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GENERALIZED SOLUTIONS OF IMPULSIVE CONTROL SYSTEMS AND REACHABLE SETS

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ABSTRACT. This paper is concerned with the impulsive Cauchy problem

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}_i,\ t \in [0,T],\ x(0) = \bar{x},$$

where u is a possibly discontinuous vector-valued function and $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$ are suitably smooth functions. We show that the input-output map is Lipschitz continuous and investigate compactness of reachable sets.

1. Introduction

Consider the Cauchy problem for an impulsive control system of the form

$$(1.1)\qquad \begin{cases} \dot{x}(t) = F(t,x,u) + \sum_{i=1}^{m} G_i(t,x,u)\dot{u}_i(t), & t \in [0,T], \\ x(0) = \bar{x} \in \mathbb{R}^n, \end{cases}$$

where $u = (u_1, \cdots, u_m)$ is a control function and the dot denotes the derivative with respect to time. We assume that the vector field F is bounded and Lipschitz continuous, the vector fields $G_i (i = 1, \cdots, m)$

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are Lipschitz continuous and bounded C^2- functions, and control functions u have values in a compact set in $\mathbb{R}^m$.

By adding the variables $x_0, x_{n+1}, \cdots, x_{n+m}$ with equations

$$x_0 = t,\ x_{n+1} = u_1,\ \cdots,\ x_{n+m} = u_m,$$

the system (1.1) is expressed as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}_i, \quad t \in [0, T], \tag{1.2}$$

$$x(0) = (0, \bar{x}_1, \cdots, \bar{x}_n, u_1(0), \cdots, u_m(0)). \tag{1.3}$$

To define the generalized solution of (1.2) and (1.3) corresponding to a control function u, we can consider the impulsive control system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}_i, \quad t \in [0, T], \quad x(0) = \bar{x} \in \mathbb{R}^n. \tag{1.4}$$

If u is a C^1- function, then problem (1.4) has a unique solution in the sense of the Carathéory solution. When u is just measurable, the generalized solution is defined in [3] under the commutative assumption of $g_i{}'s$. When each g_i depends on time and g_i is not smooth with respect to time, the generalized solution of (1.4) corresponding to scalar controls is defined in [13].

We assume that f is Lipschitz continuous, bounded, and g_i are bounded, Lipschitz continuous, twice continuously differentiable and commutative. In this paper, we define the generalized solutions of (1.4) corresponding to bounded measurable functions u (Eventually the definition in this paper is the same as the one in [3], however to prove the continuity of input-output map the generalized solution here is defined in a slightly different way.) and investigate the continuity of the input-output map of the system (1.4).

Consider the optimal problem

(1.5) $$\min_{x(u,T)\in R(T)} C(x(u,T)),$$

where $C : I\!R^n \to I\!R$ is a continuous function and $R(T)$ is a reachable set at time $t = T$ of the system (1.4). If $R(T)$ is compact, then the optimal value of (1.5) exists. The compactness of a reachable set plays an important role on the existence problem of optimal control. We show that the reachable set is compact when u varies in the set of measurable functions whose total variations are uniformly bounded, and provide an example that the reachable set is not compact when u varies in the set of uniformly bounded functions.

2. Generalized Solution and Continuity of the input-output map

Throughout this paper, e_i^n denotes the vector in $I\!R^n$ whose components are all zero but i-th component which is 1, and $\bar{B}_n(0,R)$ is the closed ball in $I\!R^n$ of radius R centered at the origin. For $M > 0$, let

$$\mathcal{U} = \{u = (u_1, \cdots, u_m) | u : [0,T] \to \bar{B}_m(0,M), u \in C^1\}.$$

Let the vector fields $f, g_i : \mathbb{R}^n \to \mathbb{R}^n$ be such that f is Lipschitz continuous and bounded by M_1, and g_i are bounded by M_1, twice continuously differentiable, Lipschitz continuous of rank L, that is,

$$|g_i(x) - g_i(y)| < L|x - y| \text{ for any } x, y \in I\!R^n,\ i = 1, \cdots, m,$$

and commutative, that is,

$$[g_i, g_j](x) \equiv 0, \quad \text{for any } x \in I\!R^n \text{ and } i, j = 1, \cdots, m.$$

Recall that $[f, g]$ is the Lie bracket defined as

$$[f, g] = (D_x g) \cdot f - (D_x f) \cdot g,$$

where $D_x f$ is the Jacobian matrix of the first derivatives of f.

