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THE CATEGORY OF INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods. We investigate the properties of continuous, open and closed maps in the intuitionistic fuzzy topological spaces, and show that the category of Chang's fuzzy topological spaces is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces.

1. Introduction and Preliminaries

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [3,4,5] introduced intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets.

In this paper, we introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods. We investigate the properties of continuous, open and closed maps in the intuitionistic fuzzy topological spaces, and show that the category of Chang's fuzzy topological spaces [2] is a bireflective full subcategory of that of intuitionistic fuzzy topological spaces.

Let X be a nonempty set. An *intuitionistic fuzzy set* (IFS for short) A is an ordered pair

$$A = (\mu_A, \gamma_A)$$

where the functions $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively, and $\mu_A + \gamma_A \leq 1$.

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Obviously every fuzzy set μ_A on X is an IFS of the form $(\mu_A, 1-\mu_A)$.

DEFINITION 1.1 ([1]). Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be IFSs on X. Then

(1) $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\gamma_A \geq \gamma_B$.
(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
(3) $A^c = (\gamma_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.
(6) $0_\sim = (\tilde{0}, \tilde{1})$ and $1_\sim = (\tilde{1}, \tilde{0})$.

Let f be a map from a set X to a set Y. Let $A = (\mu_A, \gamma_A)$ be an IFS of X and $B = (\mu_B, \gamma_B)$ an IFS of Y. Then $f^{-1}(B)$ is an IFS of X defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$$

and $f(A)$ is an IFS of Y defined by

$$f(A) = (f(\mu_A), 1 - f(1 - \gamma_A)).$$

DEFINITION 1.2 ([3]). An *intuitionistic fuzzy topology* (IFT for short) on X is a family $\mathcal{T}$ of IFSs in X which satisfies the following properties:

(1) $0_\sim, 1_\sim \in \mathcal{T}$.
(2) If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$.
(3) If $A_i \in \mathcal{T}$ for each i, then $\bigcup A_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called an *intuitionistic fuzzy topological space* (IFTS for short).

Let $(X, \mathcal{T})$ be an IFTS. Then any member of $\mathcal{T}$ is called an *intuitionistic fuzzy open set* (IFOS for short) of X and the complement an *intuitionistic fuzzy closed set* (IFCS for short).

DEFINITION 1.3 ([3]). Let $(X, \mathcal{T})$ be an IFTS and A an IFS in X. Then the *fuzzy closure* is defined by

$$\mathrm{cl}(A) = \bigcap\{F \mid A \subseteq F, F^c \in \mathcal{T}\}$$

and the *fuzzy interior* is defined by

$$\mathrm{int}(A) = \bigcup\{G \mid A \supseteq G, G \in \mathcal{T}\}.$$

THEOREM 1.4 ([3]). *For an* IFS A *of an* IFTS $(X, \mathcal{T})$, *we have:*
(1) $\text{int}(A)^c = \text{cl}(A^c)$.
(2) $\text{cl}(A)^c = \text{int}(A^c)$.

2. Intuitionistic Fuzzy Neighborhoods

We are going to introduce the concept of intuitionistic fuzzy points and intuitionistic fuzzy neighborhoods.

DEFINITION 2.1. Let $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. An *intuitionistic fuzzy point* (IFP for short) $x_{(\alpha,\beta)}$ of X is an IFS of X defined by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x. \end{cases}$$

In this case, x is called the *support* of $x_{(\alpha,\beta)}$ and α and β are called the *value* and the *nonvalue* of $x_{(\alpha,\beta)}$, respectively. An IFP $x_{(\alpha,\beta)}$ is said to *belong* to an IFS $A = (\mu_A, \gamma_A)$ of X, denoted by $x_{(\alpha,\beta)} \in A$, if $\alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x)$.

Clearly an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy points as follows:

$$x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{1-\beta}).$$

THEOREM 2.2. *Let* $A = (\mu_A, \gamma_A)$ *be an* IFS *of* X. *Then* $x_{(\alpha,\beta)} \in A$ *if and only if* $x_\alpha \in \mu_A$ *and* $x_{1-\beta} \in 1 - \gamma_A$.

Proof. Let $x_{(\alpha,\beta)} \in A \Leftrightarrow \alpha \leq \mu_A(x)$ and $\beta \geq \gamma_A(x) \Leftrightarrow \alpha \leq \mu_A(x)$ and $1 - \beta \leq 1 - \gamma_A(x) \Leftrightarrow x_\alpha \in \mu_A$ and $x_{1-\beta} \in 1 - \gamma_A$. □

THEOREM 2.3. *Let* $A = (\mu_A, \gamma_A)$ *and* $B = (\mu_B, \gamma_B)$ *be* IFSs *of* X. *Then* $A \subseteq B$ *if and only if* $x_{(\alpha,\beta)} \in A$ *implies* $x_{(\alpha,\beta)} \in B$ *for any* IFP $x_{(\alpha,\beta)}$ *in* X.

