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DEPTH OF TOR

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ABSTRACT. Using spectral sequences we calculate the highest nonvanishing index of Tor for modules of finite projective dimension. The result is applied to compute the depth of the highest nonvanishing Tor. This is one of the cases when a problem of Auslander is positive.

1. Introduction

Throughout this paper, every ring is assumed to be commutative and noetherian with identity. For an R-module M, the projective dimension of M is written as pd$M$.

The purpose of this paper is to investigate depth of Tor. Especially, the depth of the highest nonvanishing Tor. From now on we let

$$s := \sup\{i \mid \mathrm{Tor}_i(M,N) \neq 0\}.$$

In his paper [1], Auslander has initiated the computation of depth of Tor and suggested the following problem.

PROBLEM 1.1. *Let (R,m) be a local ring and M, N be finite nonzero R-modules. Suppose that M is of finite projective dimension. Is it true that*

$$pdM - depthN = j - depth(Tor_j^R(M,N))$$

for some j?

Regarding the highest nonvanishing Tor, the problem was partially answered.

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THEOREM 1.2. [1, Theorem 1.2] *Let M and N be nonzero finite modules over a local ring R such that $pdM < \infty$. If either $depth(Tor_s^R(M,N)) \leq 1$ or $s = 0$, then*

$$s = pdM - depthN + depth(Tor_s^R(M,N)).$$

First we study the highest nonvanishing Tor for modules of finite projective dimension. One of our main result (Theorem 2.4) generalizes a formula due to Serre (Theorem 1.3). The result is applied to compute the depth of the highest nonvanishing Tor. This is one of the cases when Problem 1.1 is positive.

THEOREM 1.3. [5, V. Theorem 4] *Let (R,m) be a regular local ring and M and N be finitely generated nonzero R-modules with $l(M \otimes N) < \infty$. Then*

$$s = pdM + pdN - \dim R.$$

2. Nonvanishing of Tor

In this section we generalize Theorem 1.3 for any local ring. Spectral sequences of double(triple) complexes are main tools of computation.

DEFINITION 2.1. Let L, M and N be R-modules, $P.$, $F.$ and $G.$ be projective resolutions of L, M and N respectively. Define

$$\mathrm{Tor}_i^R(L,M,N) := H_i(P. \otimes F. \otimes G.).$$

Since a projective module is a direct summand of a free module, we can formulate the following lemma.

LEMMA 2.2. *If $C.$ is a complex and P is a projective, then $H_i(P \otimes C.) \cong P \otimes H_i(C.)$.*

Applying Lemma 2.2 to compute Tor of the double complexes in Definition 2.1, we obtain the following spectral sequence.

THEOREM 2.3. $Tor_p^R(L, Tor_q^R(M,N)) \Longrightarrow Tor_{p+q}^R(L,M,N)$.

THEOREM 2.4. *Let (R, m) be a local ring and M, N be finite nonzero R-modules of finite projective dimension. Then*

$$s \geq pdM + pdN - depthR.$$

If $\mathrm{Tor}_s^R(M, N)$ has an associated prime whose grade is equal to depthR, then the equality holds in the above formula.

Proof. Let $\mathrm{depth}R = n$ and $\underline{x} = (x_1, \cdots, x_n)$ be a maximal R-sequence. Consider the spectral sequences in Theorem 2.3.

$$\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^R(R/\underline{x}, N)) \Rightarrow \mathrm{Tor}_{p+q}^R(R/\underline{x}, M, N),$$

$$\mathrm{Tor}_p^R(R/\underline{x}, \mathrm{Tor}_q^R(M, N)) \Rightarrow \mathrm{Tor}_{p+q}^R(R/\underline{x}, M, N).$$