Let $u \in \mathcal{U}$. Consider the Cauchy problem

$$(2.1_1) \qquad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\dot{u}_i, \ t \in [0, T],$$

$$(2.1_2) \qquad x(0) = \bar{x} \in I\!R^n.$$

The solution of (2.1) is uniquely defined and we denote by $x(u, \cdot)$ the solution of (2.1) corresponding to u. We define the generalized solution $x(w, \cdot)$ of (2.1) corresponding to a bounded measurable function w and show that the input-output map $\phi : w \to x(w, \cdot)$ is Lipschitz continuous on the set of uniformly bounded measurable functions.

We write by $\exp(tf)(\bar{x})$ the value at time t of the Cauchy problem

$$\dot{x} = f(x), \quad x(0) = \bar{x}.$$

Due to the commutative assumption of $g_i{}'s$, for $\alpha_i \in I\!R$ $(i = 1, \cdots, m)$,

$$\exp\left(\sum_{i=1}^{m} \alpha_i g_i\right)(\bar{x}) = \exp(\alpha_m g_m) \circ \cdots \circ \exp(\alpha_1 g_1)(\bar{x}).$$

Joining an equation $\dot{z}(t) = \dot{u}(t)$ to the system (2.1). We have the system in $I\!R^{m+n}$

$$(2.2) \qquad \dot{X} = \tilde{f}(X) + \sum_{i=1}^{m} \tilde{g}_i(X)\dot{u}_i, \qquad X(0) = (u(0), \bar{x}) \in \mathbb{R}^{m+n}$$

where $X = (z, x) \in I\!R^{m+n}$, $\tilde{f}(X) = (0, f(x))$ and $\tilde{g}_i(X) = (e_i^m, g_i(x))$.

We introduce a C^2- transformation T and show that system (2.2) is transformed by T to the control system of the form

$$(2.3) \qquad \dot{X} = \bar{f}(X) + \sum_{i=1}^{m} e_i^{m+n}\dot{u}_i, \qquad X(0) = (u(0), \tilde{x}) \in I\!R^{m+n},$$

for some Lipschitz continuous function $\bar{f}$ and $\tilde{x} \in I\!R^n$.

Define a C^2- homeomorphism $T : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ by

$$(z,x) = T(w,y) \tag{2.4}$$

where $T(w,y) = (T_1(w,y), T_2(w,y))$,

$$T_1(w,y) = w \quad \text{and} \quad T_2(w,y) = \exp\left(\sum_{i=1}^{m} w_i g_i\right)(y).$$

For any compact subset $K \subset \mathbb{R}^n$, the map T is Lipschitz continuous on $\bar{B}_m(0,M) \times K$ and the inverse of T is

$$T^{-1}(z,x) = \left(z, \exp\left(\sum_{i=1}^{m} -z_i g_i\right)(x)\right).$$

If $(z(t), x(t))$ is a solution of (2.2) and $(w(t), y(t)) = T^{-1}(z(t), x(t))$, then by [3], $(w(t), y(t))$ satisfies the Cauchy problem

$$\begin{cases} \dot{w} = \dot{u} \\ \dot{y} = F^*(w,y) \\ w(0) = u(0) \\ y(0) = \exp(\sum_{i=1}^{m} -u_i(0) g_i)(\bar{x}), \end{cases} \tag{2.5}$$

where $F^*(w,y)$ is the map from $\bar{B}_m(0,M) \times \mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$F^*(w,y) = D_x\left(\exp\left(\sum_{i=1}^{m} -w_i g_i\right)\right) \cdot f(T_2(w,y)), \tag{2.6}$$

for $w = (w_1, \cdots, w_m)$. Here $D_x(\exp(\sum_{i=1}^{m} w_i g_i))$ is the $n \times n$ Jacobian matrix of the diffeomorphism $x \to \exp(\sum_{i=1}^{m} w_i g_i)(x)$. Thus system (2.2) is transformed into (2.3) by T where $\tilde{f}(X)$ in (2.3) is $(0, F^*(X))$ and $\tilde{x} = \exp(\sum_{i=1}^{m} -u_i(0) g_i)(\bar{x})$.