Proof. Let $A \subseteq B$ and $x_{(\alpha,\beta)} \in A$. Then $\alpha \leq \mu_A(x) \leq \mu_B(x)$ and $\beta \geq \gamma_A(x) \geq \gamma_B(x)$. Thus $x_{(\alpha,\beta)} \in B$. Conversely, take any $x \in X$. Let $\alpha = \mu_A(x)$ and $\beta = \gamma_A(x)$. Then $x_{(\alpha,\beta)}$ is an IFP in X and $x_{(\alpha,\beta)} \in A$. By the hypothesis, $x_{(\alpha,\beta)} \in B$. Thus $\mu_A(x) = \alpha \leq \mu_B(x)$ and $\gamma_A(x) = \beta \geq \gamma_B(x)$. Hence $A \subseteq B$. □

THEOREM 2.4. *Let* $A = (\mu_A, \gamma_A)$ *be an* IFS *of* X. *Then* $A = \bigcup\{x_{(\alpha,\beta)} \mid x_{(\alpha,\beta)} \in A\}$.

Proof. Since $x_{(\alpha,\beta)} = (x_\alpha, 1 - x_{1-\beta})$,

$$\begin{aligned}&\bigcup\{x_{(\alpha,\beta)} \mid x_{(\alpha,\beta)} \in A\}\\&= (\bigvee\{x_\alpha \mid x_\alpha \in \mu_A\}, \bigwedge\{1 - x_{1-\beta} \mid x_{1-\beta} \in 1 - \gamma_A\}).\end{aligned}$$

Clearly $\bigvee\{x_\alpha \mid x_\alpha \in \mu_A\} = \mu_A$. On the other hand $\bigvee\{x_{1-\beta} \mid x_{1-\beta} \in 1 - \gamma_A\} = 1 - \gamma_A$ and hence

$$\gamma_A = 1 - \bigvee\{x_{1-\beta} \mid x_{1-\beta} \in 1 - \gamma_A\} = \bigwedge\{1 - x_{1-\beta} \mid x_{1-\beta} \in 1 - \gamma_A\}.$$

Hence $\bigcup\{x_{(\alpha,\beta)} \mid x_{(\alpha,\beta)} \in A\} = (\mu_A, \gamma_A) = A$. □

DEFINITION 2.5. Let $x_{(\alpha,\beta)}$ be an IFP of an IFTS $(X, \mathcal{T})$. An IFS A of X is called an *intuitionistic fuzzy neighborhood* (IFN for short) of $x_{(\alpha,\beta)}$ if there is an IFOS B in X such that $x_{(\alpha,\beta)} \in B \subseteq A$.

THEOREM 2.6. *Let* $(X, \mathcal{T})$ *be an* IFTS. *Then an* IFS A *of* X *is an* IFOS *if and only if* A *is an* IFN *of* $x_{(\alpha,\beta)}$ *for every* IFP $x_{(\alpha,\beta)} \in A$.

Proof. Let A be an IFOS of X. Clearly A is an IFN of any $x_{(\alpha,\beta)} \in A$. Conversely, let $x_{(\alpha,\beta)} \in A$. Since A is an IFN of $x_{(\alpha,\beta)}$, there is an IFOS $B_{x_{(\alpha,\beta)}}$ in X such that $x_{(\alpha,\beta)} \in B_{x_{(\alpha,\beta)}} \subseteq A$. So we have

$$A = \bigcup\{x_{(\alpha,\beta)} \mid x_{(\alpha,\beta)} \in A\} \subseteq \bigcup\{B_{x_{(\alpha,\beta)}} \mid x_{(\alpha,\beta)} \in A\} \subseteq A$$

and hence $A = \bigcup\{B_{x_{(\alpha,\beta)}} \mid x_{(\alpha,\beta)} \in A\}$. Since each $B_{x_{(\alpha,\beta)}}$ is an IFOS, A is an IFOS. □

3. Continuous, Open and Closed Maps

DEFINITION 3.1 ([3]). Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be IFSTs. Then a map $f : X \to Y$ is said to be

(1) *continuous* if $f^{-1}(B)$ is an IFOS of X for each IFOS B of Y, or equivalently, $f^{-1}(B)$ is an IFCS of X for each IFCS B of Y,
(2) *open* if $f(A)$ is an IFOS of Y for each IFOS A of X,
(3) *closed* if $f(A)$ is an IFCS of Y for each IFCS A of X,
(4) a *homeomorphism* if f is bijective, continuous and open.

THEOREM 3.2. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map. Then f is continuous if and only if for any* IFP $x_{(\alpha,\beta)}$ *in X and any* IFN B *of* $f(x_{(\alpha,\beta)})$, *there is an* IFN A *of* $x_{(\alpha,\beta)}$ *such that* $x_{(\alpha,\beta)} \in A$ *and* $f(A) \subseteq B$.

