}^{t}\ln\frac{1}{m^{i-\phi}} - \frac{1}{2}\sum_{i=\phi+1}^{t}\sum_{j=1}^{n}\frac{1}{m^{i-\phi}}Z_{ij}^2 + \frac{1}{2}\sum_{i=\phi+1}^{t}\sum_{j=1}^{n}Z_{ij}^2 \qquad (3.9)$$

By inspection, the first term on the right hand side of Eq 3.9 is expressed in logarithm, which is not simplified with other terms of Eq 3.9. Thus, that term can act as k, the "reference value" in the test obtained. The estimated value of m from Eq. 3.4 is substituted into the second term on the right hand side of Eq 3.9. Thus, the RHS(Right Hand Side) of Eq 3.9 is simplified to Eq 3.9.1.

$$\text{RHS of Eq 3.9} = \frac{n}{2}\sum_{i=\phi+1}^{t}\left\{\sum_{j=1}^{n}Z_{ij}^2/n - 1 - \ln m^{(i-\phi)}\right\} \qquad (3.9.1)$$

Therefore, the test obtained from the LRT reduces to the following. Reject H_0 when

$$\max_{0\le\phi<t}\left(\sum_{i=\phi+1}^{t}[(Y_i-1)-k]\right) > h \qquad (3.10)$$

where $Y_i = \frac{\sum_{j=1}^{n}Z_{ij}^2}{n}$, $k = \ln(m^{(i-\phi)})$, and h is the "decision interval"

3.1 Choice of m in the reference value

The value of the parameter k is determined by using the following expression[8][11].

$$k = \frac{\ln E(Y_i/\sigma_0^2) + \ln E(Y_i/m^{(i-\phi)}\sigma_0^2)}{2} \qquad (3.1.1)$$

The value of m is set equal to 1.002 in order to detect small drifts. The reasoning behind this approach is that it would be desirable that the chart be optimal in detecting small drifts since most charts should be reasonably quick in detecting large drifts.

Due to the enormous computational time involved in calibrating the DVCC for comparison purposes, this work involves the investigation into the performance of the chart for just this specific value of m=1.002. In order to investigate completely the performance of the DVCC, it would be desirable to evaluate the performance of the chart relative to other control charts for other values of m as well.

4. Results and Discussions

4.1. Performance of the DVCC

A Monte Carlo simulation approach is used to investigate the ARL performance of the DVCC. A simulation program was written in C. In order to facilitate easy comparison with other control charts, the DVCC is calibrated to have an in-control ARL of 400. The value of the decision interval, h is adjusted in order to obtain this desired sensitivity to false alarms. Once the chart is calibrated, the ARL performance of the chart is investigated for different sizes of gradual drift, m.

This procedure is repeated for different sizes of sample size, n and the results are tabulated in Tables 4.1, 4.2 and 4.3. The results are for a one-sided chart and out of control ARL values are provided for drifting increases in the process variance. The values of h, the decision interval which yield the desired sensitivity to false alarms is also shown.

Table 4.1. Steady state ARL values for n=4, h=12.9

m	ARL	Std. ERR
1.000	399.76	3.8089
1.001	116.75	0.5413
1.002	85.51	0.3491
1.003	70.47	0.2711
1.004	61.50	0.2215
1.005	55.45	0.1930
1.010	39.33	0.1229
1.050	16.86	0.0441
1.100	11.54	0.0282
1.200	7.84	0.0186

Table 4.2 Steady state ARL values for n=5, h=11.5

m	ARL	Std. ERR
1.000	396.83	3.8516
1.001	111.39	0.4023
1.002	81.21	0.2625
1.003	67.62	0.2501
1.004	58.29	0.2097
1.005	53.05	0.1797
1.010	37.32	0.1155
1.050	16.00	0.0410
1.100	10.97	0.0263
1.200	7.51	0.0177

Table 4.3 Steady state ARL values for n=10, h=8.0

m	ARL	Std. ERR
1.000	403.09	3.9046
1.001	95.46	0.4023
1.002	69.52	0.2625
1.003	57.24	0.2024
1.004	49.81	0.1668
1.005	44.47	0.1447
1.010	31.65	0.0923
1.050	13.66	0.0331
1.100	9.39	0.0214
1.200	6.46	0.0140

All the ARL values tabulated above are steady state values. The Monte Carlo simulation approach is used to obtain steady state ARL values, by introducing the drift after 50 subgroups, and resetting the chart if it signals in the period between the first and the fiftieth subgroups.

