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STRONG AND WEAK CONVERGENCE OF THE ISHIKAWA ITERATION METHOD FOR A CLASS OF NONLINEAR EQUATIONS

M. O. OSILIKE

ABSTRACT. Let E be a real q-uniformly smooth Banach space which admits a weakly sequentially continuous duality map, and K a nonempty closed convex subset of E. Let $T : K \to K$ be a mapping such that $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and $(I - T)$ satisfies the accretive-type condition:

$$\langle x - Tx, j(x - x^*)\rangle \geq \lambda ||x - Tx||^2,$$

for all $x \in K$, $x^* \in F(T)$ and for some $\lambda > 0$. The weak and strong convergence of the Ishikawa iteration method to a fixed point of T are investigated. An application of our results to the approximation of a solution of a certain linear operator equation is also given. Our results extend several important known results from the Mann iteration method to the Ishikawa iteration method. In particular, our results resolve in the affirmative an open problem posed by Naimpally and Singh (J. Math. Anal. Appl. 96 (1983), 437-446).

1. Introduction

Let H be a real Hilbert space. A mapping $T : D(T) \subseteq H \to H$ is said to satisfy condition (A) in the terminology of Maruster [13] if $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and for all $x \in D(T)$ and $x^* \in F(T)$, there exists $\lambda > 0$ such that

$$\langle x - Tx, x - x^*\rangle \geq \lambda ||x - Tx||^2. \tag{1}$$

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Regular Associate of The Abdus Salam ICTP, Trieste, Italy. Research supported by a grant from TWAS (94-224 RG/MATHS/AF/AC).

The class of Mappings satisfying condition (A) has been studied by several authors (see for example [1-4],[8],[13],[16]). Observe that if T satisfies (1), then

$$(2) \quad \begin{aligned} ||Tx - x^*||^2 &= ||x - x^* - (x - Tx)||^2 \\ &= ||x - x^*||^2 - 2\langle x - Tx, x - x^*\rangle + ||x - Tx||^2 \\ &\leq ||x - x^*||^2 + (1 - 2\lambda)||x - Tx||^2. \end{aligned}$$

Thus the class of mappings satisfying condition (A) includes the important class of *quasi-nonexpansive* mappings (i.e., mappings $T : D(T) \subseteq H \to H$ such that $F(T) \neq \emptyset$ and $||Tx - x^*|| \leq ||x - x^*||$, $\forall\, x \in D(T), x^* \in F(T)$). If $F(T) \neq \emptyset$, then the class of mappings satisfying condition (A) also includes the class of mappings studied by Goëbel *et al* [6] since these mappings are shown to be quasi-nonexpansive in [17]. If the mappings studied by Kannan [10] and Wong [21] have nonempty fixed-point sets, then they are also contained in the class of mappings satisfying condition (A).

In [8] Hicks and Kubicek studied independently the class of mappings which they called *Demicontractive*. They called a mapping $T : D(T) \subseteq H \to H$ demicontractive if $F(T) \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$ there exists $k \in [0, 1)$ such that

$$(3) \qquad ||Tx - x^*||^2 \leq ||x - x^*||^2 + k||x - Tx||^2.$$

Observe that if T satisfies (3), then

$$(4) \quad \begin{aligned} ||Tx - x^*||^2 &= ||x - x^*||^2 - 2\langle x - Tx, x - x^*\rangle + ||x - Tx||^2 \\ &\leq ||x - x^*||^2 + k||x - Tx||^2. \end{aligned}$$

It follows from (4) that

$$(5) \qquad \langle x - Tx, x - x^*\rangle \geq \frac{1}{2}(1 - k)||x - Tx||^2,$$

so that T satisfies condition (A). In view of (2) and (5), the class of mappings satisfying condition (A) introduced by Maruster [13] in 1977 coincide with the class of demicontractive mappings introduced independently in the same year by Hicks and Kubicek [8].

Let K be a nonempty closed convex subset of H and $T : K \to K$ a mapping satisfying condition (A). In [13] Maruster studied the convergence (to fixed points of T) of the *Mann iteration method* [12] generated from an arbitrary $x_0 \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \; n \geq 0, \tag{6}$$

where $\{\alpha_n\}$ is a suitable sequence in $[0,1]$. He obtained weak and strong convergence results under the basic assumption that $(I-T)$ is *demiclosed* at zero in K, where a mapping T is said to be demiclosed at a point p if weak convergence of any sequence $\{x_n\}$ to a point x and the strong convergence of $\{Tx_n\}$ to p implies that $Tx = p$. His results generalize a known result of Dotson [5] which in turn is a generalization of a result of Schaefer [19]. Some convergence results similar to the results obtained by Maruster were also obtained independently by Hicks and Kubicek [8]. In ([2],[3]) Chidume extended the definition of mappings satisfying condition (A) to arbitrary real Banach spaces. Let E be a real Banach space, and let J_q, $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q \text{ and } ||f|| = ||x||^{q-1}\},$$

where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see for example [23]) that $J_q(x) = ||x||^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single-valued. In the sequel we shall denote single-valued generalized duality mapping by j_q.

