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COINCIDENCE AND SADDLE POINT THEOREMS ON GENERALIZED CONVEX SPACES

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ABSTRACT. We give a new coincidence theorem for multimaps on generalized convex spaces and apply it to deduce ε-saddle point and saddle point theorems.

1. Introduction and Preliminaries

In [8], some ε-saddle point and saddle point theorems for convex sets in topological vector spaces were obtained. These new results generalize the corresponding ones of Komiya [2].

Now it is well-known that convex subsets of topological vector spaces are generalized to convex spaces due to Lassonde [3], which are further extended to the generalized convex spaces or G-convex spaces due to Park [4,5,6,7]. This new class of spaces contains many known spaces having certain abstract convexity without linear structure; see [5].

In the present paper, we deduce a new coincidence theorem for multimaps on G-convex spaces, and use it to deduce new ε-saddle point and saddle point theorems. Consequently, we show that main results in [8] holds for much larger class of spaces.

A *multimap* $T : X \multimap Y$ is a function from X into the power set 2^Y of Y with *fibers* $T^{-}y := \{x \in X : y \in Tx\}$ for $y \in Y$. A function $f : X \to \mathbb{R}$ on a topological space X is said to be *lower* (resp. *upper*) *semicontinuous* if the set $\{x \in X : f(x) > \alpha\}$ (resp. $\{x \in X : f(x) < \alpha\}$) is open in X for every real number α.

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Given a set A, let $\langle A\rangle$ denote the collection of all nonempty finite subsets of A and $|A|$ the cardinality of A. Let Δ_n be the standard n-simplex.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D\rangle$ with $|A| = n+1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $\phi_A(\Delta_J) \subset \Gamma(J)$ for every $J \in \langle A\rangle$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $\Delta_n = \mathrm{co}\{e_0, e_1, \cdots, e_n\}$, $A = \{a_0, a_1, \cdots, a_n\}$, and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$.

Examples of G-convex spaces [6] are convex spaces [3], C-spaces [1], and many others; see [5]. Given a G-convex space $(X, D; \Gamma)$ with $D \subset X$, a subset K of X is said to be Γ*-convex* if for each $A \in \langle D\rangle$, $A \subset K$ implies $\Gamma(A) \subset K$. For a nonempty subset K of X we define the Γ*-convex hull* of K

$$\Gamma\text{-co } K := \bigcap\{B \subset X : B \text{ is } \Gamma\text{-convex and } K \subset B\}.$$

Then the Γ-convex hull of K is the smallest Γ-convex set containing K.

If $D = X$, then $(X, D; \Gamma)$ will be denoted by (X, Γ). Let $\mathrm{Int}_K A$ denote the interior of A in K.

Given $\varepsilon > 0$, a function $f : X \times Y \to \mathbb{R}$ has an ε*-saddle point* $(x^*_\varepsilon, y^*_\varepsilon)$ if

$$f(x, y^*_\varepsilon) - \varepsilon < f(x^*_\varepsilon, y^*_\varepsilon) < f(x^*_\varepsilon, y) + \varepsilon$$

for all $x \in X$ and $y \in Y$; and a point (x^*, y^*) is a *saddle point* of f if

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y)$$

for all $x \in X$ and $y \in Y$; see [8].

Let X and Y be topological spaces, K a subset of X and L a subset of Y. A function $f : X \times Y \to \mathbb{R}$ is said to be α*-transfer lower* (resp. *upper*) *semicontinuous* on K relative to L if for each $(x, y) \in K \times L$, $f(x, y) > \alpha$ (resp. $f(x, y) < \alpha$) implies that there exists an open neighborhood $N(x)$ of x in K and a point $y' \in L$ such that $f(z, y') > \alpha$ (resp. $f(z, y') < \alpha$) for all $z \in N(x)$; and *transfer lower* (resp. *upper*) *semicontinuous* on K relative to L if f is α-transfer lower (resp.

upper) semicontinuous on K relative to L for each $\alpha \in \mathbb{R}$; see Tian [9]. These concepts are proper generalizations of lower (resp. upper) semicontinuous real-valued functions.