REMARK 2.1. For $u \in C^1$, x is the solution of (2.1) corresponding to u if and only if $T^{-1}(u,x)$ is the solution of (2.5), and (u,y) is the solution of (2.5) if and only if $Proj \circ T(u,y)$ is the solution of (2.1) corresponding to u, where $Proj$ is the projection from $\mathbb{R}^{n+m}$ to $\mathbb{R}^n$ such that $Proj(z_1,\cdots,z_m,x_1,\cdots,x_n) = (x_1,\cdots,x_n)$.

Let us recall the value of $F^*(w,y)$. If, for $\alpha \in [-M,M]$ and $, v_0, y \in \mathbb{R}^n$, we define $F_1^*(g,\alpha,y,v_0)$ as the value at time α of the solution of the initial valued linear problem

$$\dot{v}(t) = D_x g(\exp(tg)(y)) \cdot v(t), \ \ v(0) = v_0,$$

then by the commutative assumption of g_i

$$\begin{aligned} F^*(w,y) =& F_1^*\Bigg(g_m, -w_m, \exp\Bigg(\sum_{i=1}^{m-1} -w_i g_i\Bigg)(y), \\ & F_1^*\Bigg(g_{m-1}, -w_{m-1}, \exp\Bigg(\sum_{i=1}^{m-2} -w_i g_i\Bigg)(y), \cdots, \\ & F_1^*(g_1, -w_1, y, f(T_2(w,y)))\Bigg)\Bigg). \end{aligned} \tag{2.7}$$

In the next lemma, the existence of the solution of system (2.5) is guaranteed.

LEMMA 2.2. *For any $R > 0$, F^* is Lipschitz continuous on $\bar{B}_m(0,M) \times \bar{B}_n(0,R)$, bounded by $M_1 e^{mnLM}$ and the Lipschitz constant depends only on R when m, n, L, M and M_1 are fixed.*

Proof. We first show that for any $i = 1,\cdots,m$, the map $(\alpha, y, v_0) \to F_1^*(g_i,\alpha,y,v_0)$ is Lipschitz continuous on $[-M,M] \times \bar{B}_n(0, R+(m-1)MM_1) \times \bar{B}_n(0, M_1 e^{(m-1)nLM})$.

Let $i \in \{1,\cdots,m\}$, $y_1, y_2 \in \bar{B}_n(0,R+(m-1)MM_1)$ and $v_0 \in \bar{B}_n(0,M_1 e^{(m-1)nLM})$ and for $j = 1,2$, let v_j be the solution of

$$\dot{v}_j(t) = D_x g_i(\exp(tg_i)(y_j)) \cdot v_j(t), \ \ v_j(0) = v_0.$$

Then for $t \in [-M,M]$, $|v_j(t)| \le |v_0| e^{nLt}$. Since g_i is a C^2- function, every second derivative of g_i is bounded on $\bar{B}_n(0, R+mMM_1)$ and there exists $L_1(R) > 0$ such that for any $\bar{y}_1, \bar{y}_2 \in \bar{B}_n(0, R + mMM_1)$ and $\bar{v} \in I\!R^n$,

$$|D_x g_i(\bar{y}_1) \cdot \bar{v} - D_x g_i(\bar{y}_2) \cdot \bar{v}| \leq L_1 |\bar{y}_1 - \bar{y}_2||\bar{v}|.$$

For any $t \in [-M, M]$,

$$\begin{aligned}
&|\dot{v}_1(t) - \dot{v}_2(t)| \\
&\leq |D_x g_i(\exp(tg_i))(y_1) \cdot v_1(t) - D_x g_i(\exp(tg_i))(y_1) \cdot v_2(t)| \\
&\quad + |D_x g_i(\exp(tg_i))(y_1) \cdot v_2(t) - D_x g_i(\exp(tg_i))(y_2) \cdot v_2(t)| \\
&\leq nL|v_1(t) - v_2(t)| + L_1|y_1 - y_2|M_1 e^{mnLM}.
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
|v_1(t) - v_2(t)| &\leq \int_0^t L_1|y_1 - y_2|M_1 e^{mnLM} e^{nL(t-s)} ds \\
&\leq \frac{L_1}{nL} M_1 e^{mnLM + nLT} |y_1 - y_2|.
\end{aligned}$$

Hence F_1^* is Lipschitz continuous w.r.t. y. By similar computation, F_1^* is Lipschitz continuous w.r.t. α, v_0 and the Lipschitz constant depends only on R.

