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GRÖBNER-SHIRSHOV BASES FOR REPRESENTATION THEORY

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ABSTRACT. In this paper, we develop the Gröbner-Shirshov basis theory for the representations of associative algebras by introducing the notion of Gröbner-Shirshov pairs. Our result can be applied to solve the reduction problem in representation theory and to construct monomial bases of representations of associative algebras. As an illustration, we give an explicit construction of Gröbner-Shirshov pairs and monomial bases for finite dimensional irreducible representations of the simple Lie algebra sl_3. Each of these monomial bases is in 1-1 correspondence with the set of semistandard Young tableaux with a given shape.

1. Introduction

The *Gröbner basis theory* for commutative algebras was introduced by Buchberger [7] and provides a solution to the *reduction problem* for commutative algebras. More precisely, it gives an effective algorithm of computing a set of generators for a given ideal of a commutative ring which can be used to determine the *reduced elements* with respect to the relations given by the ideal. In [1], Bergman generalized the Gröbner basis theory to associative algebras by proving the *Diamond Lemma*.

On the other hand, the parallel theory of Gröbner bases was developed for Lie algebras by Shirshov [12]. The key ingredient of the theory is the so-called *Composition Lemma* which characterizes the leading terms of elements in the given ideal. In [2], Bokut noticed that Shirshov's method works for associative algebras as well, and for this reason, Shirshov's

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theory for Lie algebras and their universal enveloping algebras is called the *Gröbner-Shirshov basis theory.*

The Gröbner-Shirshov bases for finite dimensional simple Lie algebras were constructed explicitly in a series of papers by Bokut and Klein [4, 5, 6]. In [3], Bokut, Kang, Lee and Malcolmson developed the theory of Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras. They unified the Gröbner-Shirshov basis theories for Lie superalgebras and for associative algebras and gave an explicit construction of Gröbner-Shirshov bases for classical Lie superalgebras.

In this paper, we develop the Gröbner-Shirshov basis theory for the representations of associative algebras. More precisely, let $\mathcal{A}$ be a free associative algebra and consider a pair (S,T) of subsets of $\mathcal{A}$. Let J be the two-sided ideal of $\mathcal{A}$ generated by S and let $A = \mathcal{A}/J$ be the algebra defined by S. Also let I be the left ideal of A generated by (the image of) T and set $M = A/I$. Then M becomes a left A-module and we would like to solve the reduction problem for the A-module M. The main problem lies in the fact that we should deal with two-sided ideals and left ideals in a unified manner to produce a generalized version of Shirshov's Composition Lemma for the representations of associative algebras. We overcome this difficulty by introducing the notion of *Gröbner-Shirshov pairs.* It is a pair of subsets (S,T) of the free associative algebra $\mathcal{A}$ that are closed under certain compositions and the set of (S,T)*-reduced words* forms a monomial basis of the A-module M.

We also show how to apply our Gröbner-Shirshov basis theory for representations to solve the reduction problem and to construct monomial bases of integrable highest weight modules over symmetrizable Kac-Moody algebras. As an application, we give an explicit construction of Gröbner-Shirshov pairs and monomial bases for finite dimensional irreducible representations of the simple Lie algebra sl_3. Each of these monomial bases is in 1-1 correspondence with the set of semistandard Young tableaux with a given shape.

Our result is a very general one that can be applied to the representation theory of various interesting algebras such as finite dimensional simple Lie (super)algebras, Kac-Moody (super)algebras, (affine) Hecke algebras, and so on. The work on the representation theory of classical Lie algebras and Hecke algebras is in progress and will be published elsewhere.

