Bull. Korean Math. Soc. **37** (2000), No. 1, pp. 21–36

FIXED POINTS OF NONEXPANSIVE MAPS ON LOCALLY CONVEX SPACES

JOSEPH M. LING

ABSTRACT. In this article we study the relation between subinvariant submean and normal structure in a locally convex topological vector space. This extends in a natural way a result obtained recently by Lau and Takahashi. Our approach also follows closely theirs.

1. Introduction

Let S be a semitopological semigroup, i.e., S is a semigroup with Hausdorff topology such that for each $a \in S$, the maps $s \mapsto as$ and $s \mapsto sa$ from S into S are continuous. Let $l^{\infty}(S)$ be the Banach space of all bounded real-valued functions on S with the supremum norm, and let $RUC(S)$ denote the subspace of *bounded right uniformly continuous* real-valued functions on S. Thus, a bounded real-valued function $f \in RUC(S)$ if and only if the map $a \mapsto r_a f$ from S into $l^{\infty}(S)$ is continuous, where $(r_a f)(s) = f(sa)$ for all $s \in S$. It is well-known that $RUC(S)$ is a norm-closed in $l^{\infty}(S)$. It is also easy to see that $RUC(S)$ contains the constant functions and is left translation invariant, i.e., whenever $f \in RUC(S)$ and $a \in S$, the map $s \mapsto (l_a f)(s) = f(as)$ again belongs to $RUC(S)$. We say that S is left reversible if any two closed right ideals of S have non-empty intersection.

Let E be a separated locally convex topological vector space, and Q a fixed family of continuous seminorms which generates the topology of E. For any $p \in Q$ and $A \subseteq E$, let $\delta_p(A)$ denote the p-diameter of A, i.e.,

$$\delta_p(A) = \sup\{p(x-y) : x, y \in A\}.$$

Received January 14, 1999.

1991 Mathematics Subject Classification: 47H10, 47H20, 43A07.

Key words and phrases: fixed point property, nonexpansive maps, locally convex spaces.

A closed convex subset K of E is said to have normal structure with respect to Q if for each bounded closed convex subset W of K containing more than one element and for each $p \in Q$ with $\delta_p(W) > 0$, there is a point $x \in W$ which is not a diametrical point of W with respect to p, i.e.,

$$\sup\{p(x-y) : y \in W\} < \delta_p(W).$$

Belluce and Kirk [2] first proved that if K is a non-empty weakly compact convex subset of a Banach space and if K has complete normal structure, then every family of commuting nonexpansive self-maps on K has a common fixed point. Later Lim [6] extended this result to a continuous representation of a left reversible semitopological semigroup S as nonexpansive maps on a weakly compact convex set K with normal structure. Recently, Lau and Takahashi [5] used an entirely different approach to obtain a generalization of Lim's fixed point theorem. They proved that if K is a non-empty weakly compact convex subset of a Banach space E which has normal structure, and if S is a semitopological semigroup for which the space $RUC(S)$ admits a so-called left subinvariant submean, then every continuous representation of S as nonexpansive maps on K has a common fixed point. In this article, we shall prove that Lau and Takahashi's result remains valid in the more general setting of a locally convex space E.

2. Some Preliminaries

All topologies in this article are assumed to be Hausdorff. Throughout this article, E will denote a separated locally convex topological vector space, and Q a fixed family of continuous seminorm which generates the topology of E. If K is a closed convex subset of E, and if $p \in Q$, a map $T : K \to K$ is said to be *p-nonexpansive* if for all $x, y \in K$,

$$p(Tx - Ty) \leq p(x-y).$$

If T is p-nonexpansive for all $p \in Q$, we say that it is *Q-nonexpansive.*

A subset C of E is said to be *bounded* if for all $p \in Q$, there is some $\varepsilon > 0$ such that $C \subseteq \{x \in E : p(x) < 1\}$.