Proof. Let $x_{(\alpha,\beta)}$ be any IFP in X and B any IFN of $f(x_{(\alpha,\beta)})$. Then there is an IFOS C in Y such that $f(x_{(\alpha,\beta)}) \in C \subseteq B$. Since f is continuous, $f^{-1}(C)$ is an IFOS and

$$x_{(\alpha,\beta)} \in f^{-1}f(x_{(\alpha,\beta)}) \subseteq f^{-1}(C) \subseteq f^{-1}(B).$$

Put $A = f^{-1}(C)$. Then A is an IFN of $x_{(\alpha,\beta)}$ and $x_{(\alpha,\beta)} \in A \subseteq f^{-1}(B)$. Thus $x_{(\alpha,\beta)} \in A$ and $f(A) \subseteq ff^{-1}(B) \subseteq B$.

Conversely, let B be any IFOS of Y. If $f^{-1}(B) = 0_\sim$, then it is obvious. Suppose $x_{(\alpha,\beta)} \in f^{-1}(B)$. Then B is an IFN of $f(x_{(\alpha,\beta)})$. By hypothesis, there is an IFN $A_{x_{(\alpha,\beta)}}$ of $x_{(\alpha,\beta)}$ such that $x_{(\alpha,\beta)} \in A_{x_{(\alpha,\beta)}}$ and $f(A_{x_{(\alpha,\beta)}}) \subseteq B$. Since $A_{x_{(\alpha,\beta)}}$ is an IFN of $x_{(\alpha,\beta)}$, there is an IFOS $C_{x_{(\alpha,\beta)}}$ in X such that

$$x_{(\alpha,\beta)} \in C_{x_{(\alpha,\beta)}} \subseteq A_{x_{(\alpha,\beta)}} \subseteq f^{-1}f(A_{x_{(\alpha,\beta)}}) \subseteq f^{-1}(B).$$

So we have

$$\begin{aligned} f^{-1}(B) &= \bigcup\{x_{(\alpha,\beta)} \mid x_{(\alpha,\beta)} \in f^{-1}(B)\} \\ &\subseteq \bigcup\{C_{x_{(\alpha,\beta)}} \mid x_{(\alpha,\beta)} \in f^{-1}(B)\} \\ &\subseteq f^{-1}(B) \end{aligned}$$

and hence $f^{-1}(B) = \bigcup\{C_{x_{(\alpha,\beta)}} \mid x_{(\alpha,\beta)} \in f^{-1}(B)\}$. Thus $f^{-1}(B)$ is an IFOS of X. Therefore f is continuous. □

THEOREM 3.3. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map. Then the following statements are equivalent:*

(1) *f is a continuous map.*
(2) $f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ *for each* IFS A *of* X.
(3) $\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$ *for each* IFS B *of* Y.
(4) $f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{int}(f^{-1}(B))$ *for each* IFS B *of* Y.

Proof. We already know that (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)(See [3]). (1) $\Rightarrow$ (2) Let f be a continuous map and A any IFS of X. Since $\mathrm{cl}(f(A))$ is an IFCS of Y, $f^{-1}(\mathrm{cl}(f(A)))$ is an IFCS of X. Thus

$$\mathrm{cl}(A) \subseteq \mathrm{cl}(f^{-1}f(A)) \subseteq \mathrm{cl}(f^{-1}(\mathrm{cl}(f(A)))) = f^{-1}(\mathrm{cl}(f(A))).$$

Hence

$$f(\mathrm{cl}(A)) \subseteq ff^{-1}(\mathrm{cl}(f(A))) \subseteq \mathrm{cl}(f(A)).$$

(2) $\Rightarrow$ (3) Let B be any IFS of Y. By (2),

$$f(\mathrm{cl}(f^{-1}(B))) \subseteq \mathrm{cl}(ff^{-1}(B)) \subseteq \mathrm{cl}(B).$$

Thus

$$\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}f(\mathrm{cl}(f^{-1}(B))) \subseteq f^{-1}(\mathrm{cl}(B)). \qquad \square$$

The conditions in Theorem 3.3 are not equivalent to the condition that $\mathrm{int}(f(A)) \subseteq f(\mathrm{int}(A))$ for each IFS A of X. This is shown by the following two examples.

EXAMPLE 3.4. Let $X = \{x, y\}$ and A and B be IFSs of X defined as

$$A(x) = (0.3, 0.4), \quad A(y) = (0.3, 0.4);$$

and

$$B(x) = (0.3, 0.4), \quad B(y) = (0, 1).$$

Define $\mathcal{T}_1 = \{0_\sim, 1_\sim, A\}$ and $\mathcal{T}_2 = \{0_\sim, 1_\sim, B\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on X. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Then $f^{-1}(0_\sim) = 0_\sim$, $f^{-1}(1_\sim) = 1_\sim$ and $f^{-1}(B) = A$ are IFOSs of $(X, \mathcal{T}_1)$ and hence f is continuous. But $\mathrm{int}(f(B)) = \mathrm{int}(B) = B$ in $(X, \mathcal{T}_2)$ and $f(\mathrm{int}(B)) = f(0_\sim) = 0_\sim$ in $(X, \mathcal{T}_1)$. Thus $\mathrm{int}(f(B)) \not\subseteq f(\mathrm{int}(B))$.