It can be seen that there is an inverse relationship between the design parameter h, and the subgroup size n when the in-control ARL is a constant. This is expected because of the nature of the statistic Y_i that is and integral part of the test. Since $Y_i = \sum_{j=1}^{n} Z_{ij}^2 / n$, we expect that Y_i will decrease and approach the true population variance with an increase in the subgroup size n. Since smaller values now accumulate on the test, the value of the decision interval h will have to be smaller in order to have the same sensitivity to false alarms.

4.2 *Comparisons with other Control Schemes*

The other control charts which the DVCC is compared with are briefly described below. The steady state ARL performance of the charts is investigated when the charts are applied to a drifting process variance.

<u>Shewhart S^2 chart</u>

The one-sided Shewhart S^2 chart is implemented using Monte Carlo simulation The center line (CL) and upper control limit (UCL) is given by

$$CL = \sigma_0^2, \qquad UCL = \frac{\sigma_0^2}{n-1}\chi^2_{n-1,\alpha} \tag{4.2.1}$$

where $\chi^2_{n-1,\alpha}$ denotes the percentage point of the chi-distribution such that $\Pr(\chi_{n-1} > \chi_{n-1,\alpha}) = \alpha$ and σ_0^2 is the in-control process variance. In our case, $\sigma_0^2 = 1$. If the S^2 chart is calibrated analytically to have an in-control ARL value of 400, α is 0.0025. However, since we used Monte Carlo simulation, $\chi^2_{n-1,\alpha}$ is treated as a constant which is varied in order to calibrate the chart.

EWMA *on* $\ln(S^2)$

The ARL performance of the EWMA control chart on $\ln(S^2)$[4] is investigated. Successive values of the proposed EWMA at successive times t are given by

$$EWMA_t = \max\{(1-\lambda)WWMA_{t-1} + \lambda y_t \ln(\sigma_0^2)\} \tag{4.2.2}$$

where $EWMA_0 = \ln(\sigma_0^2)$, λ is a smoothing constant satisfying $0 < \lambda \le 1$, and $y_t = \ln(S_t^2)$, where S_t^2 values are successive sample variances. The upper control limit is

$$UCL = \ln(\sigma_0^2) + k\left[\left(\frac{\lambda}{2-\lambda}\right)\left\{\frac{2}{n-1} + \frac{2}{(n-1)^2} + \frac{4}{3(n-1)^3} - \frac{16}{15(n-1)^5}\right\}\right]^{1/2} \tag{4.2.3}$$

where k is a constant.

CUSUM on S

The one-sided CUSUM control chart on S[13] is implemented and the ARL performance is investigated. The cumulative sum is computed using

$$S_i = \max\{0,\ SD_i - k\overline{SD} + S_{i-1}\} \tag{4.2.4}$$

where SD_i denotes the ith standard deviation, and $\overline{SD}$ denotes the value at which the standard deviation is to be controlled. The chart signals if S_i exceeds h, the decision interval of the chart.

Tables 4.2.1, 4.2.2 and 4.2.3 give the comparative ARL values for the DVCC and the various chart described in this section when the charts are calibrated to have in-control ARL values of 400.

Table 4.2.1 Comparison of control charts (n=4)

m	DVCC h=12.9	Shewhart S^2 $\chi^2_{3,\alpha}$=12.2	EWMA on $\ln(S^2)$ k=1.432	CUSUM on S σ_1=1.2, h=2.08	CUSUM on S σ_1=1.4, h=1.38
1.000	399.76	400.75	400.37	401.25	400.12
1.001	116.75	162.84	140.67	123.54	134.80
1.003	70.47	95.13	79.59	70.66	75.68
1.005	55.45	71.27	59.71	53.05	56.73
1.050	16.86	16.54	14.34	14.18	14.10
1.100	11.54	10.47	9.30	9.57	9.25

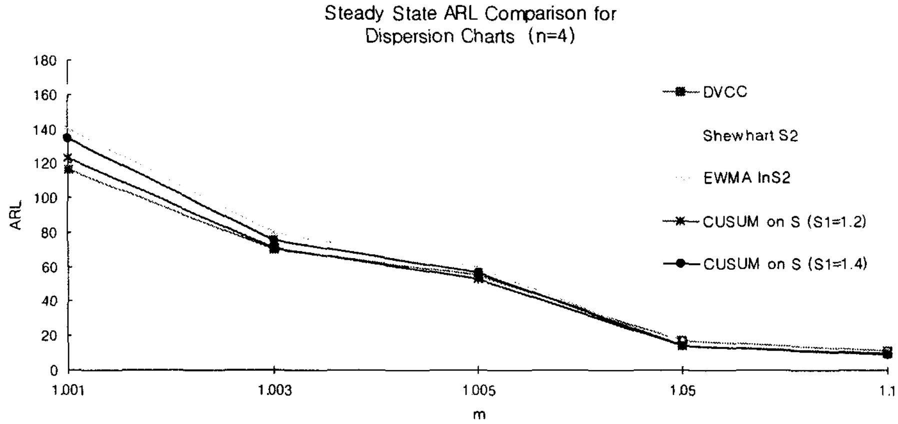

Figure 4.2.1 Comparison of control charts, n=4

Table 4.2.2. Comparison of control charts (n=5)