Let X be a closed subspace of $l^\infty(S)$ containing constants. By a *submean* on X we mean a real-valued function μ on X satisfying the following properties:

1. $\mu(f+g) \leq \mu(f) + \mu(g)$ for all $f, g \in X$;
2. $\mu(\alpha f) = \alpha\mu(f)$ for all $f \in X$ and $\alpha \geq 0$;
3. for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant function c.

The value of a submean μ on X at f will be denoted by $\mu(f)$ or $\mu_t(f(t))$. We note that the above construction depends only on the set S and not on its semigroup structure.

A subspace X of $l^\infty(S)$ is *left translation invariant* if $l_a(X) \subseteq X$ for all $a \in S$, where $(l_a f)(s) = f(as)$ for all $f \subset X$, and $a, s \in S$. A submean μ on a left translation invariant subspace X of $l^\infty(S)$ is said to be *left subinvariant* if it satisfies:

5. $\mu(l_a f) \geq \mu(f)$ for all $f \in X$ and $a \in S$.

If S is a semitopological semigroup, and C is a non-empty subset of E, then a representation

$$\mathcal{S} = \{T_s : s \in S\}$$

of S as maps from C into itself is continuous if the map $(s, x) \mapsto T_s x$ is continuous from $S \times C$ with the product topology into C. The representation $\mathcal{S}$ is said to be compatible with the subspace X if for each $x, y \in C$ and for each $p \in Q$, the function $t \mapsto p(T_t x - y)$ belongs to X. The following lemma of Lim [6] will be useful for us:

LEMMA 1. *A closed convex subset of E has normal structure if and only if it does not contain a bounded sequence $\{x_n\}$ such that for some $p \in Q$ with $\delta_p(\{x_n\}) > 0$ and some real number $c > 0$,*

$$p(x_n - x_m) \leq c \text{ and } p(x_{n+1} - \overline{x_n}) \geq c - \frac{1}{n^2}$$

for all $m \geq 1, n \geq 1$, where $\overline{x_n} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

3. Normal Structure and Submeans

In this section, we shall establish some properties relating the concepts of normal structure and of submeans.

Let S be a non-empty set and f a function from S into C such that $\{f(s) : s \in S\}$ is bounded, where C is a closed convex subset of a separated locally convex space E which contains more than one point.

For each $x \in C$ and for each $p \in Q$, define

$$f_{p,x}(s) = p(f(s) - x), \; s \in S.$$

Let X be a closed subspace of $l^{\infty}(S)$ containing constants such that $f_{p.x} \in X$ for each $p \in Q$ and for each $x \in C$. Let μ be a submean on X. Define $r_p : C \to \mathbb{R}$ by

$$r_p(x) = \mu(f_{p,x}) = \mu_t(p(f(t) - x)).$$

LEMMA 2. *The function $r_p : C \to \mathbb{R}$ is continuous, convex on C and if $p(x_n) \to \infty$, then $r_p(x_n) \to \infty$.*

Proof. Let $x, y \in C$. Then for all $t \in S$,

$$p(f(t) - x) \leq p(f(t) - y) + p(y - x).$$

Applying μ_t, we get

$$\begin{aligned} r_p(x) &\leq \mu_t[p(f(t) - y) + p(y - x)] \\ &\leq \mu_t[p(f(t) - y)] + \mu_t[p(y - x)] \\ &= r_p(y) + p(y - x). \end{aligned}$$

So,

$$r_p(x) - r_p(y) \leq p(y - x).$$

Similarly,

$$r_p(y) - r_p(x) \leq p(x - y) = p(y - x).$$

It follows that

$$|r_p(x) - r_p(y)| \leq p(y - x),$$

and r_p is continuous.

Next, if $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$, then for all $x, y \in C$, $t \in S$, we have

$$\begin{aligned} p(f(t) - (\alpha x + \beta y)) &= p(\alpha(f(t) - x) + \beta(f(t) - y)) \\ &\leq \alpha p(f(t) - x) + \beta p(f(t) - y). \end{aligned}$$

Applying μ_t, we get

$$\begin{aligned} r_p(\alpha x+\beta y) &\le \mu_t[\alpha p(f(t)-x)+\beta p(f(t)-y)] \\ &\le \alpha\mu_t[p(f(t)-x)]+\beta\mu_t[p(f(t)-y)] \\ &= \alpha r_p(x)+\beta r_p(y), \end{aligned}$$

and so r_p is convex on C.