A mapping $T : D(T) \subseteq E \to E$ is said to satisfy condition (A) (see for example [2],[3],[16]) if $F(T) \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$, $j(x-x^*) \in J(x-x^*)$, there exists a constant $\lambda > 0$ such that

$$\langle x - Tx, j(x-x^*)\rangle \geq \lambda ||x - Tx||^2. \tag{7}$$

In Hilbert spaces j is the identity so that (7) reduces to (1). In [2] Chidume extended the results of Maruster [13] to real Banach spaces which are 2-*uniformly smooth* (see definition below) and which admit weakly sequentially continuous duality map, and in [3] he further extended the results to real Banach spaces which admit weakly sequentially continuous duality map and are $(m+1)$-*uniformly smooth*, m a positive integer. Let $q \in (1,\infty)$. In [16] the author extended the results of Chidume ([2],[3]) to all real *q-uniformly smooth* Banach spaces which admit weakly sequentially continuous duality map.

It is not known, even in Hilbert spaces (see for example an open problem posed by Naimpally and Singh ([14], p. 445) for the class of demicontractive maps) whether these convergence results of the Mann iterates for mappings satisfying condition (A) can be extended to the *Ishikawa iteration method* [9] generated for $x_0 \in K$, K nonempty subset of E, $T : K \to K$ by

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \ n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0, 1]$.

It is our purpose in this paper to extend the results of Maruster [13], Chidume ([2],[3]), the author [16] and the corresponding results of Hicks and Kubicek [8] from the Mann iteration method to the Ishikawa iteration method. Our $\{\alpha_n\}$ and $\{\beta_n\}$ will be chosen in such a way that these results will be special cases of our results. Moreover, our results resolve in the affirmative an open problem posed by Naimpally and Singh ([14], p. 445).

2. Preliminaries

Let E be a real Banach space. The *modulus of smoothness* of E is the function

$$\rho_E : [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq \tau\right\}.$$

E is *uniformly smooth* if and only if $\lim\limits_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. Hilbert spaces, L_p(or ℓ_p) spaces, $1 < p < \infty$ and the Sobolev spaces, W^p_m, $1 < p < \infty$ are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p(or\ \ell_p) \text{ or } W^p_m \text{ is } \begin{cases} p - \text{uniformly smooth if } 1 < p \leq 2 \\ 2 - \text{uniformly smooth if } p \geq 2 \end{cases}$$

THEOREM HKX ([23], p. 1130). *Let $q > 1$ and let E be a real Banach space. Then the following are equivalent:*

1. *E is q-uniformly smooth.*
2. *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + c_q||y||^q. \tag{8}$$

3. *There exists a constant d_q such that for all $x, y \in E$, and $t \in [0,1]$*

$$||(1-t)x+ty||^q \geq (1-t)||x||^q + t||y||^q - \omega_q(t)d_q||x-y||^q, \tag{9}$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Furthermore, it is proved in [24] (see Remark 5, p. 208) that if E is q-uniformly smooth ($q > 1$), then for all $x, y \in E$ there exists a constant $L_* > 0$ such that

$$||j_q(x) - j_q(y)|| \leq L_*||x-y||^{q-1}. \tag{10}$$

In the sequel we shall also need the following result:

LEMMA TX. ([20], p. 303) *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n, \ n \geq 1.$$

Then $\lim a_n$ exists.

3. Main Results

For the rest of this paper, E will denote a real q-uniformly smooth Banach space ($q > 1$), which admits a weakly sequentially continuous duality map. λ is the constant appearing in the definition of mappings satisfying condition (A) and c_q, d_q, $\omega_q(t)$ and L_* are the constants appearing in inequalities (8)-(10). We are now proving the following:

LEMMA 1. *Let K be a nonempty convex subset of E and $T : K \to K$ a map satisfying condition (A). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions:*

(i) $0 \leq \alpha_n, \beta_n \leq 1, \ n \geq 0$

(ii) $0 < a \leq \alpha_n^{q-1} \leq b < \frac{q\lambda^{q-1}}{c_q}(1-\beta_n), \ n \geq 0$

(iii) $\sum_{n=0}^{\infty} \beta_n^{\tau} < \infty$, *where* $\tau := \min\{1, (q-1)\}$.
Let $\{x_n\}$ *be the sequence generated from an arbitrary* $x_0 \in K$ *by*

$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \; n \geq 0 \tag{11}$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \; n \geq 0. \tag{12}$$

Then $\lim ||y_n - Ty_n|| = 0$.