Since for any $|y| \leq R$ and $j = 1, \cdots, m-1$,

$$\begin{aligned}
&\left|\exp\left(\sum_{i=1}^{m-1} -w_i g_i\right)(y)\right| \leq R + (m-1)MM_1, \\
&\left|F_1^*\left(g_j, -w_j, \exp\left(\sum_{i=1}^{j-1} -w_i g_i\right)(y), \cdots, F_1^*(g_1, -w_1, y, f(T_2(w, y)))\right)\right| \\
&\leq M_1 e^{(m-1)nLM},
\end{aligned}$$

and f, T are Lipschitz continuous on $\bar{B}_m(0, M) \times \bar{B}_n(0, R)$, F^* is Lipschitz continuous and bounded by $M_1 e^{mnLM}$. □

If $X(\cdot)$ is a solution of the system (2.3) corresponding to $u \in \mathcal{U}$, then the function $Y = X - \sum_{i=1}^{m} e_i^{m+n} u_i \in I\!R^{m+n}$ satisfies the Cauchy problem

$$
(2.8)\qquad
\begin{aligned}
\dot{Y} &= \bar{f}\Big(Y + \sum_{i=1}^{m} e_i^{m+n} u_i\Big), \\
Y(0) &= \Big(0, \exp\Big(\sum_{i=1}^{m} -u_i(0) g_i\Big)(\bar{x})\Big),
\end{aligned}
$$

In the above system the differentiation of u does not appear. Thus the Carathéory solution $Y(u,t)$ of (2.8) corresponding to u exists when u is bounded and measurable. Hence relying on Remark 2.1 we can define the generalized solution of (2.1) corresponding to a bounded measurable function u via the solution of (2.8).

DEFINITION 2.3. For a bounded measurable function u on $[0,T]$, $x(u,\cdot)$ is a generalized solution of (2.1) if

$$
x(u,t) = Proj \circ T \left(Y(u,t) + \sum_{i=1}^{m} e_i^{m+n} u_i(t) \right),
$$

where $Y(u,t)$ is a Carathéodory solution of (2.8) corresponding to u.

If $\xi_1(u,t) = (\xi_0(u,t), \xi(u,t))$ is a solution of (2.8) corresponding to a bounded measurable function u with $\xi_0(u,t) \in \mathbb{R}^m$ and $\xi(u,t) \in \mathbb{R}^n$, then $\xi_0 \equiv 0$ and $\xi(u,\cdot)$ satisfies the Cauchy problem

$$
(2.9_1)\qquad \dot{\xi} = F^*(u,\xi),
$$

$$
(2.9_2)\qquad \xi(0) = \exp\Big(\sum_{i=1}^{m} -u_i(0) g_i\Big)(\bar{x}).
$$

Moreover, $\xi_1(u,t) + \sum_{i=1}^{m} e_i^{m+n} u_i(t) = (u(t), \xi(u,t))$ and

$$
Proj \circ T\Big(\xi_1(u.t) + \sum_{i=1}^{m} e_i^{m+n} u_i(t)\Big) = \exp\Big(\sum_{i=1}^{m} u_i(t) g_i\Big)(\xi(u,t)).
$$

Thus $x(u,t) = \exp\left(\sum_{i=1}^{m} u_i(t)g_i\right)(\xi(u,t))$ is the generalized solution of (2.1) corresponding to u.

