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NOTE ON CONTACT STRUCTURE AND SYMPLECTIC STRUCTURE

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ABSTRACT. Let (X, J) be a closed, connected almost complex four-manifold. Let X_1 be the complement of an open disc in X and let ξ_1 be the contact structure on the boundary ∂X_1 which is compatible with a symplectic structure on X_1. Then we show that (X, J) is symplectic if and only if the contact structure ξ_1 on ∂X_1 is isomorphic to the standard contact structure on the 3-sphere S^3 and ∂X_1 is J-concave. Also we show that there is a contact structure ξ_0 on $S^2 \times S^1$ which is not strongly symplectically fillable but symplectically fillable, and that $(S^2 \times S^1, \sigma)$ has infinitely many non-diffeomorphic minimal fillings whose restrictions on $S^2 \times S^1$ are σ where σ is the restriction of the standard symplectic structure on $S^2 \times D^2$.

1. Introduction

Contact geometry has recently come to the foreground of low dimensional topology. Not only have there been striking advances in the understanding of contact structures on 3-manifolds, but there has been significant interplay with symplectic geometry and Seiberg-Witten theory. In 1971 Martinet [12] showed how to construct a contact structure on any 3-manifold. Later it became clear that contact structures fell into two distinct classes: tight and overtwisted.

It is the tight contact structures that carry significant geometric information. It was known that for any 3-manifold there are only finitely many elements in its second cohomology that can be realized by tight contact structures [7]. More recently Kronheimer and Mrowka [10] have

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shown that only finitely many homotopy types of plane fields can be realized by semi-fillable (and hence tight) contact structures.

We apply them to the simplest class of 3-manifolds. Recall lens spaces $L(p, q)$ are 3-manifolds that can be written as the union of two solid tori, or in other words, lens spaces are Heegaard genus one manifolds. Recently Etnyre have shown that there is a unique tight contact structure on $L(0, q) = S^1 \times S^2$, $L(1, q) = S^3$, and $L(2, q) = \mathbb{RP}^3$.

The purpose of this paper is to introduce a relationship between the symplectic structure on a 4-manifold with boundary and the contact structure on its boundary. Let (X, J) be a closed, connected almost complex 4-manifold. Let X_1 be the complement of an open disc in X and let ξ_1 be the contact structure on the boundary ∂X_1 which is compatible with symplectic structure on X_1. Then we show that (X, J) is symplectic if and only if the contact structure ξ_1 on ∂X_1 is isomorphic to the standard contact structure on $L(1, q) = S^3$ and ∂X_1 is J-concave. Also in section 3, we show that there is an example that which is not strongly symplectically fillable but symplectically fillable contact structure on $L(0, q) = S^1 \times S^2$ and we show that $L(0, q) = S^1 \times S^2$ has infinitely many non-diffeomorphic minimal fillings.

2. Symplectic Structures on Almost Complex 4-Manifolds

2.1 Symplectic structures on open manifolds

Let X be a closed, connected, smooth 4-manifold with almost complex structure J. Let g be a Riemannian metric on X on which J is an isometry. In this case we say that g is compatible with J. The almost complex structure J and the metric g define a nondegenerate 2-form ω' by $\omega'(v_1, v_2) = g(Jv_1, v_2)$ for any $v_1, v_2 \in TX$.

In this case the nondegenerate 2-form ω' is called compatible with J. In fact there is a one-to-one correspondence of nondegenerate 2-forms compatible with J and Riemannian metrics compatible with J.

If X is not symplectic, then ω' is not closed. Let N be a small neighborhood of a point p in X which is diffeomorphic to the standard open 4-disc D^4. Then $X_1 \equiv X - N$ is an almost complex manifold with boundary ∂X_1 and has a nondegenerate 2-form $\omega_0{}'$ which is the restriction of ω'.

A manifold is called open if each component is either non compact or has a non-empty boundary.

THEOREM 2.1 (GROMOV). *Let X be an open 4-manifold. Let $w \in \Omega^2(X)$ be a nondegenerate 2-form and let $a \in H^2(X;\mathbb{R})$. Then there is a smooth family of nondegerate 2-forms ω_t on X such that $\omega_0 = \omega$ and ω_1 is a symplectic form which represents the class a.*

If $a \in H^{2,+}(X_1;\mathbb{R})$ is a self-dual cohomology class then by the Gromov's Theorem there is a smooth family of nondegenerate forms ω_t on X_1 such that $\omega_0 = {\omega_0}'$ and ω_1 is a self-dual symplectic form which represents the class a.