Proof. Let $x^* \in F(T)$. Then for all $x \in K$ it follows from condition (A) that

$$||x - x^*|| \geq \lambda ||x - Tx|| \geq \lambda ||Tx - x^*|| - \lambda ||x - x^*||,$$

so that

$$||Tx - x^*|| \leq \frac{(1+\lambda)}{\lambda} ||x - x^*|| = L||x - x^*||, \tag{13}$$

where $L = \frac{(1+\lambda)}{\lambda}$. Since $||x - x^*|| \geq \lambda ||x - Tx||$, we have

$$\begin{aligned} \langle x - Tx, j_q(x - x^*)\rangle &= ||x - x^*||^{q-2} \langle x - Tx, j(x - x^*)\rangle \\ &\geq ||x - x^*||^{q-2} \lambda ||x - Tx||^2 \\ &\geq \lambda^{q-1} ||x - Tx||^q. \end{aligned} \tag{14}$$

Using inequalities (8) and (14) we obtain

$$\begin{aligned} & ||x_{n+1} - x^*||^q \\ &= ||x_n - x^* + \alpha_n(Ty_n - x_n)||^q \\ &\leq ||x_n - x^*||^q - q\alpha_n \langle x_n - Ty_n, j_q(x_n - x^*)\rangle + c_q \alpha_n^q ||x_n - Ty_n||^q \\ &= ||x_n - x^*||^q - q\alpha_n \langle x_n - y_n, j_q(x_n - x^*)\rangle \\ &\quad - q\alpha_n \langle y_n - Ty_n, j_q(x_n - x^*)\rangle + c_q \alpha_n^q ||x_n - Ty_n||^q \\ &= ||x_n - x^*||^q - q\alpha_n \beta_n \langle x_n - Tx_n, j_q(x_n - x^*)\rangle \\ &\quad - q\alpha_n \langle y_n - Ty_n, j_q(x_n - x^*)\rangle + c_q \alpha_n^q ||x_n - Ty_n||^q \\ &\leq ||x_n - x^*||^q - q\alpha_n \beta_n \lambda^{q-1} ||x_n - Tx_n||^q \\ &\quad - q\alpha_n \langle y_n - Ty_n, j_q(x_n - x^*)\rangle + c_q \alpha_n^q ||x_n - Ty_n||^q \end{aligned} \tag{15}$$

Using inequalities (9), and (14) we obtain

$$
\begin{aligned}
(16)\quad & \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\
&= \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle + \langle y_n - Ty_n, j_q(y_n - x^*)\rangle \\
&\geq \lambda^{q-1}||y_n - Ty_n||^q + \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\
&= \lambda^{q-1}||(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)||^q \\
&\quad + \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\
&\geq \lambda^{q-1}(1-\beta_n)||x_n - Ty_n||^q + \lambda^{q-1}\beta_n||Tx_n - Ty_n||^q \\
&\quad - \lambda^{q-1}\omega_q(\beta_n)d_q||x_n - Tx_n||^q \\
&\quad + \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle
\end{aligned}
$$

Using (16) in (15) we obtain

$$
\begin{aligned}
(17)\qquad & ||x_{n+1} - x^*||^q \\
&\leq ||x_n - x^*||^q - q\alpha_n\beta_n\lambda^{q-1}||x_n - Tx_n||^q \\
&\quad - q\alpha_n\Big\{\lambda^{q-1}(1-\beta_n)||x_n - Ty_n||^q + \lambda^{q-1}\beta_n||Tx_n - Ty_n||^q \\
&\quad - \lambda^{q-1}\omega_q(\beta_n)d_q||x_n - Tx_n||^q + \langle y_n - Ty_n, j_q(x_n - x^*) \\
&\quad - j_q(y_n - x^*)\rangle\Big\} + c_q\alpha_n^q||x_n - Ty_n||^q \\
&\leq ||x_n - x^*||^q - \alpha_n\big[q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1}\big]||x_n - Ty_n||^q \\
&\quad + q\lambda^{q-1}\alpha_n\omega_q(\beta_n)d_q||x_n - Tx_n||^q \\
&\quad - q\alpha_n\langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\
&\leq ||x_n - x^*||^q - \alpha_n\big[q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1}\big]||x_n - Ty_n||^q \\
&\quad + q\lambda^{q-1}\alpha_n\omega_q(\beta_n)d_q||x_n - Tx_n||^q \\
&\quad + q\alpha_n||y_n - Ty_n||||j_q(x_n - x^*) - j_q(y_n - x^*)|| \\
&\leq ||x_n - x^*||^q - \alpha_n\big[q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1}\big]||x_n - Ty_n||^q \\
&\quad + q\lambda^{q-1}\alpha_n\omega_q(\beta_n)d_q||x_n - Tx_n||^q \\
&\quad + q\alpha_n||y_n - Ty_n||L_*\beta_n^{q-1}||x_n - Tx_n||^{q-1} \text{ (using (10))}.
\end{aligned}
$$