Finally, since $\{f(s) : s \in S\}$ is bounded, for each $p \in Q$, there is some $\varepsilon_p > 0$ such that

$$p(\varepsilon_p f(s)) < 1 \text{ for all } s \in S.$$

Then for all $x \in C$, and for all $s \in S$,

$$\begin{aligned} p(x) &\le p(x-f(s))+p(f(s)) \\ &< p(x-f(s))+\frac{1}{\varepsilon_p}. \end{aligned}$$

Applying μ_s, we get

$$\begin{aligned} p(x) &< \mu_s\left[p(x-f(s))+\frac{1}{\varepsilon_p}\right] \\ &\le \mu_s[p(x-f(s))]+\mu_s\left[\frac{1}{\varepsilon_p}\right] \\ &= r_p(x)+\frac{1}{\varepsilon_p}. \end{aligned}$$

It follows that if $p(x_n) \longrightarrow \infty$, then $r_p(x_n) > p(x_n) - \frac{1}{\varepsilon_p} \longrightarrow \infty$. □

For each $p \in Q$, let R_p denote the minimum value of r_p on C :

$$R_p = \inf\{r_p(x) : x \in C\};$$

and let M_p denote the set of all points in C at which r_p attains its minimum value R_p :

$$M_p = \{x \in C : r_p(x) = R_p\}$$

LEMMA 3. *If C is non-empty, compact and convex, then for all $p \in Q$, the set M_p is non-empty, closed and convex. Furthermore, if $C = M_p$, then*

$$R_p = \inf_{y\in C}\sup_{x\in C} p(x-y).$$

Further more, if C contains more than one point, and if $C = M_p$ for all $p \in Q$, then there is at least one $p \in Q$ for which $R_p > 0$.

Proof. Since r_p is continuous and convex, r_p is lower semicontinuous. Also, as C is compact, M_p is non-empty, closed and convex.

Suppose $C = M_p$. That is, r_p is constant on C, with

$$r_p(x) = R_p \text{ for all } x \in C.$$

Let $\varepsilon > 0$. For all $x \in C$, consider the set

$$D_x = \{z \in C : p(x - z) \leq R_p + \varepsilon\}.$$

Since p is continuous and C is compact, it follows that each D_x is compact. We now show that the set $D = \cap_{x \in C} D_x$ is non-empty. By compactness, it suffices to show that the intersection of finitely many D_x is non-empty.

So, let $x_1, x_2, \cdots, x_n \in C$. For all $\alpha_1, \alpha_2, \cdots, \alpha_n \in (0,1)$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$\begin{aligned} \mu\left[\sum_{i=1}^{n} \alpha_i f_{p,x_i}\right] &\leq \sum_{i=1}^{n} \alpha_i \mu(f_{p,x_i}) \\ &= \sum_{i=1}^{n} \alpha_i r_p(x_i) \\ &= \sum_{i=1}^{n} \alpha_i R_p \\ &= R_p \\ &< R_p + \varepsilon. \end{aligned}$$

It follows that there exists some $t_0 \in S$ such that

$$\sum_{i=1}^{n} \alpha_i f_{p,x_i}(t_0) \leq R_p + \varepsilon.$$

That is, $z_0 = f(t_0)$ is a point in C which satisfies

$$\sum_{i=1}^{n} \alpha_i p(z_0 - x_i) \leq R_p + \varepsilon.$$

Now, since the maps $z \mapsto p(z - x_i)$, $1 \leq i \leq n$, are convex and lower semicontinuous on C, it follows by Fan's Theorem [3] for convex inequalities on a topological vector space that there exists some $z \in C$ such that

$$p(z - x_i) \leq R_p + \varepsilon \text{ for } i = 1, 2, \cdots, n.$$

Therefore, $z \in \cap_{i=1}^{n} D_{x_i}$ and our proof of the claim that $D \neq \emptyset$ is complete.