2.2 J-convexity

Let (X, J) be an almost complex manifold of the real dimension 4 and Σ be an oriented hypersurface in X of the real codimension 1. Each tangent plane $T_x(\Sigma)$, $x \in \Sigma$, contains a unique complex line $\xi_x \subset T_x(\Sigma)$ which we will call a complex tangency to Σ at x. The complex tangency is canonically oriented and, therefore, cooriented. Hence the tangent plane distribution ξ on Σ can be defined by an equation $\alpha = 0$ where the 1-form α is unique up to multiplication by the same positive factor. We say that Σ is J-convex (or J-concave) if $d\alpha(v, Jv) > 0$ (or $d\alpha(v, Jv) < 0$) for any non-zero vector $v \in \xi_x$, $x \in \Sigma$. We use the word pseudo-convex or pseudo-concave when the almost complex structure J is not specified.

Following Gromov (see [9]) we say that an almost complex manifold is tame if there exists a symplectic structure ω on X such that the form $\omega(v, Jv)$, $v \in T(X)$, is positive definite.

The following theorem indicates that the topology of the J-convex boundary imposes very strong restrictions on the topology of the domain.

THEOREM 2.2 [7]. *Let (X, J) be a tame symplectic manifold and $\Omega \subset X$ be a domain bounded by a J-convex 3-sphere. Then for an almost complex structure J' which is C^∞-close to J the manifold (Ω, J') is a 4-ball up to blowing up a few points. In particular, Ω is diffeomorphic to $D^4 \sharp k\overline{\mathbb{CP}^2}$.*

2.3 Condition for symplectic manifold

Let (X, J) be a closed, connected almost complex 4-manifold and g be a Riemannian metric on X on which J is an isometry. Then there is the nondegenerate 2-form ω' on X which is compatible with J. Let N be a small neighborhood of a point p in X which is diffeomorphic to the standard open 4-disc D^4 and $X_1 \equiv X - N$. Then X_1 is an almost complex manifold with boundary $\partial X_1 \simeq S^3$ and has a nondegenerate 2-form ω_0 which is the restriction of ω'. Then by the Gromov's Theorem, there is a smooth family of nondegenerate forms ω_t on X_1 such that ω_1 is symplectic on X_1. Let $\xi_1 = T(\partial X_1) \cap J_1 T(\partial X_1)$, where J_1 is a compatible almost complex structure with ω_1. Then ξ_1 is a compatible contact structure on ∂X_1 with ω_1. Let $\omega = \sum_{i=1}^{2} dx_i \wedge dy_i$ be the standard symplectic structure on $\mathbb{R}^4$. Then the standard contact structure ξ_{st} on the 3-sphere S^3 is given by the 1-form $\frac{1}{2}\sum_{i=1}^{2}(x_i dy_i - y_i dx_i)$. Now we are ready to prove the following theorem.

THEOREM 2.3. *In the above notations, a closed, connected almost complex 4-manifold (X, J) is symplectic if and only if the contact structure ξ_1 on ∂X_1 is isomorphic to the standard contact structure ξ_{st} on S^3 and ∂X_1 is J-concave.*

Proof. Suppose that X is a closed, connected symplectic 4-manifold. Darboux's theorem says that any symplectic form ω on X is locally diffeomorphic to the standard symplectic form $\omega_0 = \sum_{i=1}^{2} dx_i \wedge dy_i$ on $\mathbb{R}^4$. For any point $p \in X$ there is a coordinate chart $\phi : D^4(1+\epsilon) \to X$ such that $\phi(0) = p$ and $\omega_0 = \phi^*\omega$ where $D^4(1+\epsilon)$ is a disc in $\mathbb{R}^4$ with center 0 and radius $1+\epsilon$ for some small $\epsilon > 0$. Then the restriction $\phi : S^3 \to \phi(S^3)$ is a diffeomorphism. The contact structure ξ_1 on $\phi(S^3) = \partial(X_1 \equiv X - \phi(D^4(1)))$ compatible with the symplectic form ω is isomorphic to the standard contact structure ξ_{st} on S^3 since there is a unique fillable contact structure (up to isotopy) on S^3.