2. The Coincidence Theorem

We begin with the following lemmas due to the first author [4].

LEMMA 1. *Let X be a Hausdorff compact space and $(Y, D; \Gamma)$ a G-convex space. Let $T : X \multimap Y$ and $S : X \multimap D$ be multimaps such that the following conditions are satisfied:*

(1) *for each $x \in X$, $A \in \langle Sx \rangle$ implies $\Gamma(A) \subset Tx$; and*
(2) $X = \bigcup \{\mathrm{Int}_X\, S^- y : y \in D\}$.

Then T has a continuous selection $f : X \to Y$ such that $f = g \circ h$, where $g : \Delta_n \to Y$ and $h : X \to \Delta_n$ are continuous functions.

LEMMA 2. *Let (X, Γ) be a Hausdorff compact G-convex space and $T : X \multimap X$ a multimap such that Tx is a Γ-convex set for each $x \in X$, and $X = \bigcup \{\mathrm{Int}_X\, T^- y : y \in X\}$. Then T has a fixed point.*

The following theorem improves and extends a result in [10, Theorem 1] to the case of a G-convex space.

THEOREM 1. *Let X be a Hausdorff topological space, $(Y, D; \Gamma_Y)$ a G-convex space, M and P subsets of $X \times Y$. Suppose that there exist a compact G-convex space (K, Γ_K) with $K \subset X$ and a subset N of $K \times D$ such that*

(1) *for each $x \in K$, Γ-co $\{y \in D : (x, y) \notin N\} \subset \{y \in Y : (x, y) \notin M\}$;*
(2) *for each $x \in K$ with $\{y \in D : (x, y) \notin N\} \neq \emptyset$, there exists $y' \in D$ such that $x \in \mathrm{Int}_K \{x' \in K : (x', y') \notin N\}$;*
(3) *for each $y \in Y$, $\{x \in K : (x, y) \in P\}$ is a Γ-convex subset of (K, Γ_K);*
(4) $Y = \bigcup \{\mathrm{Int}_Y \{y \in Y : (x, y) \in P\} : x \in K\}$; *and*
(5) *for all $(x, y) \in K \times Y$, $(x, y) \in P$ implies $(x, y) \in M$.*

Then there exists a point $x_0 \in K$ such that $\{x_0\} \times D \subset N$.

Proof. Suppose that the conclusion does not hold; that is, for each $x \in K$ there is a point $y_0 \in D$ such that $(x, y_0) \notin N$. For each $x \in K$, let

$$Sx = \{y \in D : (x,y) \notin N\}, \qquad Tx = \{y \in Y : (x,y) \notin M\}.$$

Then for each $x \in K$, $\Gamma\text{-co}\, Sx \subset Tx$ by (1); $K = \bigcup\{\mathrm{Int}_K S^- y : y \in D\}$ by (2). Define a multimap $\tilde{S} : K \multimap Y$ by $\tilde{S}x := \Gamma\text{-co}\ Sx$ for $x \in K$. Since $K = \bigcup\{\mathrm{Int}_K \tilde{S}^- y : y \in Y\}$, by Lemma 1, there is a continuous function $f : K \to Y$ such that $f(x) \in \tilde{S}x \subset Tx$ for all $x \in K$. Hence, $(x, f(x)) \notin M$ for all $x \in K$.

