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HEMICOMPACTNESS AND HEMICONNECTEDNESS OF HYPERSPACES

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Abstract. We prove the following: (1) For a Hausdorff space X, the hyperspace $\mathcal{K}(X)$ of compact subsets of X is hemicompact if and only if X is hemicompact. (2) For a regular space X, the hyperspace $C_K(X)$ of subcontinua of X is hemicompact (hemiconnected) if and only if X is hemicompact (hemiconnected). (3) For a locally compact Hausdorff space X, each open set in X is hemicompact if and only if each basic open set in the hyperspace $\mathcal{K}(X)$ is hemicompact. (4) For a connected, locally connected, locally compact Hausdorff space X, $\mathcal{K}(X)$ is hemiconnected if and only if X is hemiconnected.

Introduction

Let X be a Hausdorff space. Denote 2^X the space of all nonempty closed subsets of X endowed with the Vietoris topology and for any subset A of X, let $2^A = \{E \in 2^X : E \subset A\}$. Let $\mathcal{K}(X) = \{E \in 2^X : E \text{ is compact}\}$, $\mathcal{F}_n(X) = \{E \in 2^X : E \text{ has at most n elements}\}$, $C_K(X) = \{E \in 2^X : E \text{ compact and connected}\}$, and $C(X) = \{E \in 2^X : E \text{ is connected}\}$ with the subspace topology inherited from the topology of 2^X and for any subset A of X, $2^A = \{E \in 2^X : E \subset A\}$.

The aim of this paper is to prove the intrinsic hemicompact (hemiconnected) relation between the space X and its hyperspaces $\mathcal{K}(X)$ and $C_K(X)$.

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For notational purpose, small letters will denote elements of X, capital letters will denote subsets of X and elements of 2^X, and script letters will denote subsets of 2^X. If $\mathcal{B} \subset 2^X$, $\cup\mathcal{B} = \cup\{A : A \in \mathcal{B}\}$. If $A \subset X$, $\overline{A}$, $Int(A)$, $Bd(A)$ will denote the closure, interior, boundary of A in X respectively.

1. Preliminaries

Let X be a Hausdorff space. For a collection $\{A_1, \cdots, A_n\}$ of subsets of X, let $\langle A_1, \cdots, A_n\rangle = \{E \in 2^X : E \cap A_i \neq \emptyset$ for each i=1,$\cdots$,n and $E \subset \cup_{i=1}^n A_i\}$. The collection of all sets of the form $\langle U_1, \cdots, U_n\rangle$ with $U_1, \cdots, U_n$ open in X, is a base for the finite (Vietoris) topology T_v for 2^X. When we restrict T_v on each of $C(X)$, $\mathcal{K}(X)$, $\mathcal{F}_n(X)$, and $C_K(X)$, then each one of these spaces is also called hyperspaces of X. For each n, $\mathcal{F}_n(X)$ is closed subspace of 2^X and in particular $\mathcal{F}_1(X)$ and X are homeomorphic.

It is known that:

LEMMA 1.1 [4]. (a) $\overline{\langle U_1, \cdots, U_n\rangle} = \langle \overline{U}_1, \cdots, \overline{U}_n\rangle$.
(b) $\langle V_1, \cdots, V_m\rangle \subset \langle U_1, \cdots, U_n\rangle$ *if and only if* $\cup_{j=1}^m V_j \subset \cup_{i=1}^n U_n$ *and for each* U_i *there is a* V_j *such that* $V_j \subset U_i$.
(c) *Let* X *be a space. Then for each* $\mathcal{B} \in \mathcal{K}(\mathcal{K}(X))$, $\cup\mathcal{B} \in \mathcal{K}(X)$.
(d) *If* $\mathcal{B}$ *is a connected subset of* 2^X *which also contains at least one connected element, then* $\cup\mathcal{B}$ *is connected in* X.
(e) X *is compact if and only if* 2^X *is compact.*
(f) *The map* $f : [2^X]^n \to 2^X$ *defined by* $f(A_1, \cdots, A_n) = \cup_{i=1}^n A_i$ *is continuous.*

LEMMA 1.2. [3] *If* X *is a compact Hausdorff space, then* 2^X *and* $C(X)$ *are both arcwise connected compact Hausdorff spaces.*

LEMMA 1.3. *If* $\mathcal{U}$ *is an open set in the subspace* $\mathcal{K}(X)$, *then* $\cup\mathcal{U}$ *is open in* X.