Since ξ_1 is a contact structure on $\partial X_1 \simeq S^3$, there is a contact 1-form α_1 such that $d\alpha_1|_{\xi_1} \neq 0$. If $d\alpha_1|_{\xi_1} > 0$, then ∂X_1 is J-convex. Since X_1 is minimal, by Theorem 2.2, $X_1 \simeq D^4$ and

$$X = X_1 \bigcup_{\partial} D^4 = D^4 \bigcup_{\partial} D^4 \simeq S^4.$$

Since S^4 cannot have any almost complex structure, this contradicts the assumption. Therefore $d\alpha_1|_{\xi_1} < 0$. Hence ∂X_1 is J-concave.

Conversely, suppose that (X_1, ω_1) is a symplectic 4-manifold with boundary ∂X_1 on which the compatible contact structure ξ_1 with ω_1 is isomorphic to the standard contact structure ξ_{st} on S^3.

Let U_1 be a small collared neighborhood of ∂X_1 in X_1. Let $\phi : U_1 \to D^4$ be diffeomorphic onto its image $\phi(U_1)$ such that $\phi(\partial X_1) = S^3$ and $\phi^*(\xi_{st}) = \xi_1$. Let θ_{st} be the 1-form on S^3 whose kernel is ξ_{st} and let θ_1' be the 1-form on ∂X_1 whose kernel is ξ_1. Then $\phi^*\theta_{st} = f\theta_1'$, where f is a negative function on ∂X_1.

Extend θ_1' and θ_{st} on their collared neighborhoods respectively. Then $\phi^*\omega_0 = \phi^* d\theta_{st} == d\phi^*(\theta_{st}) = d(f\theta_1')$. Since $d\omega_1 = 0, (d\omega_1)|_{U_1} = d(\omega_1|_{U_1}) = 0$. Since ∂X_1 is a strong deformation retract of U_1 and ∂X_1 is diffeomorphic to S^3, ω_1 is exact on U_1.

There is a 1-form θ_0 on U_1 such that $d\theta_0 = \omega_1$ and $\phi^*\omega_0 - \omega_1 = d(f\theta_1') - d\theta_0 = d(f\theta_1' - \theta_0)$. Let $\theta_1 = f\theta_1'$. Since the cotangent bundle on U_1 is trivial there is a smooth 1-parameter family of 1-form θ_t joining θ_0 and θ_1. Thus we may extend smooth 1-parameter family $d\theta_t = \omega_t$ joining ω_1 to $\phi^*\omega_0$. So by attaching D^4 to X_1 via ϕ we have a symplectic 4-manifold $X = X_1 \cup_\phi D^4$. □

COROLLARY 2.4. *Let X be a closed, connected smooth 4-manifold with almost complex structure. According to the notations of the above Theorem if X is not symplectic, then the boundary $(\partial X_1, \xi_1)$ and the 3-sphere (S^3, ξ_{st}) are diffeomorphic but not contactomorphic.*

3. Fillable Contact Structure

In this chapter, we introduce some definitions on the boundary of a 4-manifold. Using this, we have some results about symplectic manifolds with contact-type boundaries. A contact structure on a 3-dimensional manifold M is a 2-plane field ξ in TM which is nowhere integrable. It determines an orientation which must agree with the given one on M^3.

(M^3, ξ) is called to be symplectically fillable if M bounds a compact, symplectic 4-manifold (X^4, ω) such that $\omega|_\xi \neq 0$. And (M^3, ξ) is called to be strongly symplectically fillable if M bounds a compact, symplectic

4-manifold (X^4, ω) such that $\omega|_\xi \neq 0$ and there exists a vector field V near M, which is outward pointing and transverse to M at M and has the property that its flow expands ω, i.e., $\mathcal{L}_V\omega = \omega$. If (M^3, ξ) is strongly symplectically fillable, then (M^3, ξ) is a contact-type boundary of a symplectic 4-manifold (X, ω).

It is known that if (M^3, ξ) is strongly symplectically fillable, then it is symplectically fillable. But the converse is not known yet. Hence we will construct a contact structure which is symplectically fillable but not strongly symplectically fillable.

Consider the symplectic manifold (S^2, ω_0). The standard symplectic form ω_0 on $S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is given by

$$\omega_0 = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

in the usual coordinates $x + iy$ on $\mathbb{C}$. Let $M = S^2 \times S^1$ and let $\alpha = ydx + d\theta$ be a 1-form on $M = S^2 \times S^1$, where $e^{i\theta}$ is the coordinate on S^1. Let $\xi_0 = \mathrm{Ker}\alpha$. Then ξ_0 is a contact structure on M and α is a contact form on $M = S^2 \times S^1$.