Using (13) we obtain

$$
(18)\qquad ||x_n - Tx_n||^q \leq (1+L)^q||x_n - x^*||^q,
$$

and

$$(19)\quad \begin{aligned} ||y_n - Ty_n|| &\leq (1+L)||y_n - x^*|| \\ &\leq (1+L)\big[(1-\beta_n)||x_n - x^*|| + \beta_n||Tx_n - x^*||\big] \\ &\leq (1+L)^2||x_n - x^*||. \end{aligned}$$

Observe that

$$(20)\qquad \omega_q(\beta_n) = \beta_n^q(1-\beta_n) + \beta_n(1-\beta_n)^q \leq 2\beta_n.$$

Using (18)-(20) and condition (i) in (17) we obtain

$$(21)\quad \begin{aligned} &||x_{n+1} - x^*||^q \\ &\leq ||x_n - x^*||^q - \alpha_n\Big[q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1}\Big]||x_n - Ty_n||^q \\ &\quad + \Big[2q\lambda^{q-1}d_q(1+L)^q\beta_n + q(1+L)^{q+1}L_*\beta_n^{q-1}\Big]||x_n - x^*||^q \\ &= [1+\sigma_n]||x_n - x^*||^q - \alpha_n\Big[q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1}\Big]||x_n - Ty_n||^q, \end{aligned}$$

where $\sigma_n = 2q\lambda^{q-1}d_q(1+L)^q\beta_n + q(1+L)^{q+1}L_*\beta_n^{q-1}$. Condition (ii) implies that

$$q\lambda^{q-1}(1-\beta_n) - c_q\alpha_n^{q-1} \geq [q\lambda^{q-1}(1-\beta_n) - c_qb] > 0,\ \forall\, n \geq 0.$$

Hence (21) reduces to

$$(22)\qquad ||x_{n+1} - x^*||^q \leq [1+\sigma_n]||x_n - x^*||^q,\ \forall\, n \geq 0.$$

If follows from condition (iii) that $\sum_{n=0}^{\infty}\sigma_n < \infty$ and hence (22) implies that $\{||x_n - x^*||\}$ is bounded. Let $||x_n - x^*|| \leq M,\ \forall\ n \geq 0$, then it follows from (21) that

(23)

$$||x_{n+1} - x^*||^q \leq ||x_n - x^*||^q - \alpha_n\Big[q\lambda^{q-1}(1-\beta_n) - c_qb\Big]||x_n - Ty_n||^q + M^q\sigma_n.$$

Since $\lim[q\lambda^{q-1}(1-\beta_n) - c_qb] = q\lambda^{q-1} - c_qb > 0$, there exists a positive integer N_0 such that $q\lambda^{q-1}(1-\beta_n) - c_qb \geq \frac{1}{2}[q\lambda^{q-1} - c_qb] > 0,\ \forall\ n \geq N_0$. Hence it follows from (23) that

$$\frac{a^{\frac{1}{q-1}}}{2}[q\lambda^{q-1} - c_qb]||x_n - Ty_n||^q \leq ||x_n - x^*||^q - ||x_{n+1} - x^*||^q + M^q\sigma_n,\ \forall n \geq N_0,$$

so that

$$\begin{aligned}\frac{a^{\frac{1}{q-1}}}{2}[q\lambda^{q-1} - c_q b] \sum_{j=N_0}^{n} ||x_j - Ty_j||^q &\leq ||x_{N_0} - x^*||^q + M^q \sum_{j=N_0}^{n} \sigma_j \\ &\leq ||x_{N_0} - x^*||^q + M^q \sum_{j=0}^{\infty} \sigma_j < \infty.\end{aligned}$$

Hence $\sum_{n=0}^{\infty} ||x_n - Ty_n||^q < \infty$, and this implies $\lim ||x_n - Ty_n|| = 0$. Since
$||x_n - Tx_n|| \leq (1 + L)||x_n - x^*|| \leq M(1 + L), \ \forall\, n \geq 0$ and

$$0 \leq ||y_n - Ty_n|| \leq ||y_n - x_n|| + ||x_n - Ty_n|| \leq \beta_n(1+L)M + ||x_n - Ty_n|| \to 0 \quad \text{as } n \to \infty,$$

we have $\lim ||y_n - Ty_n|| = 0$, completing the proof of Lemma 1. □

THEOREM 1. *Let K be a nonempty closed convex subset of E and let $T : K \to K$ be a mapping satisfying condition (A) such that: $(*)$ if any sequence $\{x_n\}$ converges weakly to x and $\{(I - T)(x_n)\}$ converges strongly to 0, then $(I - T)(x) = 0$ (i.e., $(I - T)$ is demiclosed at zero). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in* Lemma 1. *Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by (11) and (12) converges weakly to a fixed point of T.*