Proof. Without loss of generality, let $\mathcal{U} = \langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$ be an open set in $\mathcal{K}(X)$ and let $U = \cup\mathcal{U}$. Let $x \in U$. Then $x \in U_i$

for some i. Choose $x_j \in U_j$ for each $j \neq i$. For each $y \in U_i$, let $E_y = \{x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n\}$. Then $E_y \in \langle U_1, \cdots, U_n \rangle \cap \mathcal{K}(X)$ and thus $y \in E_y \subset U$. Hence $U_i \subset U$. □

LEMMA 1.4. *Suppose X is a connected, locally connected, and locally compact Hausdorff space. Then for each compact subset K of X, there exists a subcontinuum C of X which contains K.*

Proof. Let $\{U_1, \cdots, U_n\}$ be an open cover of K such that each U_i is connected and whose closure is compact. For each i, pick a point $a_i \in \overline{U}_i$. For $i > 1$, let $\mathcal{U}_i = \{V_{i,1}, \cdots, V_{i,i_k}\}$ be a simple chain from a_1 to a_i, where each V_{i,i_j} is connected open set whose closure is compact. Then $M = (\cup_{l=1}^{n} \overline{U}_l) \cup (\cup_{i=2, i_j=1}^{i=n, i_j=n_k} \overline{V}_{i,i_j})$. Then M is compact and connected. □

PROPOSITION 1.5. *X is locally compact Hausdorff if and only if $\mathcal{K}(X)$ is locally compact Hausdorff.*

Proof. Suppose X is locally compact Hausdorff. Then X is regular. Hence 2^X is Hausdorff [4, 4.9.3] so that $\mathcal{K}(X)$ is Hausdorff.

Let $E \in \mathcal{K}(X)$. Since X is locally compact, there exists an open set U containing E such that $\overline{U}$ is compact. Then $E \in \langle U \rangle \cap \mathcal{K}(X)$. For each $F \in \langle U \rangle$, F is compact. Thus $\langle U \rangle \subset \mathcal{K}(X)$. Also $\overline{\langle U \rangle} = \langle \overline{U} \rangle = 2^{\overline{U}}$. So $\langle \overline{U} \rangle$ is compact by Lemma 1.1(e). Hence $\mathcal{K}(X)$ is locally compact at E.

Conversely, suppose that $\mathcal{K}(X)$ is locally compact Hausdorff. Let $x \in X$. Let $\mathcal{U}$ be a neighborhood of $\{x\}$ in $\mathcal{K}(X)$ such that $\overline{\mathcal{U}}$ is compact. Then $U = \cup \mathcal{U}$ is open in X by Lemma 1.3, and $\overline{\mathcal{U}} \in \mathcal{K}(\mathcal{K}(X))$ implies that $\cup \overline{\mathcal{U}}$ is compact by Lemma 1.1(c). Hence $U \subset \overline{U} \subset \cup \overline{\mathcal{U}}$.