So, for all $\varepsilon > 0$, there exists $z_\varepsilon \in D \subseteq C$ such that

$$p(x - z_\varepsilon) \leq R_p + \varepsilon$$

for all $x \in C$. It follows that

$$\sup_{x \in C} p(x - z_\varepsilon) \leq R_p + \varepsilon,$$

and

$$\min_{z \in C} \sup_{x \in C} p(x - z) \leq R_p + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\min_{z \in C} \sup_{x \in C} p(x - z) \leq R_p.$$

Consequently,

$$\begin{aligned}
R_p &= \min_{y \in C} r_p(y) \\
&= \min_{y \in C} \mu_t(p(f(t) - y)) \\
&\leq \min_{y \in C} \mu_t \left[\sup_{s \in S} p(f(s) - y)\right] \\
&= \min_{y \in C} \left[\sup_{s \in S} p(f(s) - y)\right] \\
&\leq \min_{y \in C} \left[\sup_{x \in C} p(x - y)\right] \\
&\leq R_p.
\end{aligned}$$

It follows that

$$R_p = \inf_{y \in C} \sup_{x \in C} p(x - y).$$

Now, suppose that C has more than one point, $C = M_p$ for all $p \in Q$, and $R_p = 0$ for all $p \in Q$. Then we have

$$\inf_{y \in C} \sup_{x \in C} p(x - y) = 0$$

for all $p \in Q$. By compactness of C, there exists some $y_p \in C$ such that

$$\sup_{x \in C} p(x - y_p) = 0,$$

i.e., $p(z - y_p) = 0$ for all $z \in C$. But since Q is a separating family of seminorms and C contains more than one point, there exists $p \in Q$, $z_1, z_2 \in C$ such that $p(z_1 - z_2) > 0$. But then

$$p(z_1 - z_2) \leq p(z_1 - y_p) + p(y_p - z_2) = 0 + 0 = 0,$$

which is a contradiction. □

THEOREM 4. *Let C be a non-empty compact convex subset of a separated locally convex topological vector space E. Let Q be a family of continuous seminorms which generates the topology of E. If C has more than one point and if C has normal structure, then C satisfies the following condition: (P) Whenever S is a semigroup, X is a closed left translation invariant subspace of $l^\infty(S)$ containing constants with a left subinvariant submean μ, and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as Q-nonexpansive maps from C into C compatible with X, then for all $p \in Q$, there exists some $x \in C$ such that the set*

$$M_{p,x} = \{z \in C : \mu_t [p(T_t x - z)] = R_{p,x}\}$$

is a proper subset of C, where

$$R_{p,x} = \inf_{y \in C} \mu_t [p(T_t x - y)].$$

Furthermore, for all $p \in Q$ and for all $x \in C$, the set $M_{p,x}$ is non-empty, closed convex, and T_s-invariant for all $s \in S$, i.e.,

$$T_s(M_{p,x}) \subseteq M_{p,x}$$

for all $p \in Q$, $x \in C$ and $s \in S$.

Proof. Fix any $p \in Q$. Suppose there is not any x that has the property claimed in the theorem. Then for all $x \in C$, $A_{p,x} = C$. That is,

$$\mu_t [p(T_t x - y)] = R_{p,x}$$

for all $y \in C$. Now, apply the above lemma to the function $f(t) = T_t x$, we conclude that

$$R_{p,x} = \inf_{w \in C} \sup_{z \in C} p(z - w)$$

for all $x \in C$. But then this implies that $R_{p,x}$ is independent of x. So, we write $R_p = R_{p,x}$ for any $x \in C$. Thus,

$$R_p = \inf_{w \in C} \sup_{z \in C} p(z - w) = \mu_t [p(T_t x - y)]$$

for all $x, y \in C$.

Let $A_p = \left\{ z \in C : \sup_{t \in S} p(T_t x - y) \le R_p \text{ for all } x \in C \right\}$. It is not hard to see that A_p is non-empty. In fact, since C is compact, the infimum value R_p is attained, i.e., there is some $w_0 \in C$ such that $R_p = \sup_{z \in C} p(z - w_0)$, which implies that $\sup_{t \in S} p(T_t x - w_0) \le R_p$, which in turn implies that $w_0 \in A_p$.