EXAMPLE 3.5. Let $X = \{x, y\}$ and A be an IFS of X defined as

$$A(x) = (0.4, 0.2), \quad A(y) = (1, 0).$$

Define $\mathcal{T}_1 = \{0_\sim, 1_\sim\}$ and $\mathcal{T}_2 = \{0_\sim, 1_\sim, A\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on X. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Then for each IFS B of $(X, \mathcal{T}_1)$, $f(B)(y) = (0, 1)$. Thus $\mathrm{int}(f(B)) = 0_\sim \subseteq f(\mathrm{int}(B))$. But $f^{-1}(A)$ is not an IFOS in $(X, \mathcal{T}_1)$ and hence f is not continuous.

However, we obtain the following theorem for a bijection f.

THEOREM 3.6. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection. Then the following statements are equivalent:*

(1) *f is a continuous map.*
(2) *$f(\mathrm{cl}(A)) \subseteq \mathrm{cl}(f(A))$ for each* IFS *A of X.*
(3) *$\mathrm{cl}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{cl}(B))$ for each* IFS *B of Y.*
(4) *$f^{-1}(\mathrm{int}(B)) \subseteq \mathrm{int}(f^{-1}(B))$ for each* IFS *B of Y.*
(5) *$\mathrm{int}(f(A)) \subseteq f(\mathrm{int}(A))$ for each* IFS *A of X.*

Proof. By Theorem 3.3, it suffices to show that (4) is equivalent to (5). Let A be any IFS of X. Then $f(A)$ is an IFS of Y. So $f^{-1}(\mathrm{int}(f(A))) \subseteq \mathrm{int}(f^{-1}f(A))$. Since f is one-to-one,

$$f^{-1}(\mathrm{int}(f(A))) \subseteq \mathrm{int}(f^{-1}f(A)) = \mathrm{int}(A).$$

Thus $ff^{-1}(\mathrm{int}(f(A))) \subseteq f(\mathrm{int}(A))$. Since f is onto,

$$\mathrm{int}(f(A)) = ff^{-1}(\mathrm{int}(f(A))) \subseteq f(\mathrm{int}(A)).$$

Conversely, let B be any IFS of Y. Then $f^{-1}(A)$ is an IFS of X. Since f is onto,

$$\mathrm{int}(B) = \mathrm{int}(ff^{-1}(B)) \subseteq f(\mathrm{int}(f^{-1}(B))).$$

Since f is one-to-one,

$$f^{-1}(\mathrm{int}(B)) \subseteq f^{-1}f(\mathrm{int}(f^{-1}(B)) = \mathrm{int}(f^{-1}(B)).$$

Hence the theorem follows. □

THEOREM 3.7. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map. Then the following statements are equivalent:*

(1) *f is an open map.*
(2) *$f(\mathrm{int}(A)) \subseteq \mathrm{int}(f(A))$ for each* IFS *A of X.*
(3) *$\mathrm{int}(f^{-1}(B)) \subseteq f^{-1}(\mathrm{int}(B))$ for each* IFS *B of Y.*

Proof. (1) $\Rightarrow$ (2) Let A be any IFS of X. Clearly, $\text{int}(A)$ is an IFOS of X. Since f is an open map, $f(\text{int}(A))$ is an IFOS in Y. Thus

$$f(\text{int}(A)) = \text{int}(f(\text{int}(A))) \subseteq \text{int}(f(A)).$$

(2) $\Rightarrow$ (3) Let B be any IFS of Y. Then $f^{-1}(B)$ is an IFS of X. By (2),

$$f(\text{int}(f^{-1}(B)) \subseteq \text{int}(ff^{-1}(B)) \subseteq \text{int}(B).$$

Thus we have

$$\text{int}(f^{-1}(B)) \subseteq f^{-1}f(\text{int}(f^{-1}(B))) \subseteq f^{-1}(\text{int}(B)).$$

(3) $\Rightarrow$ (1) Let A be any IFOS of X. Then $\text{int}(A) = A$ and $f(A)$ is an IFS of Y. By (3),

$$A = \text{int}(A) \subseteq \text{int}(f^{-1}f(A)) \subseteq f^{-1}(\text{int}(f(A))).$$

Hence we have

$$f(A) \subseteq ff^{-1}(\text{int}(f(A))) \subseteq \text{int}(f(A)) \subseteq f(A).$$

Thus $f(A) = \text{int}(f(A))$ and hence $f(A)$ is an IFOS in Y. Therefore f is an open map. □

THEOREM 3.8. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a map. Then the following statements are equivalent:*

(1) *f is a closed map.*
(2) $\text{cl}(f(A)) \subseteq f(\text{cl}(A))$ *for each* IFS A *of* X.