Since $\mathcal{F}_1(X) = \{\{x\} : x \in X\}$ is Hausdorff and homeomorphic to X, X is Hausdorff. □

PROPOSITION 1.6. *If X is a normal space, then $C(X)$ is closed in 2^X.*

Proof. Suppose $E \in 2^X$ is a limit point of $C(X)$ such that $E \in 2^X \setminus C(X)$. Let $E \in \langle U_1, \cdots, U_n \rangle$ and $U = \cup_{i=1}^{n} U_i$. Since E is disconnected and closed, E is the union of two nonempty disjoint closed sets E_1 and E_2 in X. Since X is normal, there exist two disjoint nonempty open sets W_1 and W_2 in X containing E_1 and E_2 respectively such that $W_1 \cup W_2 \subset U$. Since $E \cap U_i = (E_1 \cup E_2) \cap U_i \neq \phi$ for each $i = 1, \cdots, n$, let $\{U_{i_1}^1, \cdots, U_{i_k}^1\}$ be the collection of all $U_i \in \{U_1, \cdots, U_n\}$ such that $U_i \cap E_1 \neq \emptyset$, and $\{U_{i_1}^2, \cdots, U_{i_p}^2\}$ be the collection of all $U_i \in \{U_1, \cdots, U_n\}$ such that $U_i \cap$

$E_2 \neq \emptyset$. Now let $V_j^1 = W_1 \cap U_{i_j}^1$ for $j = 1, \cdots, k$, and $V_l^2 = W_2 \cap U_{i_l}^2$ for $l = 1, \cdots, p$. Then $E_1 \subset \cup_{j=1}^k V_j^1 = V^1$ and $E_2 \subset \cup_{l=1}^p V_l^2 = V^2$ and $V^1 \cap V^2 = \emptyset$. It is easy to see that $E \in \langle V_1^1, \cdots, V_k^1, V_1^2, \cdots, V_p^2 \rangle \subset \langle U_1, \cdots, U_n \rangle$. Since E is a limit point of $C(X)$, there exists an element $C \in C(X)$ such that $C \in \langle V_1^1, \cdots, V_k^1, V_1^2, \cdots, V_p^2 \rangle$. This would mean that $C \subset V^1 \cup V^2$ and $C \cap V^i \neq \emptyset$ for each $i = 1, 2$ which contradicts the connectedness of C. So $C(X)$ is closed in 2^X. □

PROPOSITION 1.7. *X is compact Hausdorff if and only if $C(X)$ is compact Hausdorff.*

Proof. Suppose X is compact Hausdorff. Then clearly $C(X)$ is compact Hausdorff by Lemma 1.2.

Conversely, suppose $C(X)$ is compact Hausdorff. Then the closed subspace $\mathcal{F}_1(X) = \{\{x\} : x \in X\}$ of $C(X)$ is compact Hausdorff. Since X and $\mathcal{F}_1(X)$ are homeomorphic, X is compact Hausdorff. □

2. Hemicompactness and Hemiconnectedness of Hyperspaces

DEFINITION [1]. A subset E of a Hausdorff space X is called *hemicompact* if there exists a sequence $\{K_n\}_{n=1}^\infty$ of compact subsets of E such that each compact subset of E is contained in some K_n.

LEMMA. 2.1. *If Y is a closed subset of a hemicompact Hausdorff space X, then Y is also hemicompact.*

Proof. Let $\{K_n\}_{n=1}^\infty$ be a sequence of compact subsets of X such that each compact subset of X is contained in some K_m. Now let $L_n = Y \cap K_n$ for each $n = 1, 2, \cdots$. Then $\{L_n\}_{n=1}^\infty$ is a sequence of compact subsets of Y. Let L be a compact subset of Y. Then it is a compact subset of X so that there is a K_n such that $L \subset K_n$ Hence $L \subset Y \cap K_n = L_n$. □

PROPOSITION 2.2. *Let X be a Hausdorff space. Then X is hemicompact if and only if $\mathcal{K}(X)$ is hemicompact.*