On the other hand, we define a multimap $H : Y \multimap K$ by

$$Hy := \{x \in K : (x,y) \in P\} \qquad \text{for } y \in Y.$$

By (3), Hy is Γ-convex for every $y \in Y$, and $Y = \bigcup\{\mathrm{Int}_Y H^- x : x \in K\}$ by (4). A multimap $F : K \multimap K$ defined by $Fx := H \circ f(x)$ for $x \in K$ has Γ-convex values and $K = \bigcup\{\mathrm{Int}_K F^- y : y \in K\}$. In fact, for every $x \in K$, there is a $y \in K$ such that $f(x) \in \mathrm{Int}_Y H^- y$ and so $x \in f^-(\mathrm{Int}_Y H^- y) \subset \mathrm{Int}_K f^-(H^- y) = \mathrm{Int}_K F^- y$ by the continuity of f. Since (K, Γ_K) is a Hausdorff compact G-convex space, by Lemma 2, there is a point $x_0 \in K$ such that $x_0 \in Fx_0 = H(f(x_0))$; and hence by (5), $(x_0, f(x_0)) \in M$. This contradiction proves the theorem. □

Note that, if X and Y are C-spaces, Theorem 1 reduces to [10, Theorem 1].

Now we give a Fan-Browder type coincidence theorem for G-convex spaces which generalizes [1, Corollary 4.2] and [10, Theorem 5] for C-spaces.

THEOREM 2. *Let X be a Hausdorff topological space, $(Y, D; \Gamma_Y)$ a G-convex space, and $T : X \multimap Y$ and $S : Y \multimap X$ multimaps. Suppose that there exist a compact G-convex space (K, Γ_K) with $K \subset X$ and a multimap $A : K \multimap D$ such that*

(1) *for each $x \in K$, $Ax \subset Tx$, and Tx is* Γ-convex;
(2) $K = \bigcup\{\mathrm{Int}_K A^- y : y \in D\}$;
(3) *for each $y \in Y$, $Sy \cap K$ is Γ-convex in (K, Γ_K); and*
(4) $Y = \bigcup\{\mathrm{Int}_Y S^- x : x \in K\}$.

Then there exist points $x_0 \in K$ and $y_0 \in Y$ such that $y_0 \in Tx_0$ and $x_0 \in Sy_0$.

Proof. Let

$$P = \bigcup{}_{x\in X}\{x\} \times S^{-}x, \quad M = \{(x,y) \in X \times Y : y \notin Tx\} \text{ and}$$
$$N = \{(x,y) \in K \times D : y \notin Ax\}.$$

Suppose that $Tx \cap S^{-}x = \emptyset$ for all $x \in K$. Then for all $(x,y) \in K \times Y$, $(x,y) \in P$ implies $(x,y) \in M$. Since $\{y \in D : (x,y) \notin N\} \subset \{y \in Y : y \in Tx\} = \{y \in Y : (x,y) \notin M\}$, and Tx is Γ-convex for each $x \in K$, condition (1) of Theorem 1 is satisfied. By (2) it is clear that condition (2) of Theorem 1 holds.

For each $y \in Y$, since $\{x \in K : (x,y) \in P\} = Sy \cap K$, by assumption (3), condition (3) of Theorem 1 is also satisfied. By (4), $Y = \bigcup\{\text{Int}_Y\{y \in Y : (x,y) \in P\} : x \in K\}$, that is, condition (4) of Theorem 1 holds. By Theorem 1, there exists a point $x_0 \in K$ such that $\{x_0\} \times D \subset N$; that is, $y \notin Ax_0$ for all $y \in D$. Consequently, we have $Ax_0 = \emptyset$, which contradicts assumption (2) (since $y_0 \in Ax_0$ for some $y_0 \in D$). This completes the proof. □

Note that, even if X and Y are C-spaces, Theorem 2 improves [10, Theorem 5].

3. Main Results

Using our coincidence theorem, we obtain a new ε-saddle point theorem for G-convex spaces which generalizes [8, Theorem 1] for topological vector spaces.