Proof. (1) $\Rightarrow$ (2) Let A be any IFS of X. Clearly, $\text{cl}(A)$ is an IFCS in X. Since f is a closed map, $f(\text{cl}(A))$ is an IFCS in Y. Thus we have

$$\text{cl}(f(A)) \subseteq \text{cl}(f(\text{cl}(A))) = f(\text{cl}(A)).$$

(2) $\Rightarrow$ (1) Let A be any IFCS of X. Then $\text{cl}(A) = A$. By (2),

$$\text{cl}(f(A)) \subseteq f(\text{cl}(A)) = f(A) \subseteq \text{cl}(f(A)).$$

Thus $f(A) = \text{cl}(f(A))$ and hence $f(A)$ is an IFCS in Y. Therefore f is a closed map. □

The conditions in Theorem 3.8 are not equivalent to the condition that $f^{-1}(\text{cl}(B)) \subseteq \text{cl}(f^{-1}(B))$ for each IFS B of Y. This is shown by the following two examples.

EXAMPLE 3.9. Let $X = \{x, y\}$ and A, B and C be IFSs of X defined as

$$A(x) = (0.3, 0.4), \quad A(y) = (0.3, 0.4);$$
$$B(x) = (0.3, 0.4), \quad B(y) = (0, 1);$$

and

$$C(x) = (1, 0), \quad C(y) = (0, 1).$$

Define $\mathcal{T}_1 = \{0_\sim, 1_\sim, A^c\}$ and $\mathcal{T}_2 = \{0_\sim, 1_\sim, B^c, C^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on X. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Then $f(0_\sim) = 0_\sim$, $f(1_\sim) = C$ and $f(A) = B$ are IFCSs of $(X, \mathcal{T}_2)$ and hence f is a closed map. On the other hand, $f^{-1}(\mathrm{cl}(A)) = f^{-1}(1_\sim) = 1_\sim$ in $(X, \mathcal{T}_2)$ and $\mathrm{cl}(f^{-1}(A)) = \mathrm{cl}(A) = A$ in $(X, \mathcal{T}_1)$. Thus $f^{-1}(\mathrm{cl}(A)) \not\subseteq \mathrm{cl}(f^{-1}(A))$.

EXAMPLE 3.10. Let $X = \{x, y\}$ and A be an IFS of X defined as

$$A(x) = (0, 1), \quad A(y) = (1, 0).$$

Define $\mathcal{T}_1 = \{0_\sim, 1_\sim\}$ and $\mathcal{T}_2 = \{0_\sim, 1_\sim, A^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are IFTs on X. Consider the constant map $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ defined by $f(x) = x$ and $f(y) = x$. Let B be any IFS of Y. If $B(x) = (0, 1)$, then

$$f^{-1}(\mathrm{cl}(B)) = f^{-1}(A) = 0_\sim \subseteq \mathrm{cl}(f^{-1}(B)).$$

Suppose $B(x) \neq (0, 1)$. Then $f^{-1}(B) \neq 0_\sim$ and hence

$$f^{-1}(\mathrm{cl}(B)) \subseteq 1_\sim = \mathrm{cl}(f^{-1}(B)).$$

But $f(1_\sim) = A^c$ (note that f is not onto) is not an IFCS in $(X, \mathcal{T}_2)$ and hence f is not a closed map.

However, in case f is a bijection, we have the following theorem.

THEOREM 3.11. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection. Then the following statements are equivalent:*

(1) *f is a closed map.*
(2) $\mathrm{cl}(f(A)) \subseteq f(\mathrm{cl}(A))$ *for each* IFS A *of* X.
(3) $f^{-1}(\mathrm{cl}(B)) \subseteq \mathrm{cl}(f^{-1}(B))$ *for each* IFS B *of* Y.

Proof. By Theorem 3.8, it suffices to show that (2) is equivalent to (3). Let B be any IFS of Y. Then $f^{-1}(B)$ is an IFS of X. Since f is onto,

$$\mathrm{cl}(B) = \mathrm{cl}(ff^{-1}(B)) \subseteq f(\mathrm{cl}(f^{-1}(B)).$$

So $f^{-1}(\mathrm{cl}(B)) \subseteq f^{-1}f(\mathrm{cl}(f^{-1}(B)))$. Since f is one-to-one,

$$f^{-1}(\mathrm{cl}(B)) \subseteq f^{-1}f(\mathrm{cl}(f^{-1}(B))) = \mathrm{cl}(f^{-1}(B)).$$

Conversely, let A be any IFS of X. Then $f(A)$ is an IFS of Y. Since f is one-to-one,

$$f^{-1}(\mathrm{cl}(f(A))) \subseteq \mathrm{cl}(f^{-1}f(A)) = \mathrm{cl}(A).$$

So $ff^{-1}(\mathrm{cl}(f(A))) \subseteq f(\mathrm{cl}(A))$. Since f is onto,

$$\mathrm{cl}(f(A)) = ff^{-1}(\mathrm{cl}(f(A))) \subseteq f(\mathrm{cl}(A)).$$

Hence the theorem follows. □

From Theorem 3.6, Theorem 3.7 and Theorem 3.11 we have the following result.