Proof. Suppose X is hemicompact. Let $\{K_n\}_{n=1}^\infty$ be a sequence of compact subsets of X such that for each compact subset K of X, $K \subset K_n$ for some n. Let $\mathcal{K}_n = 2^{K_n}$ for each $n = 1, 2, \cdots$. Then $\mathcal{K}_n \subset \mathcal{K}(X)$ for each $n = 1, 2, \cdots$ and is compact by Lemma 1.1(e). So $\{\mathcal{K}_n\}_{n=1}^\infty$

is a sequence of compact subsets of $\mathcal{K}(X)$. Now let $\mathcal{K}$ be a compact subset of $\mathcal{K}(X)$. Then $\cup\mathcal{K}$ is compact by Lemma 1.1(c). Thus there is a $K_m \in \{K_n\}_{n=1}^{\infty}$ such that $\cup\mathcal{K} \subset K_m$. Hence $\mathcal{K} \subset \mathcal{K}_m$.

Suppose that $\mathcal{K}(X)$ is hemicompact. Let $\{\mathcal{K}_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of $\mathcal{K}(X)$ such that each compact subset of $\mathcal{K}(X)$ is contained in some $\mathcal{K}_n$. Let $K_n = \cup\mathcal{K}_n$ for each $n = 1, 2, \cdots$. Then by Lemma 1.1(c) each K_n is a compact subset of X. Let K be a compact subset of X. Then 2^K is compact by Lemma 1.1(e). Also it is a compact subset of $\mathcal{K}(X)$. Thus there is a $\mathcal{K}_m$ which contains 2^K. Hence $K = \cup 2^K \subset \cup\mathcal{K}_m = K_m$. □

PROPOSITION 2.3. *Let X be a Hausdorff space. Then X is hemicompact if and only if $\mathcal{F}_n(X)$ is hemicompact.*

Proof. Suppose X is hemicompact. Then $\mathcal{K}(X)$ is hemicompact by Proposition 2.2. Since $\mathcal{F}_n(X)$ is a closed subspace of $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ is hemicompact by Lemma 2.1.

Suppose $\mathcal{F}_n(X)$ is hemicompact. Let $\{\mathcal{L}_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of $\mathcal{F}_n(X)$ such that each compact subset of $\mathcal{F}_n(X)$ is contained in some $\mathcal{L}_m$. Let $L_n = \cup\mathcal{L}_n$ for each $n = 1, 2, \cdots$. Then each L_n is compact by Lemma 1.1(c). Let L be a compact subset of X. Then 2^L is compact. Hence $\mathcal{S} = 2^L \cap \mathcal{F}_n(X)$ is compact subset of $\mathcal{F}_n(X)$. Let $\mathcal{L}_k$ be an element of the sequence $\{\mathcal{L}_n\}_{n=1}^{\infty}$ such that $\mathcal{S} \subset \mathcal{L}_k$. Then $L = \cup\mathcal{S} \subset \cup\mathcal{L}_k = L_k$. □

COROLLARY 2.4. *If X is a hemicompact first countable Hausdorff space, then $\mathcal{K}(X)$ is locally compact Hausdorff.*

Any hemicompact first countable Hausdorff space is locally compact [1]. So by Proposition 1.5 the conclusion follows.

PROPOSITION 2.5. *Let X be a regular space. Then X is hemicompact if and only if $C_K(X)$ is hemicompact.*

Proof. Suppose X is hemicompact, and let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of X such that each compact subset of X is contained in some K_m. Let $C(K_n) = \{E \in C_K(X) : E \subset K_n\}$ for each n. Then each $C(K_n)$ is compact by Proposition 1.7 and is contained in $C_K(X)$. Thus $\{C(K_n)\}_{n=1}^{\infty}$ is a sequence of compact subsets of $C_K(X)$. Let $\mathcal{K}$

be a compact subset of $C_K(X)$. Since $\mathcal{K} \subset \mathcal{K}(X)$, $K = \cup\mathcal{K}$ is compact subset of X by Lemma 1.1(c). Hence there is an element K_m such that $K \subset K_m$. Therefore $\mathcal{K} \subset C(K_m)$.

