Bull. Korean Math. Soc. **37** (2000), No. 1, pp. 85–90

# COMPARISON OF EINSTEIN MANIFOLDS WITH THORPE MANIFOLDS

HOBUM KIM AND JAEMAN KIM*

ABSTRACT. On Riemannian manifolds of dimension 4 the Einstein condition is equivalent to the Thorpe condition. In this paper, we construct a few metrics which are Einstein but not Thorpe, and vice versa in dimensions larger than 4.

## 1. Introduction

A Riemannian manifold (M,g) is said to be an Einstein manifold if it has constant Ricci curvature - i.e., if its Ricci tensor $r$ is a constant multiple of the metric:

$$r = cg$$

we call this condition an Einstein condition and this metric an Einstein metric. On 4k dimensional Riemannian manifolds we can define a generalized $2k^{th}$ curvature operator $R_{2k}$ and if $R_{2k}$ commutes with $*$, i.e., $R_{2k} * = * R_{2k}$, then we call this condition a Thorpe condition and this metric a Thorpe metric and this Riemannian manifold a Thorpe manifold.

In the 4-dimensional case the Thorpe condition is equivalent to the Einstein condition [1]. It is an interesting fact that in the 4−dimensional case the Einstein condition, which is the extremal condition of the total scalar curvature of suitable normalized metrics on compact Riemannian

Received March 8, 1999.
1991 Mathematics Subject Classification: 53B20, 53B21, 53C25.
Key words and phrases: Einstein condition, Thorpe condition.
*The present studies were supported by the Basic Science Research Institute Program, Ministry of Education, 1998, BSRI-98-1422 and were supported by a postdoctoral grant to Jaeman from the Natural Science Research Institute, Yonsei University in the year 1999.

manifolds [1], can be read off from a purely algebraic condition. For dimensions higher than 4 nothing is known about topological conditions for the existence of an Einstein metric on a manifold. However, a topological obstruction is known about the existence of a Thorpe metric on compact oriented Riemannian manifolds [4]. This is reflected the fact that the Thorpe condition does not imply the Einstein condition in dimensions higher than 4. In fact, we shall see some examples of manifolds whose given metrics are Thorpe metrics but not Einstein metrics, and vice versa.

## 2. The $p^{th}$ Curvature Operator and Thorpe Manifolds

Let $M$ be a Riemannian manifold of dimension $n$ and let $\bigwedge^p(M)$ denote the bundle of $p-$vectors of $M$. $\bigwedge^p(M)$ is a Riemannian vector bundle, with inner product on the fiber $\bigwedge^p(x)$ over the point $x$ [4]. Let $R$ denote the covariant curvature tensor of $M$. For each even $p > 0$, we define the $p^{th}$ curvature tensor $R_p$ of $M$ to be the covariant tensor field of order $2p$ given by

$$\begin{aligned} &R_p(u_1, \dots, u_p, v_1, \dots, v_p) \\ &= \frac{1}{2^{\frac{p}{2}} p!} \sum_{\alpha,\beta \in S_p} \varepsilon(\alpha)\, \varepsilon(\beta)\, R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \\ &\qquad\qquad \cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}) \end{aligned}$$

where $u_i, v_j \in T_xM$, and $S_p$ denotes the group of permutations of $(1, \dots, p)$ and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of the permutation $\alpha$.

The tensor $R_p$ has the following properties: it is alternating in the first $p$ variables, alternating in the last $p$ variables and is invariant under the operation of interchanging the first $p$ variables with the last $p$ variables. Hence, at each point $x \in M$, $R_p$ can be regarded as a symmetric bilinear form on $\bigwedge^p(x)$. By use of the inner product on $\bigwedge^p(x)$, $R_p$ at $x$ may then be identified with a self-adjoint linear operator $R_p$ on $\bigwedge^p(x)$. Explicitly, this identification is given by

$$\langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p \rangle \equiv R_p(u_1, \dots, u_p, v_1, \dots, v_p)$$

with $u_i, v_j \in T_xM$. From now on, we will use the same notations for the $p^{th}$ curvature operators and the $p^{th}$ curvature tensors . Let $P \in G_p(M)$, where the Grassmann bundle $G_p(M)$ of oriented tangent $p-$planes of $M$ shall be viewed as a subbundle of the unit sphere bundle of $\bigwedge^p(M)$ by identifying $P \in G_p(M)$ with $e_1 \wedge \cdots \wedge e_p \in \bigwedge^p(M)$, where $\{e_1, \cdots, e_p\}$ is any oriented orthonormal basis for $P$. Then