Note that $\mathrm{Tor}_p^R(M, \mathrm{Tor}_q^R(R/\underline{x}, N)) = 0$ if $p \geq \mathrm{pd}M + 1$ or $q \geq \mathrm{pd}N + 1$. As $x_1, \cdots, x_n$ are a maximal regular sequence, $(0 :_{R/\underline{x}} m) \neq 0$. Computing $\mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N)$ from the minimal free resolution of N, we obtain $\mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N) \neq 0$, and it is a submodule of a finite free $R/(\underline{x})$-module. Hence

$$\mathrm{Tor}_{\mathrm{pd}M}^R(M, \mathrm{Tor}_{\mathrm{pd}N}^R(R/\underline{x}, N)) \neq 0.$$

On the other hand, $\mathrm{Tor}_p^R(R/\underline{x}, \mathrm{Tor}_q^R(M, N)) = 0$ if $p \geq n+1$ or $q \geq s+1$. It is due to the maximal cycle principle [4] that

$$\mathrm{pd}M + \mathrm{pd}N = \sup\{i \mid \mathrm{Tor}_i^R(R/\underline{x}, M, N) \neq 0\} \leq n + s.$$

Suppose that $\mathrm{Tor}_s^R(M, N)$ has an associated prime P of grade n. Choose a maximal R-sequence $\underline{x} = (x_1, \cdots, x_n)$ in P. Then

$$\mathrm{Tor}_n^R(R/\underline{x}, \mathrm{Tor}_s^R(M, N)) = (0 :_{\mathrm{Tor}_s^R(M,N)} \underline{x}) \neq 0.$$

Therefore

$$n + s = \mathrm{pd}M + \mathrm{pd}N = \sup\{i \mid \mathrm{Tor}_i^R(R/\underline{x}, M, N) \neq 0\}.$$

This concludes the proof of Theorem 2.4. □

Notice that the equality, $n+s = \text{pd}M+\text{pd}N$, does *not* depend on the choice of the maximal R-sequence. Thus if $\text{Tor}_n^R(R/\underline{x}, \text{Tor}_s^R(M,N)) \neq 0$, for a maximal R-sequence $\underline{x} = (x_1, \cdots, x_n)$, then for any R-sequence $\underline{y} = (y_1, \cdots, y_n)$,

$$\text{Tor}_n^R(R/\underline{y}, \text{Tor}_s^R(M,N)) \cong \text{Tor}_{\text{pd}M}^R(M, \text{Tor}_{\text{pd}N}^R(R/\underline{y}, N)) \neq 0.$$

We ask whether there is a natural map between $\text{Tor}_n^R(R/\underline{x}, \text{Tor}_s^R(M,N))$ and

$\text{Tor}_n^R(R/\underline{y}, \text{Tor}_s^R(M,N))$ for two maximal R-sequence $\underline{x}$ and $\underline{y}$.

C. Huneke has pointed out that $s \geq \text{pd}M - \text{depth}N$ without assuming that N is of finite projective dimension (cf. [2]). Suppose that $s < \text{pd}M - \text{depth}N$. Let $\text{pd}M = m$, $\text{pd}M - \text{depth}N = l$ and $F.$ be a minimal free resolution of M. Note that

$$0 \longrightarrow F_m \xrightarrow{\phi} \cdots \longrightarrow F_l \longrightarrow F_{l-1}$$

is exact. Since $s < l$,

$$0 \longrightarrow F_m \otimes N \xrightarrow{\phi\otimes 1} \cdots \longrightarrow F_l \otimes N \longrightarrow F_{l-1} \otimes N$$

is also exact. Due to the Buchsbaum-Eisenbud criterion of exactness[3], $\text{depth}_{I(\phi)}N \geq m - l + 1$. Hence $\text{depth}N \geq m - l + 1$. This is a contradiction and $s \geq \text{pd}M - \text{depth}N$.

If $M \otimes N$ has the maximal grade then so does $\text{Tor}_s^R(M,N)$. So we obtain the following corollary

COROLLARY 2.5. *Let (R,m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If grade $(M \otimes N) =$ depth$R$, then*

$$s = pdM + pdN - depthR.$$

If $l(M \otimes N) < \infty$, then $\text{ann}M + \text{ann}N$ is m-primary and its grade is equal to depth$R$. Hence we obtain a corollary similar to Theorem 1.1.