Now, fix $x_0 \in A_p$ and $s \in S$. Then for all $x \in C$, since μ is subinvariant, we have

$$\begin{aligned} R_p &= \mu_t [p(T_t x - x_0)] \\ &\le \mu_t [l_s (p(T_t x - x_0))] \\ &= \mu_t [p(T_{st} x - x_0)] \\ &\le \sup_{t \in S} p(T_{st} x - x_0) \\ &\le \sup_{t \in S} p(T_t x - x_0) \\ &\le R_p. \end{aligned}$$

This means that we have equalities throughout.

Next, since each T_s is p-nonexpansive, we have

$$\begin{aligned} R_p &= \mu_t [p(T_t x - T_s x_0)] \\ &\le \mu_t [l_s (p(T_t x - T_s x_0))] \\ &= \mu_t [p(T_{st} x - T_s x_0)] \\ &= \mu_t [p(T_s (T_t x - x_0))] \\ &\le \mu_t [p(T_t x - x_0)] \\ &\le \sup_{t \in S} p(T_{st} x - x_0) \end{aligned}$$

$$\begin{aligned} &\leq \sup_{t\in S} p\left(T_t x - x_0\right) \\ &\leq R_p, \end{aligned}$$

and we have equalities throughout. Thus, we have established that for all $s \in S$, $x \in C$, and $x_0 \in A_p$,

$$R_p = \mu_t\left[p\left(T_t x - x_0\right)\right] = \mu_t\left[p\left(T_t x - T_s x_0\right)\right].$$

Now, to finish the proof, we would like to invoke Lim's Lemma to derive a contradiction by constructing a sequence $\{x_n\}$ whose existence is supposedly prohibited by the normal structure of C.

With $x_0 \in A_p$, it follows from the equality $R_p = \mu_t\left[p\left(T_t x_0 - x_0\right)\right]$ that there is some $s_1 \in S$ such that

$$p\left(T_{s_1} x_0 - x_0\right) \geq R_p - 1.$$

Let $x_1 = x_0$ and $x_2 = T_{s_1} x_0$. We have

$$p\left(x_2 - x_1\right) \geq R_p - 1.$$

Consider $\overline{x_2} = \frac{1}{2}\left(x_1 + x_2\right)$. We have

$$\begin{aligned} R_p &\leq \mu_t\left[p\left(T_t x_0 - \overline{x_2}\right)\right] \\ &= \mu_t\left[p\left(\frac{1}{2}\left(T_t x_0 - x_0\right) + \frac{1}{2}\left(T_t x_0 - T_{s_1} x_0\right)\right)\right] \\ &\leq \frac{1}{2}\mu_t\left[p\left(\left(T_t x_0 - x_0\right)\right)\right] + \frac{1}{2}\mu_t\left[p\left(T_t x_0 - T_{s_1} x_0\right)\right] \\ &= \frac{1}{2}R_p + \frac{1}{2}R_p \\ &= R_p, \end{aligned}$$

which implies that $R_p = \mu_t\left[p\left(T_t x_0 - \overline{x_2}\right)\right]$. So, by left subinvariance of μ, we get

$$\begin{aligned} R_p &= \mu_t\left[p\left(T_t x_0 - \overline{x_2}\right)\right] \\ &\leq \mu_t\left[l_{s_1}\left(p\left(T_t x_0 - \overline{x_2}\right)\right)\right] \\ &= \mu_t\left[p\left(T_{s_1 t} x_0 - \overline{x_2}\right)\right]. \end{aligned}$$