Proof. It follows from Lemma 1 that $\{x_n\}$ is bounded. Hence it has a weakly convergent subsequence $\{x_{n_j}\}$. Suppose $\{x_{n_j}\}$ converges weakly to some p. Then since $\{x_{n_j}\}$ is in K and K is weakly closed, we have that $p \in K$. Since $||x_n - x^*|| \leq M, \ \forall\, n \geq 0, \ x^* \in F(T)$, then $||Tx_n - x^*|| \leq L||x_n - x^*|| \leq ML, \ \forall\, n \geq 0$ so that $\{Tx_n\}$ is bounded. Let f be an arbitrary element of E^*. Then

$$f(y_{n_j}) = (1 - \beta_{n_j})f(x_{n_j}) + \beta_{n_j} f(Tx_{n_j})$$

and

$$\begin{aligned}0 \leq |f(y_{n_j}) - f(p)| &\leq (1 - \beta_{n_j})|f(x_{n_j}) - f(p)| + \beta_{n_j}|f(Tx_{n_j}) - f(p)| \\ &\leq |f(x_{n_j}) - f(p)| + \beta_{n_j}||f||\,||Tx_{n_j} - p|| \to 0 \\ &\quad \text{as } j \to \infty.\end{aligned}$$

Hence $\{y_{n_j}\}$ converges weakly to p, and from Lemma 1 $\{(I - T)(y_{n_j})\}$ converges strongly to 0. It now follows from condition (*) that $(I -$

$T)(p) = 0$, i.e., p is a fixed point of T. Suppose $\{x_n\}$ does not converge weakly to p. Then $\{x_n\}$ has at least one other cluster point $q \neq p$. Suppose $\{x_{m_i}\}$ converges weakly to q. Then as for p, $Tq = q$. From (23) we have

$$||x_{n+1} - p||^q \leq ||x_n - p||^q + M^q\sigma_n, \ \forall n \geq 0,$$

and

$$||x_{n+1} - q||^q \leq ||x_n - q||^q + M^q\sigma_n, \ \forall n \geq 0.$$

Thus it follows from Lemma TX that both $\lim ||x_n - p||$ and $\lim ||x_n - q||$ exist. Since E admits a weakly sequentially continuous duality map, it follows from Theorem 1 of [7] that if a sequence $\{x_n\}$ converges weakly in E to x_0, then

$$\liminf ||x_n - x|| > \liminf ||x_n - x_0||, \ \forall\, x \neq x_0.$$

Hence

$$\begin{aligned} \lim_n ||x_n - p|| &= \liminf_j ||x_{n_j} - p|| < \liminf_j ||x_{n_j} - q|| \\ &= \liminf_i ||x_{m_i} - q|| < \liminf_i ||x_{m_i} - p|| \\ &= \lim_n ||x_n - p||, \end{aligned}$$

a contradiction. Thus $\{x_n\}$ converges weakly to $p \in F(T)$, completing the proof of Theorem 1. □

REMARK 1. Let E,K,T,$\{\alpha_n\}$,$\{\beta_n\}$ and $\{x_n\}$ be as in Theorem 1. Suppose $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges strongly to some point p, and T is continuous at p. Then

$$\begin{aligned} 0 &\leq ||y_{n_j} - p|| \leq (1 - \beta_{n_j})||x_{n_j} - p|| + \beta_{n_j}||Tx_{n_j} - p|| \\ &\leq ||x_{n_j} - p|| + \beta_{n_j}||Tx_{n_j} - p|| \to 0 \text{ as } j \to \infty, \end{aligned}$$

so that $y_{n_j} \to p$ as $j \to \infty$. since T is continuous at p, then $Ty_{n_j} \to Tp$ as $j \to \infty$. Hence $\lim_j ||y_{n_j} - Ty_{n_j}|| = ||p - Tp|| = 0$, so that $p \in F(T)$. From (23) we obtain

$$||x_{n+1} - p||^q \leq ||x_n - p||^q + M^q\sigma_n, \ n \geq 0.$$

It follows from Lemma TX that $\lim ||x_n - p||$ exists, and since $\lim_j ||x_{n_j} - p|| = 0$, we have $\lim ||x_n - p|| = 0$. Thus we can conclude that if T is continuous, then either $\{x_n\}$ converges strongly to a fixed point of T or $\{x_n\}$ *has no subsequence which converges strongly.*