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2. Gröbner-Shirshov pair

Let $X = \{x_1, x_2, \cdots\}$ be a set of alphabets indexed by positive integers. Define a linear ordering on $X \prec$ by setting $x_i \prec x_j$ if and only if $i < j$. Let X^* be the semigroup of associative words on X. We denote the empty word by 1 and the *length* of a word u by $l(u)$ with $l(1) = 0$. We consider two linear orderings $<$ and $\ll$ on X^* defined as follows:

(i) $u < 1$ for any nonempty word u; and inductively, $u < v$ whenever $u = x_i u'$, $v = x_j v'$ and $x_i \prec x_j$ or $x_i = x_j$ and $u' < v'$ with $x_i, x_j \in X$,

(ii) $u \ll v$ if $l(u) < l(v)$ or $l(u) = l(v)$ and $u < v$.

The ordering $<$ (resp. $\ll$) is called the *lexicographic ordering* (resp. *length-lexicographic ordering*).

Let $\mathcal{A}_X$ be the free associative algebra generated by X over $\mathbb{C}$. Given a nonzero element $p \in \mathcal{A}_X$ we denote by $\overline{p}$ the maximal monomial appearing in p under the ordering $\ll$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{C}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \ll \overline{p}$. The α is called the *leading coefficient* of p and if $\alpha = 1$, p is said to be *monic*.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$. Let S be a set of monic elements of $\mathcal{A}_X$. We define the *composition* of p and q as follows.

DEFINITION 2.1. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the **composition of intersection** is defined to be $(p, q)_w = pa - bq$.

(b) If there exist a and b in X^* such that $a\overline{p}b = \overline{q} = w$, then the **composition of inclusion** is defined to be $(p, q)_w = apb - q$.

(c) A composition $(p, q)_w$ is called **right-justified** if $\overline{p} = a\overline{q} = w$ for some $a \in X^*$.

EXAMPLE 2.2. Let $X = \{x_1, x_2, x_3, x_4\}$.

(a) If $p = x_4x_2^2 + x_2x_4x_3$ and $q = x_2^2x_3 - x_4$, then we have two possible compositions of intersection:

$$\begin{aligned}(p,q)_{x_4x_2^2x_3} &= (x_4x_2^2 + x_2x_4x_3)x_3 - x_4(x_2^2x_3 - x_4)\\ &= x_2x_4x_3^2 + x_4^2,\\ (p,q)_{x_4x_2^3x_3} &= (x_4x_2^2 + x_2x_4x_3)x_2x_3 - x_4x_2(x_2^2x_3 - x_4)\\ &= x_2x_4x_3x_2x_3 + x_4x_2x_4.\end{aligned}$$

(b) If $p = x_2 - 1$ and $q = x_4x_2x_3 - x_1x_2$, then we have a composition of inclusion:

$$(p,q)_{x_4x_2x_3} = x_4(x_2 - 1)x_3 - (x_4x_2x_3 - x_1x_2) = -x_4x_3 + x_1x_2.$$

(c) If $p = x_1x_2x_1x_4 + x_2x_3$ and $q = x_1x_4 + x_4$, then we have a right-justified composition:

$$(p,q)_{x_1x_2x_1x_4} = x_1x_2x_1x_4 + x_2x_3 - x_1x_2(x_1x_4 + x_4) = x_2x_3 - x_1x_2x_4.$$

Let (S, T) be a pair of subsets of $\mathcal{A}_X$. We define a *congruence relation* on $\mathcal{A}_X$ as follows.

DEFINITION 2.3. Let $p, q \in \mathcal{A}_X$ and $w \in X^*$. We say that p and q are **congruent** to each other with respect to the pair (S, T) and w, denoted by $p \equiv q \bmod (S, T; w)$, if $p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in \mathbb{C}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$ with $a_i\overline{s_i}b_i \ll w$ and $c_j\overline{t_j} \ll w$ for each i and j.

REMARK. If $T = \emptyset$, we simply write $p \equiv q \bmod (S; w)$. In this case, the congruence relation is defined by the two-sided ideal of $\mathcal{A}_X$ generated by S as we can see in the discussion below.

EXAMPLE 2.4. Let $S = \{x_1x_2 + x_1\}$ and $T = \{x_3x_4 - x_1\}$.