m	DVCC h=11.5	Shewhart S^2 $\chi^2_{4,\alpha}$=14.6	EWMA on $\ln(S^2)$ k=1.595	CUSUM on S σ_1=1.2 h=1.75	CUSUM on S σ_1=1.4 h=1.13
1.000	396.83	402.07	400.13	402.77	402.07
1.001	111.39	129.72	132.90	119.00	129.72
1.003	67.62	74.21	75.24	67.55	74.21
1.005	53.05	54.18	56.10	50.81	54.18
1.050	16.00	13.18	13.41	13.17	13.18
1.100	10.92	8.72	8.72	8.89	8.72

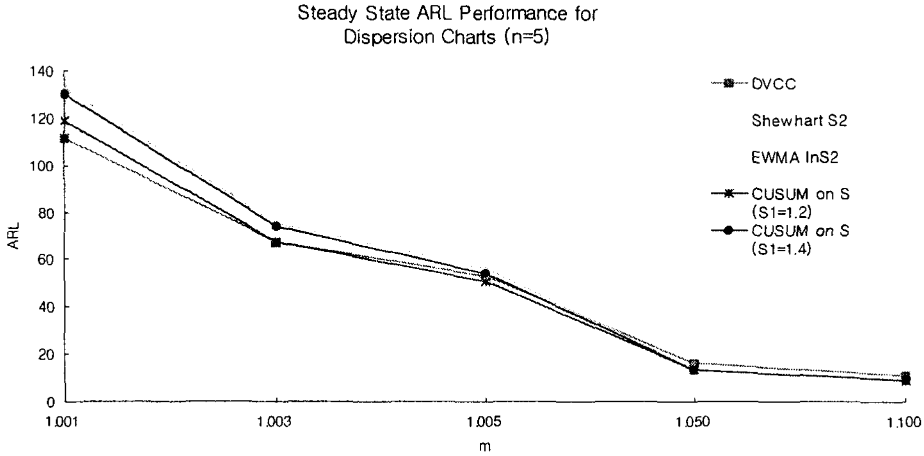

Figure 4.2.2 Comparison of control charts (n=5)

It is seen from Table 4.2.1 and 4.2.2 that for small drifts of size m=1.001 and 1.003, the DVCC outperforms the other charts. The only competitor is the CUSUM on S designed for small shifts, and even this chart is outperformed at m=1.001. It must be remembered, while making inferences about performance, that the DVCC is designed for m=1.002, and it seems to perform best in this range These observations hold good for larger subgroup sizes also as can be observed from Table 4.2.3

Table 4.2.3 Comparison of control charts,(n=10)

m	DVCC h=8.0	Shewhart S^2 $\chi^2_{9,\alpha}$=24.3	EWMA on $\ln(S^2)$ k=1.987	CUSUM on S σ_1=1.2, h=0.9	CUSUM on S σ_1=1.4, h=0.5
1.000	403.09	399.81	401.41	403.43	402.77
1.001	95.46	141.11	114.08	108.35	120.31
1.003	57.24	79.53	62.23	58.39	65.73
1.005	44.47	58.92	46.34	43.39	48.33
1.050	13.66	13.08	10.84	10.83	11.06
1.100	0.021	8.07	7.02	7.14	7.15