Conversely, if $x(u,t)$ is the generalized solution of (2.1) corresponding to a bounded measurable function u, then $T^{-1}(u, x(u,t)) - \sum_{i=1}^{m} e_i^{m+n} u_i(t) = (0, \exp(\sum_{i=1}^{m} -u_i(t)g_i)(x(u,t)))$ is the solution of (2.8). Hence $\xi(u,t) = \exp(\sum_{i=1}^{m} -u_i(t)g_i)(x(u,t))$ is the solution of (2.9).

REMARK 2.4. For a bounded measurable function u on $[0,T]$, the function $x(u,t)$ is a solution of (2.1) if and only if $\xi(u,t) = \exp(\sum_{i=1}^{m} u_i(t)g_i)(x(u,t))$ is the Carathéodory solution of (2.9).

Let

$$\mathcal{U}_2 = \{u : [0,T] \to \bar{B}_m(0,M) | \, u \text{ is measurable}\}.$$

For $\tau \in [0,T]$, define the distance on $\mathcal{U}_2$ by

$$d_\tau(u,v) = |u(0) - v(0)| + |u(\tau) - v(\tau)| + \int_0^T |u(s) - v(s)| ds.$$

Now, we prove that the input-output map $\phi : u \to x(u,\cdot)$ is Lipschitz continuous on $\mathcal{U}_2$.

THEOREM 2.5. (a) *There exists a positive constant* $\bar{M}$ *such that*

$$\begin{aligned} &|x(u,\tau) - x(\tilde{u},\tau)| \\ &\le \bar{M}\left[|u(0) - \tilde{u}(0)| + |u(\tau) - \tilde{u}(\tau)| + \int_0^T |u(s) - \tilde{u}(s)| ds\right], \end{aligned} \tag{3.10}$$

for all $u, \tilde{u} \in \mathcal{U}_2$, *and* $\tau \in [0,T]$.

(b) *For* $u \in \mathcal{U}_2$ *and* $n = 0, 1, \cdots$, *let* x_n *be the generalized solution of* (2.1_1) *with* $x_n(0) = \bar{x}_n$. *If* $\bar{x}_n$ *converges to* $\bar{x}_0$, *then* $x_n(\cdot)$ *converges uniformly to* $x_0(\cdot)$.

Proof. (a) Let $u, \tilde{u} \in \mathcal{U}_2$. Since F^* is bounded by $M_1 e^{mnML}$ and $|\exp(-\sum_{i=1}^{m} u_i(0)g_i)(\bar{x})|$, $|\exp(-\sum_{i=1}^{m} \tilde{u}_i(0)g_i)(\bar{x})|$ are bounded by $|\bar{x}|mM_1M$, the solutions of (2.9) are bounded by $|\bar{x}|mM_1M + TM_1e^{mnLM}$ By Lemma 2.1, F^* is Lipschitz continuous of rank L_1 for some L_1 and

$\bar{f} = (0, F^*)$ is also Lipschitz continuous of rank L_1 on $\bar{B}_{m+n}(0, \bar{R})$ where $\bar{R} = M + |\bar{x}|mM_1M + TM_1e^{mnLM}$.

For $\tau \in [0, T]$,

$$\begin{aligned}
&\frac{d}{dt}|Y(u,\tau) - Y(\tilde{u},\tau)| \\
&\leq \left| \bar{f}\Big(Y(u,\tau) + \sum_{i=1}^{m} e_i^{m+n} u_i(\tau)\Big) - \bar{f}\Big(Y(\tilde{u},\tau) + \sum_{i=1}^{m} e_i^{m+n} \tilde{u}_i(\tau)\Big) \right| \\
&\leq L_1(|Y(u,\tau) - Y(\tilde{u},\tau)| + |u(\tau) - \tilde{u}(\tau)|).
\end{aligned}$$

Observing that $|Y(u,0) - Y(\tilde{u},0)| \leq m|u(0) - \tilde{u}(0)|M_1e^{mLM}$, by Gronwall's inequality

$$\begin{aligned}
&|Y(u,\tau) - Y(\tilde{u},\tau)| \\
&\leq |Y(u,0) - Y(\tilde{u},0)|e^{L_1\tau} + \int_0^\tau L_1|u(s) - \tilde{u}(s)|e^{L_1|\tau - s|}ds \\
&\leq |u(0) - \tilde{u}(0)|mM_1e^{mLM + L_1\tau} + \int_0^\tau L_1|u(s) - \tilde{u}(s)|e^{L_1|\tau - s|}ds.
\end{aligned}$$