So, there exists some $s_2 \in S$ such that

$$p\left(T_{s_1 s_2} x_0 - \overline{x_2}\right) \geq R_p - \frac{1}{2^2}.$$

Let $x_3 = T_{s_1 s_2} x_0$. This means

$$p(x_3 - \overline{x_2}) \geq R_p - \frac{1}{2^2}.$$

Moreover, regarding the mutual distances between x_1, x_2, x_3, we have

$$\begin{aligned} p(x_1 - x_2) &= p(T_{s_1} x_0 - x_0) \\ &\leq \sup_{t \in S} p(T_t x_0 - x_0) \\ &= R_p; \end{aligned}$$

$$\begin{aligned} p(x_2 - x_3) &= p(T_{s_1} x_0 - T_{s_1 s_2} x_0) \\ &= p(T_{s_1}(x_0 - T_{s_2} x_0)) \\ &\leq p(x_0 - T_{s_2} x_0) \\ &\leq \sup_{t \in S} p(T_t x_0 - x_0) \\ &= R_p; \end{aligned}$$

and

$$\begin{aligned} p(x_1 - x_3) &= p(T_{s_1 s_2} x_0 - x_0) \\ &\leq \sup_{t \in S} p(T_t x_0 - x_0) \\ &= R_p. \end{aligned}$$

Therefore, we have constructed $x_1, x_2, x_3 \in C$ such that

$$p(x_i - x_j) \leq R_p \text{ and } p(x_{k+1} - \overline{x_k}) \geq R_p - \frac{1}{k^2},$$

for $i, j \in \{1, 2, 3\}$ and $k \in \{1, 2\}$.

For the sake of clarity, we shall illustrate one more step in the inductive construction. Now, $\overline{x_3} = \frac{1}{3}(x_1 + x_2 + x_3)$. Then

$$\begin{aligned} R_p &\leq \mu_t [p(T_t x_0 - \overline{x_3})] \\ &= \mu_t \left[p\left(\frac{1}{3} \sum_{i=1}^{3} (T_t x_0 - x_i) \right) \right] \\ &\leq \frac{1}{3} \sum_{i=1}^{3} \mu_t [p((T_t x_0 - x_i))] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3}\left(R_p + R_p + R_p\right) \\ &= R_p, \end{aligned}$$

from which it follows that

$$\begin{aligned} R_p &= \mu_t\left[p\left(T_t x_0 - \overline{x_3}\right)\right] \\ &\leq \mu_t\left[l_{s_1 s_2} p\left(T_t x_0 - \overline{x_3}\right)\right] \\ &= \mu_t\left[p\left(T_{s_1 s_2 t} x_0 - \overline{x_3}\right)\right]. \end{aligned}$$

Therefore, there exists some $s_3 \in S$ such that

$$R_p - \frac{1}{3^2} \leq p\left(T_{s_1 s_2 s_3} x_0 - \overline{x_3}\right).$$

Let $x_4 = T_{s_1 s_2 s_3} x_0$. Then

$$p\left(x_4 - \overline{x_3}\right) \geq R_p - \frac{1}{3^2}.$$

Furthermore,

$$\begin{aligned} \rho\left(x_1 - x_4\right) &= p\left(T_{s_1 s_2 s_3} x_0 - x_0\right) \\ &\leq \sup_{t \in S} p\left(T_t x_0 - x_0\right) \\ &= R_p; \end{aligned}$$

$$\begin{aligned} p\left(x_2 - x_4\right) &= p\left(T_{s_1} x_0 - T_{s_1 s_2 s_3} x_0\right) \\ &= p\left(T_{s_1}\left(x_0 - T_{s_2 s_3} x_0\right)\right) \\ &\leq p\left(x_0 - T_{s_2 s_3} x_0\right) \\ &\leq \sup_{t \in S} p\left(T_t x_0 - x_0\right) \\ &= R_p; \end{aligned}$$

and

$$\begin{aligned} p\left(x_3 - x_4\right) &= p\left(T_{s_1 s_2} x_0 - T_{s_1 s_2 s_3} x_0\right) \\ &= p\left(T_{s_1 s_2}\left(x_0 - T_{s_3} x_0\right)\right) \\ &\leq p\left(x_0 - T_{s_3} x_0\right) \\ &\leq \sup_{t \in S} p\left(T_t x_0 - x_0\right) \\ &= R_p. \end{aligned}$$