REMARK 2. If we set $\beta_n = 0$, $\forall\ n \geq 0$ in Corollary 1, we obtain Theorem 1 of the author [16]. Furthermore since spaces satisfying condition $(U, \alpha, m+1, m)$ are $(m+1)$-uniformly smooth and satisfy (8) with $c_q = \frac{\alpha}{2^m - 1}$, if we set $q = (m+1)$, $c_q = \frac{\alpha}{2^m-1}$ and $\beta_n = 0$, $\forall n \geq 0$ in Corollary 1 we obtain Theorem 1 of Chidume [3]. Hilbert spaces are 2-uniformly smooth and satisfy inequality (8) with $c_q = 1$. If we set $q = 2$, $c_q = 1$ and $\beta_n = 0$, $\forall\ n \geq 0$ in Corollary 1 we obtain Theorem 1 of Maruster [13]. Furthermore, if we set $q = 2$, $c_q = 1$ and $\beta_n = 0$, $\forall\ n \geq 0$ in Theorem 1, we obtain Theorem 2 of Hicks and Kubicek [8]. Theorem 1 resolves in the affirmative an open problem posed by Naimpally and Singh [14] on whether the convergence theorem of Hicks and Kubicek [8] for the Mann iteration method for demicontractive maps can be extended to the Ishikawa iteration method.
From the point of view of applications it is interesting to obtain additional conditions such that the sequence $\{x_n\}$ converges strongly to a fixed point of T. We consider the condition considered by Maruster [13], Chidume ([2],[3]) and the author [16] and apply it to the approximation of solutions of certain operator equations.

THEOREM 2. *Let E, T, and K be as in* Theorem 1 *and let $\{\alpha_n\}$, $\{\beta_n\}$ be as in* Lemma 1. *If in addition there is $h \in K$, $h \neq 0$ such that $\langle x - Tx, j(h)\rangle \leq 0$ for all $x \in K$. Then for a suitable $x_0 \in K$, the sequence $\{x_n\}$ generated from x_0 by* (11) *and* (12) *converges strongly to a fixed point of T.*

Proof. By Theorem 1, $\{x_n\}$ converges weakly to some $x^* \in F(T)$. Choose x_0 such that $\langle x_0, j(h)\rangle > \langle x^*, j(h)\rangle$. Thus $\langle x_0 - x^*, j(h)\rangle > 0$. Observe that

$$
\begin{aligned}
(24)\quad & \langle x_{n+1} - x^*, j(h)\rangle \\
&= \langle x_n - x^*, j(h)\rangle - \alpha_n\langle x_n - Ty_n, j(h)\rangle \\
&= \langle x_n - x^*, j(h)\rangle - \alpha_n\langle x_n - y_n, j(h)\rangle - \alpha_n\langle y_n - Ty_n, j(h)\rangle \\
&= \langle x_n - x^*, j(h)\rangle - \alpha_n\beta_n\langle x_n - Tx_n, j(h)\rangle - \alpha_n\langle y_n - Ty_n, j(h)\rangle \\
&\geq \langle x_n - x^*, j(h)\rangle \geq \ldots \geq \langle x_0 - x^*, j(h)\rangle > 0.
\end{aligned}
$$

Since $\{x_n\}$ converges weakly to x^*, we have $\lim\limits_n\langle x_n - x^*, j(h)\rangle = 0$, so that given any $\epsilon > 0$ there exists a positive integer N_1 such that

$$|\langle x_n - x^*, j(h)\rangle| \leq \epsilon^q,\ \forall n \geq N_1.$$

Furthermore, $\sum_{n=0}^{\infty} \sigma_n < \infty$ implies that there exists a positive integer N_2 such that

$$\sum_{j=0}^{\infty} \sigma_{n+j} < \frac{\epsilon^q}{M^q},\ \forall\, n \geq N_2.$$

Let $N = \max\{N_1, N_2\}$. Then $\langle x_N - x^*, j(h)\rangle > 0$ and hence there exists $\epsilon_0 > 0$ such that

$$\langle x_N - x^*, j(h)\rangle \geq \epsilon_0 ||x_N - x^*||^q.$$

We prove that

$$\langle x_{N+p} - x^*, j(h)\rangle \geq \epsilon_0 ||x_{N+p} - x^*||^q - \epsilon_0 M^q \sum_{j=0}^{p-1} \sigma_{N+j},\ \forall \text{ integers } p \geq 1.$$