THEOREM 3.12. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ be a bijection. Then the following statements are equivalent:*

(1) *f is a homeomorphism.*
(2) *f is continuous and closed.*
(3) *$f(\mathrm{cl}(A)) = \mathrm{cl}(f(A))$ for each* IFS *A of X.*
(4) *$\mathrm{cl}(f^{-1}(B)) = f^{-1}(\mathrm{cl}(B))$ for each* IFS *B of Y.*
(5) *$f^{-1}(\mathrm{int}(B)) = \mathrm{int}(f^{-1}(B))$ for each* IFS *B of Y.*
(6) *$\mathrm{int}(f(A)) = f(\mathrm{int}(A))$ for each* IFS *A of X.*

4. The Category of Intuitionistic Fuzzy Topological Spaces

Let **CFt** be the category of all Chang's fuzzy topological spaces and fuzzy continuous maps and **IFt** the category of all IFTSs and continuous maps.

Recall ([3]) that there are functors $G_1, G_2 : \mathbf{IFt} \to \mathbf{CFt}$ defined by

$$G_1(X, \mathcal{T}) = (X, G_1(\mathcal{T})) \quad \text{and} \quad G_1(f) = f,$$

where $G_1(\mathcal{T}) = \{\mu_A \mid A = (\mu_A, \gamma_A) \in \mathcal{T}\}$,

$$G_2(X, \mathcal{T}) = (X, G_2(\mathcal{T})) \quad \text{and} \quad G_2(f) = f,$$

where $G_2(\mathcal{T}) = \{1 - \gamma_A \mid A = (\mu_A, \gamma_A) \in \mathcal{T}\}$.

THEOREM 4.1. *Define* $F_1 : \mathbf{CFt} \to \mathbf{IFt}$ *by*

$$F_1(X, T) = (X, F_1(T)) \quad \text{and} \quad F_1(f) = f,$$

where $F_1(T) = \{A = (\mu_A, \gamma_A) \mid \mu_A \in T, \mu_A + \gamma_A \leq 1\}$. *Then* F_1 *is a functor.*

Proof. First, we show that $F_1(T)$ is an IFT. Clearly $0_\sim, 1_\sim \in F_1(T)$. Let $A_1 = (\mu_{A_1}, \gamma_{A_1})$, $A_2 = (\mu_{A_2}, \gamma_{A_2})$ in $F_1(T)$. Then $\mu_{A_1}, \mu_{A_2} \in T$ and $\mu_{A_1} + \gamma_{A_1} \leq 1$, $\mu_{A_2} + \gamma_{A_2} \leq 1$. So $\mu_{A_1} \leq 1 - \gamma_{A_1}$ and $\mu_{A_2} \leq 1 - \gamma_{A_2}$. Thus $\mu_{A_1} \wedge \mu_{A_2} \leq (1 - \gamma_{A_1}) \wedge (1 - \gamma_{A_2}) = 1 - (\gamma_{A_1} \vee \gamma_{A_2})$. Hence $\mu_{A_1} \wedge \mu_{A_2} \in T$ and $(\mu_{A_1} \wedge \mu_{A_2}) + (\gamma_{A_1} \vee \gamma_{A_2}) \leq 1$. Thus

$$A_1 \cap A_2 = (\mu_{A_1} \wedge \mu_{A_2}, \gamma_{A_1} \vee \gamma_{A_2}) \in F_1(T).$$

Let $A_i = (\mu_{A_i}, \gamma_{A_i}) \in F_1(T)$ for all $i \in \Gamma$. Then for each $i \in \Gamma$, $\mu_{A_i} \in T$ and $\mu_{A_i} + \gamma_{A_i} \leq 1$. So $\bigvee \mu_{A_i} \in T$ and $(\bigvee \mu_{A_i}) + (\bigwedge \gamma_{A_i}) \leq 1$. Hence

$$\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \gamma_{A_i}) \in F_1(T).$$