Since T is continuously differentiable, T is Lipschitz continuous of some rank L_2 on $\bar{B}_{m+n}(0, \bar{R})$ and for any $\tau \in [0, T]$

$$\begin{aligned}
&|x(u,\tau) - x(\tilde{u},\tau)| \\
&\quad\leq \left| T\Big(Y(u,\tau) + \sum_{i=1}^{m+n} e_i^{m+n} u_i\Big) - T\Big(Y(\tilde{u},\tau) + \sum_{i=1}^{m+n} e_i^{m+n} \tilde{u}_i\Big) \right| \\
&\quad\leq L_2\Big(|u(0) - \tilde{u}(0)|mM_1e^{mLM + L_1\tau} \\
&\qquad\quad + \int_0^\tau L_1|u(s) - \tilde{u}(s)|e^{L_1|\tau - s|}ds + |u(\tau) - \tilde{u}(\tau)|\Big) \\
&\quad\leq M_2\left[|u(0) - \tilde{u}(0)| + |u(\tau) - \tilde{u}(\tau)| + \int_0^\tau |u(s) - \tilde{u}(s)|ds\right],
\end{aligned}$$

where $M_2 = L_2(mM_1e^{mLM + L_1T} + L_1e^{2L_1T} + 1)$.

(b) For $n = 0, 1, \cdots$, let ξ_n be a solution of (2.9_1) with

$$\xi_n(0) = \exp(\sum_{i=1}^{m} -u_i(0)g_i)(\bar{x}_n).$$

As $n \to \infty$, $\xi_n(0) \to \xi_0(0)$ and $\xi_n(\cdot)$ converges uniformly to $\xi_0(\cdot)$ on $[0,T]$. Since the set $\{\xi_n(t) : n = 0, 1, \cdots, \ t \in [0,T]\}$ is bounded, $x_n(\cdot)$ converges uniformly to $x_0(\cdot)$. □

Depending on Theorem 2.5 (a), it is natural to define the generalized solution $x(u, \cdot)$ of (2.1) corresponding to a bounded measurable function u for each $t \in [0,T]$ as $x(u,t) = \lim_{n\to\infty} x(u^n, t)$, where $u^n \in C^1$, $u^n \to u$ in L^1, $\lim_{n\to\infty} u^n(0) = u(0)$ and $\lim_{n\to\infty} u^n(t) = u(t)$.

COROLLARY 2.6. *Let $\{u^n\}$ be a sequence in $\mathcal{U}_2$ such that for each $t \in [0,T]$, $u^n(t)$ converges to $u(t) \in \mathcal{U}_2$. Then for any $t \in [0,T]$, $x(u^n, t)$ converges to $x(u,t)$.*

3. Compactness of Reachable sets

Consider the impulsive control system

$$(3.1) \quad \begin{cases} \dot{x}_1(t) = F(u(t), x_1(t)) + \sum_{i=1}^{m} G_i(u(t), x_1(t))\dot{u}_i(t), \quad t \in [0,T], \\ x_1(0) = \bar{x} \in I\!R^n. \end{cases}$$

We assume that F is bounded and Lipschitz continuous in all variables, and G_i are bounded, twice continuously differentiable, Lipschitz continuous and commutative. By introducing the new variable $x_0(t) = u(t)$, (3.1) is equivalent to

$$(3.2) \quad \begin{cases} \dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\dot{u}_i(t), \\ x(0) = (u(0), \bar{x}), \quad t \in [0,T] \end{cases}$$

where $x = (x_0, x_1) \in I\!R^{m+n}$, $f = (0, F)$ and $g_i = (e_i^m, G_i)$.