Conversely, suppose that $C_K(X)$ is hemicompact, and let $\{\mathcal{K}_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of $C_K(X)$ such that each compact subset of $C_K(X)$ is contained in some $\mathcal{K}_m$. We note that each $\mathcal{K}_n$ is also a compact subset of 2^X, and is contained in $\mathcal{K}(X)$. Hence each $K_n = \cup\mathcal{K}_n$ is a compact subset of X by Lemma 1.1(c), and thus $\{K_n\}_{n=1}^{\infty}$ is a sequence of compact subsets of X. Let K be a compact subset of X. Then $C(K)$ is compact subset of $C_K(X)$. Hence there is an element $\mathcal{K}_m$ such that $C(K) \subset \mathcal{K}_m$. Let $\mathcal{K}^* = \{\{x\} : x \in K\}$. Then $K = \cup\mathcal{K}^* \subset \cup C(K) \subset \cup\mathcal{K}_m = K_m$. □

LEMMA 2.6. *Let X be a locally compact Hausdorff space. Let $\langle U_1, \cdots, U_n\rangle$ be a basic open set in 2^X and $K \in \langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$. Then there exists a finite set $\mathcal{W}_K = \{V_1, \cdots, V_p\}$ of open sets in X satisfying the following conditions:*

(1) For each $i = 1, \cdots, p$, $\overline{V}_i$ is compact and $\overline{V_i} \subset U_j$ for each $j = 1, \cdots, n$.

(2) For each $j = 1, \cdots, n$, $U_j \supset \overline{V_k}$ for some $k = 1, \cdots, p$.

(3) $K \in \mathcal{V}_K = \langle V_1, \cdots, V_p\rangle \cap K(X) \subset \langle U_1, \cdots, U_n\rangle \cap K(X)$.

Proof. For each $x \in K \cap U_i$, let V_x be an open neighborhood of x in X such that $\overline{V}_x$ is compact and $\overline{V}_x \subset U_i$. The collection of all such V_x covers the compact set K. Let $\{V_{x_1}, \cdots, V_{x_m}\}$ be a finite subcollection which covers K. Now for each $i = 1, \cdots, n$, let $y_i \in K \cap U_i$ and V_{y_i} be an open neighborhood of y_i such that $\overline{V}_{y_i} \subset U_i$. Consider the collection $\mathcal{W}_K = \{V_{y_1}, \cdots, V_{y_n}, V_{x_1}, \cdots, V_{x_m}\}$ of open sets in X. Then $\mathcal{W}_K$ satisfies the conditions (1), (2) and (3). □

PROPOSITION 2.7. *Let X be a locally compact Hausdorff space. Then each nonempty open set in X is hemicompact if and only if each basic open set in $\mathcal{K}(X)$ is hemicompact.*

Proof. Suppose each nonempty open set in X is hemicompact. Let $\langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$ be a basic open set in $\mathcal{K}(X)$. Since each U_i is hemicompact, let $\mathcal{M}_i = \{K_j^i\}_{j=1}^{\infty}$ be a sequence of compact subsets of U_i satisfying the condition that if K is a compact subset of U_i then it is contained in some K_m^i.

Let $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_n$. Then $\mathcal{M}$ is countable. For each $\mathcal{B} = (K^1, \cdots, K^n) \in \mathcal{M}$, let $f : 2^{K^1} \times \cdots \ times 2^{K^n} \to \mathcal{K}(X)$ be the restriction of the map defined in Lemma 1.1 (f). Then each $f(\mathcal{B})$ is a compact subset of $\langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$, and $\mathcal{K} = \{f(\mathcal{B}) : \mathcal{B} \in \mathcal{M}\}$ is a countable collection of compact subsets of $\langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$.