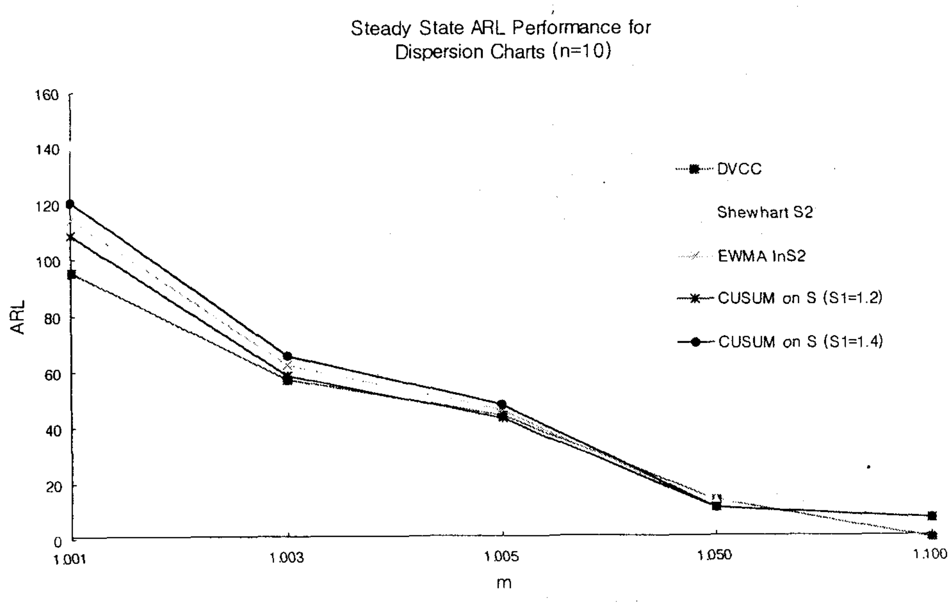

Figure 4.2.3. Comparison of control charts (n=10)

4.3. Sensitivity Analysis

Since the DVCC is designed to detect gradual drifts in the variance, it is expected to outperform the charts it was compared to in Section 4.3 because those charts are designed to detect step changes in the variance. Therefore, in order to determine the overall performance of the DVCC, it is necessary to investigate its performance when it is applied to stepped change in the variance.

The DVCC is calibrated to have an in-control initial performance ARL of 200 for comparison purposes. All of the data for the different charts in Table 4.3.1 are from Acosta[1], except for the data for the EWMA $\ln(S^2)$ chart, which is from Crowder[4]. The data are all initial performance ARL values for subgroup sizes of n=5.

Table 4.3.1 compares the ARL performance DVCC with those of 3 other control charts for the process dispersion. The best ARL performance is selected among all the dispersion charts that the DVCC is compared to. Figure 4.3.1. provides a comparison between the DVCC and some selected charts.

It is seen that the DVCC performs very well for small stepped changes. When the percentage increase in σ is 10%, the DVCC performs much better than most of the charts it is compared to. However, as the size of the step increases, all the charts start "catching up" with the DVCC. For the largest shift considered (a 100% increase in σ), the DVCC is actually the worst of the 4 charts considered. This is not unexpected since the DVCC is designed to detect gradual drifts in the process variance, and also because the particular DVCC under consideration is designed for small shifts.

Table 4.3.1 Comparison of initial performance ARL values for a stepped change in variance

% increase in σ	DVCC h=7.88	Shewhart S UCL=2.89	EWMA $\ln(S^2)$ k=1.07, λ=0.05	CUSUM S k=1.034 ,h=1.90
0.00	199.32	200.18	200.00	200.60
10.00	37.05	65.10	43.00	38.80
20.00	18.90	28.30	18.00	16.65
30.00	12.62	15.10	11.00	10.36
40.00	9.32	9.20	7.60	7.50
50.00	7.35	6.30	6.00	5.85
100.00	3.48	2.40	3.20	3.01

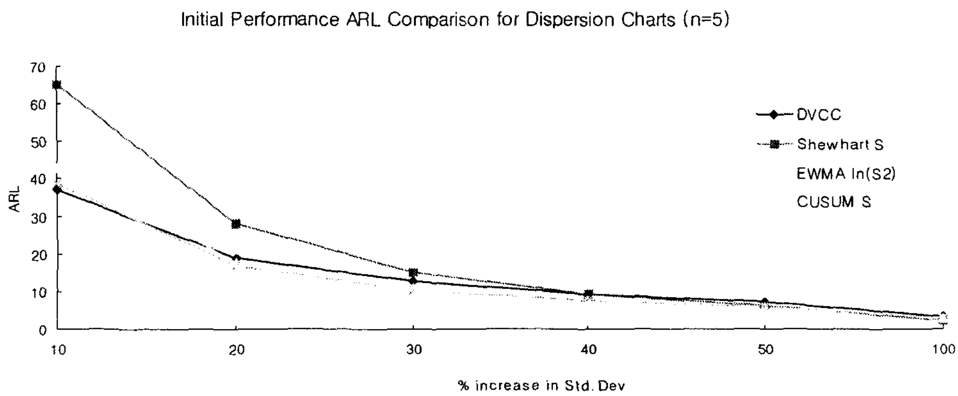

Figure 4.3.1. Comparison of initial performance ARL values for a stepped change in variance

5. Conclusions

It can be observed that the DVCC performs very well for small sizes of drift. The sensitivity analysis performed also shows that the DVCC performs quite well for small stepped changes. An overall recommendation is to use the DVCC in processes where small drifts or steps are expected. It must be kept in mind that the DVCC that is investigated here is designed for a very small drift (m=1.002) and this might play a significant role on the performance of the chart. In order to fully understand the performance of the chart, we recommend investigating the ARL performance of the DVCC when it is designed for moderate and large sizes of drift.

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