Consider the symplectic 4-manifold $(S^2 \times D^2, \omega)$ where $\omega = \omega_0 \oplus \omega^*$, ω_0 is the standard symplectic form on S^2 and ω^* is the standard symplectic form on D^2. Then $\partial(S^2 \times D^2) \simeq S^2 \times S^1$ and $\omega|_{\xi_0} \neq 0$. Therefore $(S^2 \times S^1, \xi_0)$ is symplectically fillable.

LEMMA 3.1. *$(S^2 \times S^1, \xi_0)$ is symplectically fillable and there is no ω-tame almost complex structure J on $S^2 \times D^2$ such that $S^2 \times S^1$ is J-convex.*

Proof. Suppose not. That is, there is an ω-tame almost complex structure J on $(S^2 \times D^2, \omega)$ such that $S^2 \times S^1$ is J-convex. Hence $(S^2 \times D^2, \omega)$ has an ω-tame almost complex structure J with the J-convex boundary. By the following Theorem 3.2, $S^2 \times \{p\}$ in $S^2 \times S^1 \simeq \partial(S^2 \times D^2)$ bounds an embedded ball B^3 in $S^2 \times D^2$. This is impossible. □

THEOREM 3.2 [7]. *Let (X, ω) be a symplectic 4-manifold and J be an ω-tame almost complex structure and ∂X is a J-convex boundary.*

Any closed surface $\Sigma \subset \partial X$ different from S^2 satisfies the inequality

$$\chi(\Sigma) \leq -|c_1(X)(\Sigma)|.$$

If M is diffeomorphic to S^2, then it can be filled by holomorphic disc. In particular, it bounds an embedded ball $B^3 \subset X$.

Let $\mathrm{Fill}^s(\xi_0)$ be the set of all symplectic fillings of $(S^2 \times S^1, \xi_0)$ and $\mathrm{Fill}^{s.s}(\xi_0)$ the set of all strongly symplectic fillings of $(S^2 \times S^1, \xi_0)$. Hence if $(X,\omega) \in \mathrm{Fill}^s(\xi_0)$,then (X,ω) is a compact symplectic 4-manifold with $\partial X \simeq S^2 \times S^1$ and $\omega|_{\xi_0} \neq 0$. Similarly if $(X,\omega) \in \mathrm{Fill}^{s.s}(\xi_0)$, then (X,ω) is a compact symplectic 4-manifold with a contact type boundary $(S^2 \times S^1, \xi_0)$. Also $\mathrm{Fill}^s(\xi_0) \supset \mathrm{Fill}^{s.s}(\xi_0)$.

REMARK. $(S^2 \times D^2, \omega) \notin \mathrm{Fill}^{s.s}(\xi_0)$. If not, $\omega|_{\partial(S^2 \times D^2)} = d\alpha$. Since $(S^2 \times D^2, \omega) \in \mathrm{Fill}^s(\xi_0)$, $\omega|_{\xi_0} \neq 0$. Hence we can choose an almost complex structure J on $S^2 \times D^2$ has a J-convex boundary $(S^2 \times S^1, \xi_0)$. This contradicts the above Lemma 3.1.

By Lemma 3.1, $(S^2 \times D^2, \omega) \in \mathrm{Fill}^s(\xi_0)$. If $(X,\omega) \in \mathrm{Fill}^s(\xi_0)$, then (X,ω) is a compact symplectic 4-manifold with $\partial X \simeq S^2 \times S^1$ and $\omega|_{\xi_0} \neq 0$. Then we can choose an almost complex structure $J \in \mathcal{J}_\tau(S^2 \times D^2, \omega)$ such that ξ_0 is J-invariant. Hence $\omega|_{\xi_0} > 0$. Since α is a contact form on ξ_0, $d\alpha|_{\xi_0} \neq 0$. If $d\alpha|_{\xi_0} < 0$, then $(X,\omega) \notin \mathrm{Fill}^{s.s}(\xi_0)$. If not, that is, $(X,\omega) \in \mathrm{Fill}^{s.s}(\xi_0)$, $\omega|_{\partial X} = d\alpha$. Since $\omega|_{\xi_0} > 0$ and $d\alpha|_{\xi_0} < 0$, this is impossible. If $d\alpha|_{\xi_0} > 0$, then (X,ω) has a J-convex boundary $(S^2 \times S^1, \xi_0)$. Let $F_p = S^2 \times \{p\}$ be a sphere in a J-convex boundary $(S^2 \times S^1, \xi_0)$, for all $p \in S^1$. Then by Theorem 3.2, F_p bounds an embedded ball B^3 in X. Hence $B^3 \times S^1$ is embedded in (X,ω). Let $Y \equiv X \underset{\partial X}{\cup} (B^3 \times S^1)$. Then Y is a closed 4-manifold and Y contains a closed 4-manifold $S^3 \times S^1$. Hence this is impossible unless $X \simeq B^3 \times S^1$. Therefore the only candidate which is an element of $\mathrm{Fill}^{s.s}(\xi_0)$ is $B^3 \times S^1$. Note that if $(X,\omega) \in \mathrm{Fill}^{s.s}(\xi_0)$, then the first Chern class $c_1(X)$ restricted to $\partial X \simeq S^2 \times S^1$ coincides with the Euler class $e(\xi_0)$ of the bundle ξ_0. Since $(S^2 \times D^2, \omega) \in \mathrm{Fill}^s(\xi_0)$,