Therefore $(X, F_1(T))$ is an IFTS. Next, we show that if $f : (X, T) \to (Y, U)$ is fuzzy continuous then $f : (X, F_1(T)) \to (Y, F_1(U))$ is continuous. Let $B = (\mu_B, \gamma_B) \in F_1(U)$. Then $\mu_B \in U$ and $\mu_B + \gamma_B \leq 1$. Since $f : (X, T) \to (Y, U)$ is fuzzy continuous, $f^{-1}(\mu_B) \in T$. Since $\mu_B + \gamma_B \leq 1$, $f^{-1}(\mu_B)(x) + f^{-1}(\gamma_B)(x) = \mu_B(f(x)) + \gamma_B(f(x)) \leq 1$ for any $x \in X$. So $f^{-1}(\mu_B) + f^{-1}(\gamma_B) \leq 1$ and hence

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \in F_1(T).$$

Therefore $f : (X, F_1(T)) \to (Y, F_1(U))$ is continuous. In all, F_1 is a functor. □

THEOREM 4.2. *The functor $F_1 : \mathbf{CFt} \to \mathbf{IFt}$ is a left adjoint of the functor $G_1 : \mathbf{IFt} \to \mathbf{CFt}$.*

Proof. For any (X, T) in **CFt**, $1_X : (X, T) \to G_1F_1(X, T) = (X, T)$ is a fuzzy continuous map. Consider $(Y, \mathcal{U}) \in \mathbf{IFt}$ and a fuzzy continuous map $f : (X, T) \to G_1(Y, \mathcal{U})$. In order to show that $f : F_1(X, T) = (X, F_1(T)) \to (Y, \mathcal{U})$ is a continuous map, let $B = (\mu_B, \gamma_B) \in \mathcal{U}$. Then $\mu_B \in G_1(\mathcal{U})$. Since $f : (X, T) \to G_1(Y, \mathcal{U}) = (Y, G_1(\mathcal{U}))$ is fuzzy continuous, $f^{-1}(\mu_B) \in T$. Since $\mu_B + \gamma_B \leq 1$, $f^{-1}(\mu_B) + f^{-1}(\gamma_B) \leq 1$ and hence $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \in F_1(T)$. Thus $f : F_1(X, T) \to (Y, \mathcal{U})$ is continuous. Therefore 1_X is a G_1-universal map for (X, T) in **CFt**. □

THEOREM 4.3. *Define $F_2 : \mathbf{CFt} \to \mathbf{IFt}$ by*

$$F_2(X, T) = (X, F_2(T)) \quad \text{and} \quad F_2(f) = f,$$

where $F_2(T) = \{A = (\lambda_A, \theta_A) \mid 1 - \theta_A \in T, \lambda_A + \theta_A \leq 1\}$. Then F_2 is a functor.

Proof. First, we show that $F_2(T)$ is an IFT. Clearly $0_\sim, 1_\sim \in F_2(T)$. Let $A_1 = (\lambda_{A_1}, \theta_{A_1})$, $A_2 = (\lambda_{A_2}, \theta_{A_2})$ in $F_2(T)$. Then $1 - \theta_{A_1}, 1 - \theta_{A_2} \in T$ and $\lambda_{A_1} + \theta_{A_1} \leq 1$, $\lambda_{A_2} + \theta_{A_2} \leq 1$. So $1 - (\theta_{A_1} \vee \theta_{A_2}) = (1 - \theta_{A_1}) \wedge (1 - \theta_{A_2}) \in T$ and $(\lambda_{A_1} \wedge \lambda_{A_2}) + (\theta_{A_1} \vee \theta_{A_2}) \leq 1$. Thus

$$A_1 \cap A_2 = (\lambda_{A_1} \wedge \lambda_{A_2}, \theta_{A_1} \vee \theta_{A_2}) \in F_2(T).$$

Let $A_i = (\lambda_{A_i}, \theta_{A_i}) \in F_1(T)$ for all $i \in \Gamma$. Then for each $i \in \Gamma$, $1 - \theta_{A_i} \in T$ and $\lambda_{A_i} + \theta_{A_i} \leq 1$. So $1 - \bigwedge \theta_{A_i} = \bigvee (1 - \theta_{A_i}) \in T$ and $(\bigvee \lambda_{A_i}) + (\bigwedge \theta_{A_i}) \leq 1$. Hence

$$\bigcup A_i = (\bigvee \lambda_{A_i}, \bigwedge \theta_{A_i}) \in F_2(T).$$

Therefore $(X, F_2(T))$ is an IFTS. Next, we show that if $f : (X, T) \to (Y, U)$ is fuzzy continuous then $f : (X, F_2(T)) \to (Y, F_2(U))$ is continuous. Let $B = (\lambda_B, \theta_B) \in F_2(U)$. Then $1 - \theta_B \in U$ and $\lambda_B + \theta_B \leq 1$. Since $f : (X, T) \to (Y, U)$ is fuzzy continuous, $1 - f^{-1}(\theta_B) = f^{-1}(1 - \theta_B) \in T$. Since $\lambda_B + \theta_B \leq 1$, $f^{-1}(\lambda_B)(x) + f^{-1}(\theta_B)(x) =$

$\lambda_B(f(x)) + \theta_B(f(x)) \leq 1$ for any $x \in X$. So $f^{-1}(\lambda_B) + f^{-1}(\theta_B) \leq 1$ and hence

$$f^{-1}(B) = (f^{-1}(\lambda_B), f^{-1}(\theta_B)) \in F_2(T).$$

Therefore $f : (X, F_2(T)) \to (Y, F_2(U))$ is continuous. In all, F_2 is a functor. □

THEOREM 4.4. *The functor* $F_2 : \mathbf{CFt} \to \mathbf{IFt}$ *is a left adjoint of the functor* $G_2 : \mathbf{IFt} \to \mathbf{CFt}$.