Therefore, we have constructed $x_1, x_2, x_3, x_4 \in C$ such that

$$p(x_i - x_j) \leq R_p \text{ and } p(x_{k+1} - \overline{x_k}) \geq R_p - \frac{1}{k^2},$$

for $i, j \in \{1, 2, 3, 4\}$ and $k \in \{1, 2, 3\}$. The full induction argument should now be clear. We now have a sequence $\{x_n\}$ in C such that

$$p(x_m - x_n) \leq R_p \text{ and } p(x_{n+1} - \overline{x_n}) \geq R_p - \frac{1}{n^2},$$

for all $m \geq 1, n \geq 1$. To apply Lim's Lemma, it suffices to check that the sequence $\{x_n\}$ is bounded and that $R_p > 0$ and $\delta_p(\{x_n\}) > 0$ for at least one choice of $p \in Q$. Now, the first statement follows from the compactness of C. As for the second statement, we first note that by the preceding Lemma, there is some $p \in Q$ for which $R_p > 0$. Now, for this particular p, if $\delta_p(\{x_n\}) = 0$, then for all $m \geq 1$ and $n \geq 1$,

$$p(x_m - x_n) = 0,$$

and so, for all $n \geq 1$,

$$\begin{aligned} R_p - \frac{1}{n^2} &\leq p(x_{n+1} - \overline{x_n}) \\ &= p\left(\frac{1}{n}\sum_{i=1}^{n}(x_{n+1} - x_i)\right) \\ &\leq \frac{1}{n}\sum_{i=1}^{n} p(x_{n+1} - x_i) \\ &= 0, \end{aligned}$$

which implies that $R_p \leq 0$, a contradiction.

Finally, it remains to prove the assertions about $M_{p,x}$. The only non-trivial part is T_s-invariance. So, let $y \in M_{p,x}$ and let $s \in S$. We have

$$\begin{aligned} R_{p,x} &\leq \mu_t[p(T_t x - T_s y)] \\ &\leq \mu_t[l_s p(T_t x - T_s y)] \\ &= \mu_t[p(T_{st} x - T_s y)] \\ &= \mu_t[p(T_s(T_t x - y))] \\ &\leq \mu_t[p(T_t x - y)] \\ &= R_{p,x}. \end{aligned}$$

It follows that we have equalities throughout, and in particular, $R_{p,x} = \mu_t [p(T_t x - T_s y)]$, and $T_s y \in M_{p,x}$. □

REMARK 5. We note that in Lau and Takahashi's original Banach space version, the fact that the sequence $\{x_n\}$ is bounded with positive diameter is trivial since the norm separates all distinct points. In our case, we obtain not just one sequence $\{x_n\}$, but rather, a sequence $\{x_n\}$ associated with each seminorm p. The fact that seminorms do not necessarily separate distinct points, and that we are dealing with a family of sequences parametrized by the seminorms forces us to take special care to establish the claim for positive p-diameter for AT LEAST ONE p. It turns out that this is all that is needed to invoke the power of Lim's Lemma.

4. Fixed Point Theorems

In this section, we shall obtain an extension of Lau and Takahashi's fixed point theorem for left reversible semigroups of nonexpansive maps.

THEOREM 6. *Let S be a semitopological semigroup, let D be a nonempty compact convex subset of a separated locally convex topological vector space E which has normal structure. Let Q denote a fixed family of continuous seminorms which generates the topology of E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as Q-nonexpansive maps from D into D. If $RUC(S)$ has a left subinvariant submean, then $\mathcal{S}$ has a common fixed point in D.*