$$e(\xi_0) = c_1(T(S^2 \times D^2))|_{S^2 \times S^1} \neq 0 \in H^2(S^2 \times S^1).$$

If there is a symplectic form ω' on $B^3 \times S^1$ such that $(B^3 \times S^1, \omega') \in \mathrm{Fill}^s(\xi_0)$, then $e(\xi_0) = c_1(T(B^3 \times S^1))|_{\partial(B^3 \times S^1)}$. But this is impossible because that $c_1(T(B^3 \times S^1)) = 0$ in $H^2(B^3 \times S^1) = \{0\}$. Hence $\mathrm{Fill}^{s.s}(\xi_0) = \emptyset$. Therefore we have the following proposition.

PROPOSITION 3.3. *There is a contact structure ξ_0 on $S^2 \times S^1$ which is not strongly symplectically fillable but symplectically fillable.*

Consider an oriented 3-dimensional manifold M with closed 2-form σ. We will say that (M^3, σ) has contact type if there is a positively oriented contact form α on M such that $d\alpha = \sigma$. Following Eliashberg [7], we say that the symplectic manifold (Z, ω) fills (M, σ) if there is a diffeomorphism $f : \partial X \to M$ such that $f^*\sigma = \omega|_{\partial Z}$. Further, the filling (Z, ω) is said to be minimal if Z contains no exceptional spheres in its interior. In [13], McDuff show that the lens space L_p, $p \geq 1$, all have minimal symplectic fillings and if $p \neq 4$, minimal fillings (Z, ω) of (L_p, σ) are unique up to diffeomorphism, and up to symplectomorphism if one fixes the cohomology class $[\omega]$. However (L_4, σ) has exactly two nondiffeomorphic minimal fillings.

Fix the symplectic form $\omega = \omega_0 \oplus \omega^*$ on $S^2 \times D^2$. Here ω_0 is the standard symplectic form on S^2 and ω^* is the standard symplectic form on D^2. Let $\sigma = \omega|_{S^2 \times S^1}$. Then σ is a closed 2-form on $S^2 \times S^1$. Let ω_g be the standard symplectic form on Σ_g. Then $(S^2 \times \Sigma_g, \bar{\omega} = \omega_0 \oplus \omega_g)$ is a symplectic 4-manifold. Choose a point p in Σ_g. Then there is a neighborhood $\mathcal{N}(p)$ in Σ_g such that $\omega_g|_{\mathcal{N}(p)} = \omega^*$. Therefore

$$\bar{\omega}|_{S^2 \times \mathcal{N}(p)} = \omega_0 \oplus \omega^*.$$

Let $Z_g = (S^2 \times \Sigma_g) \backslash (S^2 \times \mathcal{N}(p))$ and $\bar{\omega}_g = \bar{\omega}|_{Z_g}$. Then $(Z_g, \bar{\omega}_g)$ are symplectic 4-manifolds with $\partial Z_g \simeq S^2 \times S^1$ and

$$\bar{\omega}_g|_{S^2 \times S^1} = \omega_0 \oplus \omega^*|_{S^2 \times S^1} = \sigma.$$

Hence $(Z_g, \bar{\omega}_g)$ are minimal symplectic fillings of $(S^2 \times S^1, \sigma)$. Therefore we have the following proposition.

PROPOSITION 3.4. *$(S^2 \times S^1, \sigma)$ has infinitely many non-diffeomorphic minimal fillings whose restrictions on the boundary are σ.*

REMARK. The above $(Z_g, \bar{\omega}_g) \in \mathrm{Fill}^s(\xi_0)$, but $(Z_g, \bar{\omega}_g) \notin \mathrm{Fill}^{s.s}(\xi_0)$.

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