THEOREM 3. *Let X be a Hausdorff topological space, (Y,Γ_Y) a G-convex space, $f : X \times Y \to \mathbb{R}$ a real-valued function and $\varepsilon > 0$. Suppose that there exists a compact G-convex space (K,Γ_K) with $K \subset X$ such that*

(1) *for any $(x,y) \in X \times Y$, $\inf_{v\in Y} f(x,v) > -\infty$ and $\sup_{u\in X} f(u,y) < +\infty$;*

(2) *the function* $(x,y) \mapsto f(x,y) - \inf_{v\in Y} f(x,v)$ *is* ε*-transfer upper semicontinuous on* K *relative to* Y*, and the set* $\{x \in K : f(x,y) > t\}$ *is a nonempty* Γ*-convex set for each* $y \in Y$ *and each* $t \in \mathbb{R}$*;*

(3) *the function* $(x,y) \mapsto f(x,y) - \sup_{u\in X} f(u,y)$ *is* $(-\varepsilon)$*-transfer lower semicontinuous on* Y *relative to* K*, and* $\{y \in Y : f(x,y) < t\}$ *is a nonempty* Γ*-convex set for each* $x \in K$ *and each* $t \in \mathbb{R}$*.*

Then f *has a point* $(x^*_\varepsilon, y^*_\varepsilon) \in K \times Y$ *such that* $f(x, y^*_\varepsilon) - \varepsilon < f(x^*_\varepsilon, y^*_\varepsilon) < f(x^*_\varepsilon, y) + \varepsilon$ *for all* $x \in X$ *and* $y \in Y$*.*

Proof. Let $\varepsilon > 0$. Define multimaps $A : K \multimap Y$, $T : X \multimap Y$ and $S : Y \multimap X$ by

$$Ax = \{y \in Y : f(x,y) - \inf_{v\in Y} f(x,v) < \varepsilon\}$$
$$Tx = \{y \in Y : f(x,y) - \inf_{v\in Y} f(x,v) < \varepsilon\}$$
$$Sy = \{x \in X : f(x,y) - \sup_{u\in X} f(u,y) > -\varepsilon\}.$$

Then for each $x \in K$, $Ax = Tx$, and Tx is a nonempty Γ-convex set. For each $x \in K$, there exists a $y \in Y$ such that $f(x,y) - \inf_{v\in Y} f(x,v) < \varepsilon$. By (2), there exists an open neighborhood $N(x)$ of x in K and a point $y' \in Y$ such that $f(z,y') - \inf_{v\in Y} f(z,v) < \varepsilon$ for all $z \in N(x)$, that is, $N(x) \subset A^- y'$; and hence $x \in \operatorname{Int}_K A^- y'$. Thus $K = \bigcup\{\operatorname{Int}_K A^- y : y \in Y\}$. Moreover, $Sy \cap K$ is a nonempty Γ-convex set for each $y \in Y$ by (2). A similar argument shows by (3) that $Y = \bigcup\{\operatorname{Int}_Y S^- x : x \in K\}$. By Theorem 2, there exists $(x^*, y^*) \in K \times Y$ such that $y^* \in Tx^*$ and $x^* \in Sy^*$; that is, $f(x,y^*) - \varepsilon < f(x^*,y^*) < f(x^*,y) + \varepsilon$ for all $x \in X$ and $y \in Y$. This completes the proof. □

For the case when X and Y are convex spaces in the sense of Lassonde [3] and for mere upper (resp. lower) semicontinuous functions, Theorem 3 improves [8, Theorem 1].

From Theorem 3 we deduce the following new saddle point theorem for spaces without linear structure.