For $p = 1$ we have

$$\begin{aligned} \langle x_{N+1} - x^*, j(h)\rangle &\geq \langle x_N - x^*, j(h)\rangle \\ &\geq \epsilon_0 ||x_N - x^*||^q \\ &\geq \epsilon_0 ||x_{N+1} - x^*||^q - \epsilon_0 M^q \sigma_N \text{ (using (23))} \end{aligned}$$

so that the result holds for $p = 1$. Assume now that

$$\langle x_{N+p_0} - x^*, j(h)\rangle \geq \epsilon_0 ||x_{N+p_0} - x^*||^q - \epsilon_0 M^q \sum_{j=0}^{p_0-1} \sigma_{N+j}$$

for some integer $p_0 > 1$. Then

$$\begin{aligned} \langle x_{N+(p_0+1)} - x^*, j(h)\rangle &\geq \langle x_{N+p_0} - x^*, j(h)\rangle \\ &\geq \epsilon_0 ||x_{N+p_0} - x^*||^q - \epsilon_0 M^q \sum_{j=0}^{p_0-1} \sigma_{N+j} \\ &\geq \epsilon_0 ||x_{N+(p_0+1)} - x^*||^q - \epsilon_0 M^q \sigma_{N+p_0} - \epsilon_0 \sum_{j=0}^{p_0-1} \sigma_{N+j} \\ &= \epsilon_0 ||x_{N+(p_0+1)} - x^*||^q - \epsilon_0 M^q \sum_{j=0}^{p_0} \sigma_{N+j}. \end{aligned}$$

Hence

$$||x_{N+p} - x^*||^q \leq \frac{1}{\epsilon_0}\Big[\langle x_{N+p} - x^*, j(h)\rangle + \epsilon_0 M^q \sum_{j=0}^{p-1} \sigma_{N+j}\Big] < \frac{(1+\epsilon_0)}{\epsilon_0}\epsilon^q,$$

$$\forall \text{ intergers } p \geq 1.$$

Hence $||x_{N+p} - x^*|| \leq \left(\frac{1+\epsilon_0}{\epsilon_0}\right)^{-q}\epsilon$, for all integers $p \geq 1$. This implies $\lim_n ||x_n - x^*|| = 0$, completing the proof of Theorem 2. □

REMARK 3. It follows as in Remark 2 that Theorem 2 is a generalization and extension of Theorem 2 of ([2],[3],[13],[16]).

Application to Linear Equations. As an application of Theorem 2 we obtain the following convergence theorem for the iterative approximation of solutions of certain linear operator equation in E.

THEOREM 3. *Let $A : E \to E$ be a continuous linear map and let $f \in A(E)$. Suppose zero is an eigenvalue of A and that the following condition is satisfied:*

$$\langle Ay, j(y)\rangle \geq \lambda ||Ay||^2, \tag{24}$$

for all $y \in E$ and for some $\lambda > 0$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be as in Lemma 1. *Then the sequence $\{x_n\}$ generated from a suitable $x_0 \in E$ by*

$$y_n = (1-\beta_n)x_n + \beta_n(f + x_n - Ax_n),\ n \geq 0$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + y_n - Ay_n),\ n \geq 0$$

converges strongly to a solution of the equation $Ax = f$.

Proof. Apply Theorem 2 with $K = E$, and $T : E \to E$ defined by $Tx = f + x - Ax$. Since $f \in A(E)$, there exists $p \in E$ such that $Ap = f$. Clearly $p \in F(T)$, so that $F(T) \neq \emptyset$. Furthermore, $x - Tx = Ax - f = Ax - Ap = A(x-p)$. If we set $y = x - p$ in (24) we obtain

$$\langle x - Tx, j(x-p)\rangle \geq \lambda ||x - Tx||^2.$$

Suppose $\{x_n\}$ is any sequence which converges weakly to x^*, and $\{x_n - Tx_n\} = \{Ax_n - f\}$ converges strongly to zero, then

$$||Ax^* - f||^2 = \lim_n \langle Ax_n - f, j(Ax^* - f)\rangle = 0,$$

(since A is weakly continuous). Thus $Ax^* - f = (I - T)(x^*) = 0$, which establishes that $(I - T)$ is demiclosed at zero. Let A^* denote the adjoint of A. Then zero is an eigenvalue of A^* (since zero is an eigenvalue of A). Thus there exists $j(h) \in E^*$, $j(h) \neq 0$ such that $A^*(j(h)) = 0$. Observe that for all $x \in E$ we have

$$\langle x - Tx, j(h)\rangle = \langle Ax - f, j(h)\rangle = \langle A(x-p), j(h)\rangle = \langle x - p, A^*(j(h))\rangle = 0.$$

An application of Theorem 2 now completes the proof of Theorem 3. □

REMARK 4. J. B. Diaz and F. T. Metcalf [4] proved that in Hilbert spaces H, if $A : H \to H$ is compact, semipositive (i.e., $\langle Ax, x\rangle \geq 0$, $\forall\, x \in H$) and selfadjoint, then (24) is satisfied with $\lambda = \frac{1}{\lambda_1}$, where λ_1 is the largest eigenvalue of A. For other applications of our result, the reader may consult [13].