For $M > 0$, define the set $\mathcal{U}_1$ of control functions by

$$\mathcal{U}_1 = \{u : [0,T] \to I\!R^m | \text{the total variation of } u \text{ on } [0,T] \text{ is less than or equal to } M\},$$

and define the set $\mathcal{U}_2$ of control functions by

$$\mathcal{U}_2 = \{u : [0,T] \to \bar{B}_m(0,M) \mid u \text{ is measurable}\}.$$

Define the reachable sets $R_1(T)$ and $R_2(T)$ as

$$R_1(T) = \{x(u,T) | u \in \mathcal{U}_1\} \quad \text{and} \quad R_2(T) = \{x(u,T) | u \in \mathcal{U}_2\}.$$

We show that in Theorem 3.1 $R_1(T)$ is compact, and provide Example 3.1 in which $R_2(T)$ is not compact.

THEOREM 3.1. *$R_1(T)$ is compact.*

Proof. By Theorem 2.5 (a), there exists $\bar{M} > 0$ such that

$$|x(u,T)| \le |x(0,T)| + \bar{M}\left[|u(0)| - |u(T)| + \int_0^T |u(s)|ds\right] \text{ for any } u \in \mathcal{U}_1,$$

so the set $R_1(T)$ is bounded.

Next we show that $R_1(T)$ is closed. Choose a point Q in the closure of $R_1(T)$ and a sequence $\{Q_n\}$ in the set $R_1(T)$ converging to Q. Since for each $n \in I\!N$, $Q_n \in R_1(T)$, there exists a control function $u^n \in \mathcal{U}_1$ such that $x(u^n,T) = Q_n$. Observing that the total variations of u^n are uniformly bounded, by Theorem 2.1 in [9, p. 11] there exists a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ such that $\lim_{k\to\infty} u^{n_k}(t) = u(t)$ exists for any $t \in [0,T]$ and total variation of u is less than or equal to M. Thus $u \in \mathcal{U}_1$. By Theorem 2.5 (b), Corollary 2.6 and Lebesque dominated convergent theorem, $x(u^{n_k},T)$ converges to $x(u,T)$ as $k \to \infty$. Since $Q_{n_k} = x(u^{n_k},T)$ and Q_{n_k} converges to Q, $Q = x(u,T)$ and Q lies in the set $R_1(T)$. □

In Theorem 3.1, the assumption that control functions u have uniform total variation is essential. If control functions u are just bounded, then the set of $x(u,T)$ is not compact. Before providing an example, we review the relation between the solutions of (3.2) and (3.4).

If $x(u,\cdot)$ is a solution of (3.2) corresponding to $u \in \mathcal{U}_2$ and

$$\xi(u,t) = \exp\Big(\sum_{i=1}^{m} -u_i(t)g_i\Big)(x(u,t)), \tag{3.3}$$

then by Remark 2.4, $\xi(u,\cdot)$ satisfies

$$\begin{cases} \dot{\xi} = F^*(u,\xi) \\ \xi(0) = \exp\Big(\sum_{i=1}^{m} -u_i(0)g_i\Big)(u(0),\bar{x}). \end{cases} \tag{3.4}$$

Conversely, if $\xi(u,\cdot)$ is solution of (3.4) corresponding to u, then

$$x(u,t) = \exp\Big(\sum_{i=1}^{m} u_i(t)g_i\Big)\xi(u,t)$$

is a solution of (3.1).

EXAMPLE 3.2. Consider the impulsive control system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} u \\ 2ux_1 - x_1^2\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\dot{u}, \quad t\in[0,1], \quad x(0)=(-1,0) \tag{3.5}$$

where $u \in \mathcal{U}_2 = \{u : [0,1] \to [-1,1] \mid u \text{ is measurable}\}$. By adding a new variable $x_0 = u$, the system (3.5) is equivalent to

$$\begin{pmatrix}\dot{x}_0\\ \dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} 0 \\ x_0 \\ 2x_0x_1 - x_1^2\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}\dot{u}, \ x(0) = (u(0),-1,0). \tag{3.6}$$

The auxiliary function $\xi(u,\cdot) = (\xi_0,\xi_1,\xi_2)$ corresponding to u is defined as $x(u,\cdot) - u(\cdot)$ by (3.3) and if u is differentiable, then $\xi(u,\cdot)$ satisfies

$$\begin{aligned} \dot{\xi}_0 &= \dot{x}_0 - \dot{u} = 0 \\ \dot{\xi}_1 &= \dot{x}_1 - \dot{u} = \xi_0 + u \\ \dot{\xi}_2 &= \dot{x}_2 - \dot{u} = 2x_0x_1 - {x_i}^2 = u^2 - {\xi_1}^2. \end{aligned}$$