Proof. For each $p \in Q$, $x \in D$ and $y \in E$, define the function $h_p : S \to \mathbb{R}$ by

$$h_p(t) = p(T_t x - y), \ t \in S.$$

Since D is compact, $h_p \in l^\infty(S)$. We claim that $h_p \in RUC(S)$. Indeed, for all $s, u \in S$,

$$\begin{aligned} \|r_s h_p - r_u h_p\| &= \sup_{t\in S} |(r_s h_p)(t) - (r_u h_p)(t)| \\ &= \sup_{t\in S} |h_p(ts) - h_p(tu)| \\ &= \sup_{t\in S} |p(T_{ts}x - y) - p(T_{tu} - y)| \end{aligned}$$

$$\begin{aligned} &\leq \sup_{t\in S} p(T_{ts}x - T_{tu}y) \\ &= \sup_{t\in S} p\left(T_t\left(T_s x - T_u y\right)\right) \\ &\leq \sup_{t\in S} p\left(T_s x - T_u y\right) \\ &= p\left(T_s x - T_u y\right). \end{aligned}$$

Since for all x, the map $s \mapsto T_s x$ is continuous, it follows that the map $s \mapsto r_s h_p$ from S into $l^\infty(S)$ is continuous, i.e., $h_p \in RUC(S)$.

A simple application of the Zorn's lemma shows that there exists a minimal non-empty closed convex subset C of D which is T_s-invariant for all $s \in S$. We show that C is a singleton. Indeed, supposing C has more than one element, we are in a situation to apply the above theorem. Let μ be a left subinvariant submean on $RUC(S)$. Then, by the above theorem, for all $x \in C$, and for all $p \in Q$, the set $M_{p,x}$ is a non-empty closed convex subset of C which is T_s-invariant for all $s \in S$. In particular, $M_{p,x} \subseteq C$. Yet, by the minimality of C, we have $C \subseteq M_{p,x}$ for all x. Thus, $C = M_{p,x}$ for all x and p. This contradicts the conclusion of the preceding theorem that at least one of the $M_{p,x}$ must be a proper subset of C. This contradiction shows that C has to be a singleton, say $\{x\}$. But then the fact that C is T_s-invariant for all $s \in S$ that x is a common fixed point for $\mathcal{S}$. □

COROLLARY 7. *Let S be a left reversible semitopological semigroup. Let D be a non-empty compact convex subset of a separated locally convex topological vector space E which has normal structure. Let Q denote a fixed family of continuous seminorms which generates the topology of E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as Q-nonexpansive self maps on D. Then $\mathcal{S}$ has a fixed point in D.*

Proof. As noted in Lau and Takahashi's paper, if S is left reversible, there is a natural left subinvariant submean on $RUC(S)$, namely

$$\mu(f) = \inf_{s\in S} \sup_{t\in sS} f(t).$$

□

REMARK 8. It has been proved by Hsu [4] (see also [1]) that if S is discrete and left reversible, and $\mathcal{S} = \{T_s : s \in S\}$ is a representation of S as weakly continuous nonexpansive mappings on a weakly compact convex subset C of a Banach space, then C has a common fixed point for

S. Lau and Takahashi's result extends this to the case of semitopological semigroups. We covered the case for locally convex spaces here.

ACKNOWLEDGMENTS. Part of this article was prepared during the author's sabbatical leave. The author would like to thank the University of Calgary for granting such leave, and Professor Anthony Lau and the University of Alberta for all the support, encouragement, and hospitality author received during his visit to the University of Alberta.

References

[1] W. Bartoszek, *Nonexpansive actions of topological semigroups on strictly convex Banach space and fixed points*, Proc. Amer. Math. Soc. **104** (1988), 809-811.

[2] L. P. Belluce and W. A. Kirk, *Nonexpansive mappings and fixed points in Banach spaces*, Illinois J. Math. **11** (1967), 474-479.

[3] K. Fan, *Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations*, Math. Z. **68** (1957), 205-217.

[4] R. Hsu, *Topics on weakly almost periodic functions*, Ph.D. thesis, SUNY at Buffalo, 1985.

[5] A. Lau and W. Takahashi, *Invariant submeans and semigroups of nonexpansive mappings on Banach spaces with normal structure*, J. of Func. Anal. **142** No.1 (1996), 79-88.

[6] T. C. Lim, *Characterization of normal structure*, Proc. Amer. Math. Soc. **43** (1974), 313-319.

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA T2N 1N4
E-mail: ling@math.ucalgary.ca