$$R_p(P) = \frac{1}{p!} \sum_{\alpha \in S_p} \varepsilon(\alpha) R(e_{\alpha(1)} \wedge e_{\alpha(2)}) \wedge \cdots \wedge R(e_{\alpha(p-1)} \wedge e_{\alpha(p)})$$

and suppose $p \geq 0$ and $q \geq 0$ are even integers with $p + q \leq n$. For $P \in G_{p+q}(M)$, let $\{e_1, \ldots, e_{p+q}\}$ be an orthonormal basis for $P$ and let us consider $B = \{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq p+q\}$, then $B \subset G_p(M)$ and

$$R_{p+q}(P) = \frac{p!\ q!}{(p+q)!} \sum_{Q \in P} R_p(Q) \wedge R_q(Q^{\perp})$$

where $Q^{\perp}$ is the oriented orthogonal complement of $Q$ in $P$ [4].

Now we can consider the necessary condition for the existence of a Thorpe metric [4]:

THEOREM 2.1. *Let $M$ be a compact orientable $4k-$dimensional Riemannian manifold which admits a Thorpe metric, then*

$$\chi \geq \frac{k!\ k!}{(2k)!} |P_k|$$

*where $\chi$ is the Euler characteristic of $M$ and $P_k$ is the $k^{th}$ Pontrjagin number of $M$. And in particular $\chi \geq 0$.*

*Proof.* The de Rham representation for the $k^{th}$ Pontrjagin [3] class of $M$ is the differential $4k-$form

$$\frac{[(2k)!]^3}{(2^k\, k!)^2 (2\pi)^{2k}} \operatorname{trace}(R_{2k} \, * \, R_{2k})\, dV.$$

Since $R_{2k}$ commutes with $*$, it also commutes with $I \pm *$, where $I$ denotes the identity operator on $\bigwedge^{2k}$. Hence $R_{2k}(I \pm *)$ is self adjoint and

$$0 \leq \operatorname{trace}\, [R_{2k}(I \pm *)]^2 = 2 \left[\operatorname{trace}(R_{2k})^2 \pm \operatorname{trace}\, (R_{2k} \, * \, R_{2k})\right]$$

and so

$$\operatorname{trace}(R_{2k})^2 \geq |\operatorname{trace}\, (R_{2k} \, * \, R_{2k})|$$

that means

$$\chi \geq \frac{k!\,k!}{(2k)!}\,|P_k|$$

and this completes the proof. □

## 3. The Main Result

It is known that in the 4−dimensional case the Thorpe condition is equivalent to that of Einstein [1]. On the other hand we can prove the following facts in this section: On dimensions larger than 4 the Thorpe condition does not imply that of Einstein, and vice versa.

THEOREM 3.1. *On 4 dimensions the Thorpe condition is the Einstein condition.*

*Proof.* If we consider $R_{2k} \equiv R$ in $S^2 \bigwedge^2 T^*M^4$ as a linear map of $\bigwedge^2 T^*M^4$ and if we decompose

$$\bigwedge^2 T^*M^4 = \bigwedge^+ T^*M^4 \oplus \bigwedge^- T^*M^4$$

where $\bigwedge^+ T^*M^4$ is the $(+1)-$eigenspace (self-dual space) and $\bigwedge^- T^*M^4$ is the $(-1)-$eigen space (anti-self-dual space) of the Hodge $*$ operator, respectively, then we get the following expression for $R$ [1],

$$\begin{array}{ccc} & \textit{self-dual} \quad \textit{anti-self-dual} & \\ R \;=\; & \left(\begin{array}{c|c} W^+ + \frac{s}{12}Id & ric_o \\ \hline {}^t ric_o & W^- + \frac{s}{12}Id \end{array}\right) & \begin{array}{c} \textit{self-dual} \\ \textit{anti-self-dual} \end{array} \end{array}$$

where $s$ is the scalar curvature, $ric_o$ is the traceless Ricci curvature, $W^+$ is the self-dual Weyl curvature, and $W^-$ is the anti-self-dual Weyl curvature.