Now suppose $\mathcal{N}$ is a compact subset of $\langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$. For each $K \in \mathcal{N}$, let $\mathcal{W}_K = \{V_1, \cdots, V_p\}$ and $\mathcal{V}_K = \langle V_1, \cdots, V_p\rangle \cap \mathcal{K}(X)$ satisfying the conditions (1), (2) and (3) in Lemma 2.6. Let $\mathcal{U}$ be the collection of all $\mathcal{V}_K$, $K \in \mathcal{N}$. Then $\mathcal{U}$ covers the compact set $\mathcal{N}$. Let $\{\mathcal{V}_{K_1}, \cdots, \mathcal{V}_{K_q}\}$ be a finite subcollection of $\mathcal{U}$ which covers $\mathcal{N}$. For each i, let $\mathcal{W}_{K_i}$ be the finite set of open sets which defines $\mathcal{V}_{K_i}$ and let $\mathcal{W} = \cup_{i=1}^{q} \mathcal{W}_{K_i}$. For each i, let $\mathcal{S}_j = \{\overline{V} : V \in \mathcal{W} \text{ and } V \subset U_j\}$, $j = 1, \cdots, n$. Then, for each $j = 1, \cdots, n$, there is an element $K^j \in \mathcal{M}_j$ such that $\cup \mathcal{S}_j = \cup\{D; D \in \mathcal{S}_j\} \subset K^j$. Then $\mathcal{B} = (K^1, \cdots, K^n) \in \mathcal{M}$, and thus $f(\mathcal{B}) \in \mathcal{K}$. We show that $\mathcal{N} \subset f(\mathcal{B})$. Let $K \in \mathcal{N}$. Then $K \subset \cup_{j=1}^{p}(\cup \mathcal{S}_j) \subset \cup_{j=1}^{n} K^j$. Let $A_j = K \cap K^j$ for each $j = 1, \cdots, n$. Then $(A_1, \cdots, A_n) \in 2^{K^1} \times \cdots \times 2^{K^n}$ so that $K = \cup_{j=1}^{n} A_j \in f(\mathcal{B})$. This shows that $\mathcal{N} \subset f(\mathcal{B})$. And hence $\langle U_1, \cdots, U_n\rangle \cap \mathcal{K}(X)$ is hemicompact.

Suppose each basic open set in $\mathcal{K}(X)$ is hemicompact. Let V be a nonempty open set in X. Then $\langle V\rangle \cap \mathcal{K}(X)$ is a basic open set in $\mathcal{K}(X)$. Let $\{\mathcal{K}_n\}_{n=1}^{\infty}$ be an increasing sequence of compact subsets of $\langle V\rangle \cap \mathcal{K}(X)$ such that each compact subset of $\langle V\rangle \cap \mathcal{K}(X)$ is contained in some $\mathcal{K}_m$. Now let $K_n = \cup \mathcal{K}_n$ for each n. Then by Lemma 1.1 (c), each K_n is compact. So $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of compact subsets of V. Let K be any compact subset of V. Then 2^K is a compact subset of $\langle V\rangle \cap \mathcal{K}(X)$. Hence there exists $\mathcal{K}_m$ in the sequence $\{\mathcal{K}_n\}_{n=1}^{\infty}$ such that $2^K \subset \mathcal{K}_m$. Thus $K = \cup 2^K \subset \cup \mathcal{K}_m = K_m$. This shows that V is hemicompact. □

DEFINITION 2. A subset E of a Hausdorff space X is called *hemiconnected* if there exists an increasing sequence $\{C_n\}_{n=1}^{\infty}$ of continua in E such that each continuum in E is contained in some C_m.

It is clear that a hemiconnected space is a semi-continuum and the countable union of subcontinua, where a space X is called a *semi-continuum* if every pair of points in X can be joint by a subcontinuum of X [2]. The class of hemiconnected spaces includes compact connected Hausdorff spaces as well as Euclidean spaces.