REMARK 5. The fact that our choices of $\{\alpha_n\}$ and $\{\beta_n\}$ allow us to obtain all the results expected when $\lim ||x_n - Tx_n|| = 0$ from $\lim ||y_n - Ty_n|| = 0$ is of independent interest. Suitable choices of our real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are
$\alpha_n = q^{-(\frac{1}{q-1})}\left[\frac{q\lambda^{q-1}}{4c_q}\left(1 - \frac{1}{2(n+1)^{\frac{2}{\tau}}}\right)\right]^{\frac{1}{q-1}}, \quad \beta_n = \frac{1}{2(n+1)^{\frac{2}{\tau}}}, \quad n \geq 0,$
where $\tau = \min\{1, (q-1)\}$.

REMARK 6. Recently, some authors have studied the so called Mann and Ishikawa iteration methods with errors (see for example [11],[22]). In a recent paper, Xu [22] introduced the following:

(a) Ishikawa Iteration Method with Errors. Let K be a convex subset of a Banach space and $T : K \to K$ a given mapping. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be real sequences in $[0, 1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$. The sequence $\{x_n\}$ is generated from an arbitrary $x_0 \in K$ by

$$y_n = a_n x_n + b_n T x_n + c_n u_n,\ n \geq 0 \tag{25}$$

$$x_{n+1} = a'_n x_n + b'_n T y_n + c'_n v_n,\ n \geq 0. \tag{26}$$

The Mann Iteration Method With Errors. For K, T and x_0 as in (a) the Mann iteration method with errors $\{x_n\}$ given by

$$x_{n+1} = a'_n x_n + b'_n T y_n + c'_n v_n, \; n \geq 0$$

is a special case of the Ishikawa iteration method with errors for which $a_n = 1$, $b_n = c_n = 0$. In [22] Xu showed that his Mann and Ishikawa iteration methods with errors are better than the earlier Mann and Ishikawa iteration methods introduced by Liu [11].
Observe that if we set $\beta_n = b_n + c_n$, $\alpha_n = b'_n + c'_n$, then (25) and (26) become

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n + c_n(u_n - T x_n), \; n \geq 0 \tag{27}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + c'_n(v_n - T y_n), \; n \geq 0. \tag{28}$$

Once Convergence results have been proved for the original Mann and Ishikawa iteration methods, the extension of the results to these iteration methods with errors is usually straightforward when the necessary conditions have been imposed on the error terms.
For the above iteration methods with errors we have the following results whose proof are omitted because they follow by straightforward modifications of the proofs of the corresponding results for the original Mann and Ishikawa iteration methods.

LEMMA 2. *Let K be a nonempty convex subset of E. Let $T : K \to K$ be a mapping satisfying condition (A) and let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}$ and $\{c'_n\}$ be sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ such that*
(i) $0 < a \leq (b'_n + c'_n)^{q-1} \leq b < \frac{q\lambda^{q-1}}{c_q}\left[1 - (b_n + c_n)\right], \; n \geq 0$
(ii) $\sum_{n=0}^{\infty}(b_n + c_n)^{\tau} < \infty$, $\sum_{n=0}^{\infty} c'_n < \infty$, *where* $\tau = \min\{1, (q-1)\}$.
Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K and let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in K$ by (25) and (26). Then $\lim ||y_n - T y_n|| = 0$.

THEOREM 3. *Let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a mapping satisfying conditions (A) and $(*)$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{u_n\}, \{v_n\}$ be as in Lemma 2. Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by (25) and (26) converges weakly to a fixed point of T.*

THEOREM 4. *Let E, K and T be as in* Theorem 3 *and let* $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{u_n\}, \{v_n\}$ *be as in* Lemma 2. *If in addition there is $h \in K$, $h \neq 0$ such that $\langle x - Tx, j(h)\rangle \leq 0$ for all $x \in K$. Then for a suitable $x_0 \in K$, the sequence $\{x_n\}$ generated from x_0 by* (25) *and* (26) *converges weakly to a fixed point of T.*

THEOREM 5. *Let $A : E \to E$ be a continuous linear map and let $f \in A(E)$. Suppose zero is an eigenvalue of A and that the following condition is satisfied:*

$$\langle Ay, j(y)\rangle \geq \lambda ||Ay||^2, \tag{24}$$

for all $y \in E$ and for some $\lambda > 0$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{u_n\}, \{v_n\}$ be as in Lemma 2. Then the sequence $\{x_n\}$ generated from a suitable $x_0 \in E$ by

$$y_n = a_n x_n + b_n(f + x_n - Ax_n) + c_n u_n, \; n \geq 0$$

$$x_{n+1} = a'_n x_n + b'_n(f + y_n - Ay_n) + c'_n v_n, \; n \geq 0$$

converges strongly to a solution of the equation $Ax = f$.

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Department of Mathematics, University of Nigeria, Nsukka, NIGERIA