COROLLARY 2.6. *Let (R,m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If $l(M \otimes N) < \infty$, then*

$$s = pdM + pdN - depthR.$$

3. Depth of Tor

Associated primes of modules behave well modulo a regular sequence in the following sense.

LEMMA 3.1. *Let (R, m) be a local ring and M be a finitely generated R-module. If p is an associated prime of M and x be a nonzero divisor of M. Then there exists an associated prime p' of M/xM containing p and x.*

Proof. Note that if x, y is an M-regular sequence, then y, x is also M-regular sequence. Hence the union of the associated prime ideals of M is contained in the union of the associated prime ideals of M/xM. It is due to the prime avoidance lemma that each associated prime ideals of M is contained in an associated prime ideals of M/xM. □

THEOREM 3.2. *Let (R, m) be a local ring and M, N be finite nonzero R-modules of finite projective dimension. If $\mathrm{Tor}_s^R(M, N)$ has an associated prime whose grade is equal to depth$R$, then depth$(\mathrm{Tor}_s^R(M, N)) = 0$ and* Problem 1.1 *is true for s.*

Proof. Assume that depth$N = 0$. Compute $\mathrm{Tor}_i^R(M, N)$ using the minimal free resolution of M tensored with N. Since $\mathrm{Tor}_s^R(M, N)$ is a submodule of $\oplus N$ annihilated by a matrix with the entries in maximal ideal, $s = \mathrm{pd}M$ and $\mathrm{depth}(\mathrm{Tor}_s^R(M, N)) = 0$.

Suppose that depth$N > 0$ and $\mathrm{depth}(\mathrm{Tor}_s^R(M, N)) > 0$. Let x be a non-zerodivisor both for N and $\mathrm{Tor}_s^R(M, N)$. The short exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_s^R(M, N) &\xrightarrow{x} \mathrm{Tor}_s^R(M, N) \longrightarrow \mathrm{Tor}_s^R(M, N/xN) \\ &\longrightarrow \mathrm{Tor}_{s-1}^R(M, N) \longrightarrow \cdots . \end{aligned}$$

Thus $\bar{s} := \max\{i \mid \mathrm{Tor}_i^R(M, N/xN) \neq 0\}$ is equal to s. Let p be an associated prime of $\mathrm{Tor}_s^R(M, N)$ of the maximal grade. It is due to Lemma 2.2 that $\mathrm{Tor}_s^R(M, N)/x\mathrm{Tor}_s^R(M, N)$ has an associated prime

p' containing both p and x. So its grade is also equal to depth$R$. From the long exact sequence in the above, $\mathrm{Tor}_s^R(M,N)/x\mathrm{Tor}_s^R(M,N)$ is a submodule of $\mathrm{Tor}_s^R(M,N/xN)$. Hence p' is an associated prime $\mathrm{Tor}_s^R(M,N/xN)$. It is due to Theorem 2.1 that

$$\bar{s} = \mathrm{pd}M - \mathrm{depth}N/xN = \mathrm{pd}M - \mathrm{depth}N + 1.$$

This is a contradiction since $\bar{s} = s = \mathrm{pd}M - \mathrm{depth}N$. Therefore, $\mathrm{depth}(\mathrm{Tor}_s^R(M,N)) = 0$ and Problem 1.1 is true by Theorem 1.2. □

If $M \otimes N$ has the maximal grade then so does $\mathrm{Tor}_s^R(M,N)$. In particular, if $l(M \otimes N) < \infty$, then $M \otimes N$ has the maximal grade. Thus the following result can be concluded by Theorem 3.2.

COROLLARY 3.3. *Let (R,m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If grade $(M \otimes N) =$ depth$R$, then $s =$ pd$M -$ depth$N$, and* Problem 1.1 *is true for s.*

COROLLARY 3.4. *Let (R,m) be a local ring and M and N be finitely generated nonzero R-modules of finite projective dimension. If $l(M \otimes N) < \infty$, then $s =$ pd$M -$ depth$N$, and* Problem 1.1 *is true for s.*

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