It is possible to interpret $R* = *R$ as $R$ (self-dual) = self-dual , $R$ (anti-self-dual) = anti-self-dual:

$$R * (A_1) = R(A_1) = A_2 + B_2$$

on the other hand

$$*R(A_1) = *(A_2 + B_2) = A_2 - B_2$$

where $A_1$, $A_2 \in \bigwedge^+ T^*M^4$ and $B_2 \in \bigwedge^- T^*M^4$. Hence, $R* = *R$ implies $B_2 \equiv 0$, i.e., $R(A_1) = A_2$.

In the same way

$$R * (B_1) = R(-B_1) = -A_3 - B_3$$

on the other hand

$$* R(B_1) = *(A_3 + B_3) = A_3 - B_3$$

where $A_3 \in \bigwedge^+ T^*M^4$ and $B_1, B_3 \in \bigwedge^- T^*M^4$. Hence $R* = *R$ implies $A_3 \equiv 0$, i.e., $R(B_1) = B_3$.

Therefore we can conclude that $*R = R*$ is equivalent to the vanishing of the traceless Ricci curvature which means the Einstein condition and this completes the proof. □

Now we can see examples which are not Einstein manifolds but Thorpe manifolds, and vice versa in dimensions larger than 4.

THEOREM 3.2. (i) $S^{4k} \times H^{4k}$ *with a product metric of standard ones,*
(ii) $CP^2 \times CH^2$ *with a product metric of standard ones,*
(iii) *the canonical quaternion projective space* $HP^n$ *with* $n \geq 3$.
*Both* (i) *and* (ii) *are not Einstein manifolds but Thorpe manifolds, and* (iii) *are not Thorpe manifolds but Einstein manifolds.*

*Proof.* The curvature $R$ of a Riemannian product manifold of $M$ with $N$ is

$$R = R_M + R_N,$$

where $R_M$, $R_N$ are curvatures for $M$, $N$, respectively. And hence, in the case of (i), the only non-zero terms in the $4k^{th}$ curvature tensor are those which are products of sectional curvatures. All the other terms are zero. And hence we can verify, case by case, that the product metric is a Thorpe metric [2]. However, the product metric cannot get an Einstein constant because the Ricci curvature of a product metric is the addition of each one which has different constant signs. Hence (i) are not Einstein manifolds but Thorpe manifolds. The same kind of argument can be applied to the case of (ii) together with the property of kähler and hence we can easily verify the Thorpe condition [2]. However, the product metric is obviously not an Einstein metric because each one has different constant signs. On the other hand the curvature tensor $R$ of the canonical quaternion projective space $HP^n$ ($n \geq 3$) does not satisfy the Thorpe condition [2]. For instance, on $HP^3$,

$$R_6\,(e_1, e_2, e_3, Ie_3, Je_3, Ke_3, e_1, e_2, e_3, Ie_3, Je_3, Ke_3)$$
$$\neq R_6\,(Ie_1, Je_1, Ke_1, Ie_2, Je_2, Ke_2, Ie_1, Je_1, Ke_1, Ie_2, Je_2, Ke_2)$$

where $I, J$ and $K$ are the almost complex structures and $R_6$ is the $6^{th}$ curvature tensor.
However, the canonical quaternion projective space $HP^n$ are Einstein manifolds [1]. Therefore (iii) are not Thorpe manifolds but Einstein manifolds and this completes the proof. □

REMARK. $CP^{2n} \times CH^{2n}(n \geq 2)$ with the standard product metric are not Thorpe manifolds [2]. On the other hand the canonical quaternion projective space $HP^n(n \leq 2)$ are Thorpe manifolds [2].

## References

[1] A. L. Besse, *Einstein manifolds,* Springer Verlag, 1986.
[2] J. M. Kim, *Einstein-Thorpe Manifolds,* Thesis, SUNY at Stony Brook, 1998.
[3] J. A. Thorpe, *Sectional curvatures and characteristic classes,* Annals of Mathematics **80** (1964), 429–443.
[4] ———, *Some remarks on the Gauss-Bonnet integral,* Journal of Mathematics and Mechanics **18** (1969), 779–786.

DEPARTMENT OF MATHEMATICS, YONSEI UNIVERSITY, SHINCHON 134, SEOUL 120-749, KOREA