Proof. For any (X, T) in **CFt**, $1_X : (X, T) \to G_2F_2(X, T) = (X, T)$ is a fuzzy continuous map. Consider $(Y, \mathcal{U}) \in \mathbf{IFt}$ and a fuzzy continuous map $f : (X, T) \to G_2(Y, \mathcal{U})$. We will show that $f : F_2(X, T) = (X, F_2(T)) \to (Y, \mathcal{U})$ is a continuous map. Let $B = (\mu_B, \gamma_B) \in \mathcal{U}$. Then $1 - \gamma_B \in G_2(\mathcal{U})$. Since $f : (X, T) \to G_2(Y, \mathcal{U}) = (Y, G_2(\mathcal{U}))$ is fuzzy continuous, $1 - f^{-1}(\gamma_B) = f^{-1}(1 - \gamma_B) \in T$. Since $\mu_B + \gamma_B \leq 1$, $f^{-1}(\mu_B) + f^{-1}(\gamma_B) \leq 1$ and hence $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \in F_2(T)$. Thus $f : F_2(X, T) \to (Y, \mathcal{U})$ is continuous. Therefore 1_X is a G_2-universal map for (X, T) in **CFt**. □

Let (X, T) be a Chang's fuzzy topological space. Then $T_c = \{A = (\mu_A, 1 - \mu_A) \mid \mu_A \in T\}$ is an IFT. In this case, we call T_c a *c-intuitionistic fuzzy topology* and (X, T_c) a *c-intuitionistic fuzzy topological space*. Also **c-IFt** denote the category of all c-intuitionistic fuzzy topological spaces and continuous maps.

THEOREM 4.5. *Two categories* **CFt** *and* **c-IFt** *are isomorphic.*

Proof. Defined $F : \mathbf{CFt} \to \mathbf{c\text{-}IFt}$ by

$$F(X, T) = (X, T_c) \quad \text{and} \quad F(f) = f.$$

Consider the restriction $G_1 : \mathbf{c\text{-}IFt} \to \mathbf{CFt}$ of the functor G_1 in the beginning of this section. Then F and G_1 are functors. Clearly $G_1F(X, T) = G_1(X, T_c) = (X, G_1(T_c)) = (X, T)$. In order to show that $FG_1(X, \mathcal{T}) = (X, \mathcal{T})$, let $(X, \mathcal{T}) \in \mathbf{c\text{-}IFt}$ and $A = (\mu_A, \gamma_A) \in \mathcal{T}$. Then $\gamma_A = 1 - \mu_A$. So $\mu_A \in G_1(\mathcal{T})$ and hence $A = (\mu_A, \gamma_A) = (\mu_A, 1 - \mu_A) \in (G_1(\mathcal{T}))_c$. Thus $FG_1(X, \mathcal{T}) = F(X, G_1(\mathcal{T})) = (X, (G_1(\mathcal{T}))_c) = (X, \mathcal{T})$. □

THEOREM 4.6. *The category* **c-IFt** *is a bireflective full subcategory of* **IFt**.

Proof. Clearly **c-IFt** is a full subcategory of **IFt**. Take any $(X, \mathcal{T})$ in **IFt**. Let $\mathcal{T}^* = \{A \in \mathcal{T} \mid A = (\mu_A, 1-\mu_A)\}$. Then $(X, \mathcal{T}^*) \in$ **c-IFt** and $1_X : (X, \mathcal{T}) \to (X, \mathcal{T}^*)$ is a continuous map. Consider $(Y, \mathcal{U}) \in$ **c-IFt** and a continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$. We need only to check that $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous map. Let $B \in \mathcal{U}$. Since $(Y, \mathcal{U})$ is a c-intuitionistic fuzzy topological space, $B = (\mu_B, 1 - \mu_B)$. Since $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is continuous, $f^{-1}(B) \in \mathcal{T}$. Also,

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(1 - \mu_B)) = (f^{-1}(\mu_B), 1 - f^{-1}(\mu_B)).$$

Thus $f^{-1}(B) \in \mathcal{T}^*$. Hence $f : (X, \mathcal{T}^*) \to (Y, \mathcal{U})$ is a continuous map.□

COROLLARY 4.7. *The category* **CFt** *is a bireflective full subcategory of* **IFt**.

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