THEOREM 4. *Let* X *be a Hausdorff topological space,* (Y, Γ_Y) *a Hausdorff G-convex space and* $f : X \times Y \to \mathbb{R}$ *a real-valued function.*

Suppose that there exists a compact G-convex space (K, Γ_K) with $K \subset X$ such that

(1) *for any $(x, y) \in X \times Y$, $\inf_{v \in Y} f(x, v) > -\infty$ and $\sup_{u \in X} f(u, y) < +\infty$;*

(2) *the function $(x, y) \mapsto f(x, y) - \inf_{v \in Y} f(x, v)$ is transfer upper semicontinuous on K relative to Y, the function $x \mapsto f(x, y)$ is upper semicontinuous on K for each $y \in Y$; and the set $\{x \in K : f(x, y) > t\}$ is a nonempty Γ-convex set for each $y \in Y$ and $t \in \mathbb{R}$;*

(3) *the function $(x, y) \mapsto f(x, y) - \sup_{u \in X} f(u, y)$ is transfer lower semicontinuous on Y relative to K, and $\{y \in Y : f(x, y) < t\}$ is a nonempty Γ-convex set for each $x \in K$ and each $t \in \mathbb{R}$;*

(4) *for every sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ in $K \times Y$ such that (x_n, y_n) is an ε_n-saddle point of f and $\varepsilon_n \to 0^+$, there exist a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ and a point $y^* \in Y$ such that*

$$\liminf_{k \to \infty} f(x, y_{n_k}) \geq f(x, y^*) \quad \text{for all } x \in X.$$

Then f has a point $(x^*, y^*) \in K \times Y$ such that $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$ for all $x \in X$ and $y \in Y$.

Proof. For each $n \in \mathbb{N}$ with $\varepsilon_n \to 0^+$, by Theorem 3, there is a point $(x_n^*, y_n^*) \in K \times Y$ such that

$$f(x, y_n^*) - \varepsilon_n < f(x_n^*, y_n^*) < f(x_n^*, y) + \varepsilon_n \quad \text{for all } (x, y) \in X \times Y.$$

By (4), there exist a subsequence $\{y_{n_k}^*\}_{k \in \mathbb{N}}$ and a point $y^* \in Y$ such that

$$\liminf_{k \to \infty} f(x, y_{n_k}^*) \geq f(x, y^*) \quad \text{for each } x \in X.$$

Since K is compact, there is a subnet $\{x_\alpha^*\}$ of $\{x_{n_k}^*\}$ and $x^* \in K$ such that $\{x_\alpha^*\}$ converges to x^*.

For each $x \in X$ and each α, we have

$$\begin{aligned} f(x^*, y^*) &= f(x^*, y^*) - f(x_\alpha^*, y^*) + f(x_\alpha^*, y^*) \\ &> f(x^*, y^*) - f(x_\alpha^*, y^*) + f(x, y_\alpha^*) - 2\varepsilon_\alpha \end{aligned}$$

and hence by the uppersemicontinuity of $f(\cdot, y^*)$ on K

$$\begin{aligned} f(x^*, y^*) &\geq f(x^*, y^*) - \limsup_{\alpha} f(x^*_\alpha, y^*) + \liminf_{\alpha} f(x, y^*_\alpha) \\ &\geq f(x, y^*). \end{aligned}$$

Next, for each $y \in Y$ and each α, we have

$$\begin{aligned} f(x^*, y^*) &= f(x^*, y^*) - f(x^*, y^*_\alpha) + f(x^*, y^*_\alpha) \\ &< f(x^*, y^*) - f(x^*, y^*_\alpha) + f(x^*_\alpha, y) + 2\varepsilon_\alpha \end{aligned}$$

and hence by the uppersemicontinuity of $f(\cdot, y)$ on K

$$\begin{aligned} f(x^*, y^*) &\leq f(x^*, y^*) - \liminf_{\alpha} f(x^*, y^*_\alpha) + \limsup_{\alpha} f(x^*_\alpha, y) \\ &\leq f(x^*, y). \end{aligned}$$

Thus, $(x^*, y^*) \in K \times Y$ is a saddle point of f. This completes the proof. □

Note that Theorem 4 is a far-reaching generalization of [8, Theorem 2] and [2, Theorem 3].

Similarly, many other results for convex spaces or C-spaces can be extended to the framework of G-convex spaces. In the first author's works on G-convex spaces, he tried to restrict to write down only essential things.

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