Thus for $u \in \mathcal{U}_2$, the auxiliary function $\xi(u,\cdot)$ satisfies the Cauchy problem

$$(3.7) \qquad \begin{pmatrix} \dot{\xi}_0 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u \\ u^2 - \xi_1{}^2 \end{pmatrix}, \ \xi(0) = (0, -1-u(0), -u(0)).$$

For each $n \in \mathbb{N}$, define the control function u^n on $[0,1]$ by

$$u^n(t) = \begin{cases} -1, & \frac{2k}{2n} \le t < \frac{2k+1}{2n},\ k = 0, \cdots, n-1 \\ 1, & \frac{2k+1}{2n} \le t < \frac{2k+2}{2n},\ k = 0, \cdots, n-1 \\ 1, & t = 1. \end{cases}$$

Then for each $n \in I\!N$, $u^n(t) \in \mathcal{U}_2, u^n(0) = -1$, and $u^n(1) = 1$. Let

$$\xi(u^n,t) = (\xi_0(u^n,t), \xi_1(u^n,t), \xi_2(u^n,t))$$

be the solution of (3.7) corresponding to u^n. Then for each $n \in I\!N$

$$\begin{aligned} \xi_0(u^n,t) &= 0 \\ \xi_1(u^n,t) &= \int_0^t u^n(s)ds - 1 - u^n(0) \\ &= \int_0^t u^n(s)ds \end{aligned}$$

and

$$\begin{aligned} \xi_2(u^n,t) &= \int_0^t \Big(u^n(s)^2 - \xi_1^2(u^n,s)\Big)ds - u^n(0) \\ &= \int_0^t \Big(u^n(s)^2 - \xi_1^2(u^n,s)\Big)ds + 1. \end{aligned}$$

Since $\int_0^t u^n(s)ds$ converges uniformly to 0 and $(u^n)^2 \equiv 1$, $\xi_1(u^n,1) \to 0$ and $\xi_2(u^n,1) \to 2$. Since $x(u^n,1) = \exp(u^n(1)g)(\xi(u^n,1))$ converges to $\exp(1g)\ (0,0,2) = (1,1,3)$, $(1,1,3)$ lies in the closure of $R_2(1)$. Suppose that for some $u \in \mathcal{U}_2$, $x(u,1) = (1,1,3)$. By (3.3),

$$(3.8) \quad \xi(u,1) = \exp(-u(1)g)(1,1,3) = (1-u(1), 1-u(1), 3-u(1)).$$

Since $|u(1)| \leq 1$, we have

$$0 \leq \xi_1(u,1) \leq 2, \text{ and } 2 \leq \xi_2(u,1) \leq 4, \tag{3.9}$$

where $\xi(u,1) = (\xi_0(u,1), \xi_1(u,1), \xi_2(u,1))$. By (3.7), $\dot{\xi}_2(u,t) = u(t)^2 - \xi_1^2(u,t) \leq u(t)^2 \leq 1$ and so

$$\xi_2(u,1) \leq \int_0^1 1ds - u(0) \leq 2. \tag{3.10}$$

By (3.8)-(3.10), $\xi_2(u,1) = 2, u(1) = 1$ and so $\xi_1(u,1) = 0$.

Consequently, if $R_2(1)$ is compact, then there exists $u \in \mathcal{U}_2$ such that $\xi(u,1) = (0,0,2)$. However, this is impossible. In fact, $\xi_2(u,1) = \int_0^1 (u^2(s) - \xi_1^2(u,s))ds - u(0) = 2$ and $|u(t)| \leq 1$ for any $t \in [0,1]$ imply that

$$u(0) = -1, \ \ \xi_1(u,\cdot) \equiv 0 \text{ and } u(t)^2 = 1 \text{ a.e.} \tag{3.11}$$

On the other hand, by (3.7) $\dot{\xi}_1(u,t) = u(t)$ a.e. which contradicts (3.11). Therefore, $R_2(1)$ is not compact.

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