PROPOSITION 2.8. *Let X be a regular space. Then X is hemiconnected if and only if $C_K(X)$ is regular hemiconnected.*

Proof. Suppose X is a regular hemiconnected space. Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of subcontinua of X such that each subcontinuum of X is contained in some C_m. Then $C_K(C_n)$ is a subcontinuum lying in $C_K(X)$ by Lemma 1.2 and thus $\{C_K(C_n)\}_{n=1}^{\infty}$ a sequence of subcontinua of $C_K(X)$. Let $\mathcal{K}$ be a subcontinuum of $C_K(X)$. Then $K = \cup\mathcal{K}$ is a subcontinuum of X by Lemma 1.1(c) and (d). Let C_m be an element of $\{C_n\}_{n=1}^{\infty}$ which contains K. Then $\mathcal{K} \subset C_K(C_m)$. This proves that $C_K(X)$ is hemiconnected. Since X is regular, $\mathcal{K}(X)$ is regular by [4, 4.9.10], so its subspace $C_K(X)$ is regular.

Suppose $C_K(X)$ is a regular hemiconnected space. Let $\{\mathcal{K}_n\}_{n=1}^{\infty}$ be a sequence of subcontinua of $C_K(X)$ such that each subcontinuum of $C_K(X)$ is contained in some $\mathcal{K}_m$. Let $C_n = \cup\mathcal{K}_n$ for each n. Then $\{C_n\}_{n=1}^{\infty}$ is a sequence of subcontinua of X. Let D be a subcontinuum of X. Then $C_K(D)$ is a continuum lying in $C_K(X)$ by Lemma 1.2. So there is a $\mathcal{K}_m$ which contains $C_K(D)$. Hence $D \subset \cup C_K(D) \subset \cup\mathcal{K}_m = C_m$. This proves that $C_k(X)$ is hemiconnected.
Since $C_K(X)$ is regular, its subspace $\mathcal{F}_1(X)$ is regular. So X is regular. □

PROPOSITION 2.9. *Let X be a connected, locally connected, and locally compact Hausdorff space. Then X is hemiconnected if and only if $\mathcal{K}(X)$ is hemiconnected.*

Proof. Suppose X is hemiconnected. Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of subcontinua of X such that each subcontinuum of X is contained in some C_m. Then, for each n, 2^{C_n} is a subcontinuum in $\mathcal{K}(X)$ and thus $\{2^{C_n}\}_{n=1}^{\infty}$ is a sequence of subcontinua of $\mathcal{K}(X)$. Let $\mathcal{K}$ be a subcontinuum of $\mathcal{K}(X)$. Then, since X is connected, locally connected, and locally compact Hausdorff and $\cup\mathcal{K}$ is a compact subset of X, by Lemma 1.4 there is a subcontinuum M of X which contains $\cup\mathcal{K}$. So let C_m be an element of the sequence $\{C_n\}_{n=1}^{\infty}$ such that $M \subset C_m$. Then it is clear that $\mathcal{K} \subset 2^M \subset 2^{C_m}$. Hence $\mathcal{K}(X)$ is hemiconnected.

Suppose that $\mathcal{K}(X)$ is hemiconnected. Let $\{\mathcal{K}_n\}_{n=1}^{\infty}$ be a sequence of subcontinua of $\mathcal{K}(X)$ such that each subcontinuum of $\mathcal{K}(X)$ is contained in some $\mathcal{K}_m$. Inductively, we define a sequence $\{M_n\}_{n=1}^{\infty}$ of subcontinua of X as follows: Let M_1 be a subcontinuum of X containing $\cup\mathcal{K}_1$. Suppose that, for $k > 1$, M_k has been defined. Let M_{k+1} be a subcontinuum of X containing the compact set $M_k \cup (\cup \mathcal{K}_{k+1})$ which is provided by Lemma 1.4. Let K be a subcontinuum of X. Then 2^K is a subcontinuum of $\mathcal{K}(X)$. Thus there exists an element $\mathcal{K}_m$ of the sequence $\{\mathcal{K}_n\}_{n=1}^{\infty}$ such that $2^K \subset \mathcal{K}_m$. Hence $K \subset \cup \mathcal{K}_m \subset M_m$. Therefore X is hemiconnected. □

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