(a) If $p = x_2x_3x_2$, $q = x_2x_3x_4 + x_1^2x_2x_3 + x_1^2x_3$, $w = x_2^3x_3$, then $p \equiv q \bmod (S; w)$.

(b) If $p = x_2x_3x_4 - x_1x_2$, $q = x_1^2x_2x_3 + x_1^2x_3 + x_2x_1 - x_1x_2$, $w = x_1^2x_3^2$, then $p \equiv q \bmod (S, T; w)$.

Let (S, T) be a pair of subsets of $\mathcal{A}_X$. If J is the two-sided ideal in $\mathcal{A}_X$ generated by S, then we say that the algebra $A = \mathcal{A}_X/J$ is *defined by* S. Let I be the left ideal generated by (the image of) T in A. Then we say that the left A-module $M = A/I$ is *defined by* the pair (S, T).

The image of $p \in \mathcal{A}_X$ in A and M under the canonical quotient maps will also be denoted by p as long as there is no peril of confusion.

A set S of monic elements of $\mathcal{A}_X$ is said to be *closed under the composition* if for any $p, q \in S$ and $w \in X^*$ such that $(p,q)_w$ is defined, we have $(p,q)_w \equiv 0 \bmod (S; w)$. In this case, we say that S is a *Gröbner-Shirshov basis* for the algebra $A = \mathcal{A}_X/J$ defined by S. A set T of monic elements of $\mathcal{A}_X$ is said to be *closed under the right-justified composition with respect to S* if for any $p, q \in T$ and $w \in X^*$ such that a right-justified $(p,q)_w$ is defined, we have $(p,q)_w \equiv 0 \bmod (S, T; w)$.

DEFINITION 2.5. (a) *A pair (S,T) of subsets of monic elements of $\mathcal{A}_X$ is called a* **Gröbner-Shirshov pair** *if S is closed under the composition, T is closed under the right-justified composition with respect to S, and for any $p \in S$, $q \in T$ and $w \in X^*$ such that $(p,q)_w$ is defined, we have $(p,q)_w \equiv 0 \bmod (S,T; w)$. In this case, we say that (S,T) is a* **Gröbner-Shirshov pair** *for the A-module $M = A/I$ defined by (S,T).*

(b) *A word $u \in X^*$ is said to be (S,T)-***reduced** *if $u \neq a\bar{s}b$ and $u \neq c\bar{t}$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the word u is said to be (S,T)-***reducible**. *If $T = \emptyset$, we will simply say that u is S-***reduced** *or S-***reducible**.

3. Composition Lemma

The main ingredient of the Gröbner-Shirshov theory for Lie algebras and their universal enveloping algebras is the *Composition Lemma* proved by Shirshov [12]. It asserts that if S is a Gröbner-Shirshov basis for the algebra $A = \mathcal{A}_X/J$ defined by S and p is trivial in A, then the word $\bar{p}$ is S-reducible. In this section, we prove the following generalization of Shirshov's Composition Lemma to the representations of associative algebras.

THEOREM 3.1. *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S,T). Suppose that (S,T) is a Gröbner-Shirshov pair for the A-module M and that $p \in \mathcal{A}_X$ is trivial in M. Then the word $\bar{p}$ is (S,T)-reducible.*

Proof. Since p is trivial in M, we can write

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j,$$

where $\alpha_i, \beta_j \in \mathbb{C}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, and $t_j \in T$. Choose the maximal word w in the length-lexicographic ordering $\ll$ among the words $\{a_i\overline{s_i}b_i, c_j\overline{t_j}\}$ in the expression of p. If $\overline{p} = w$, then we are done. Suppose this is not the case. Then $\overline{p} \ll w$ and without loss of generality, we may assume that one of the following three cases holds:

(I) $w = a_1\overline{s_1}b_1 = a_2\overline{s_2}b_2$, (II) $w = c_1\overline{t_1} = c_2\overline{t_2}$, (III) $w = a_1\overline{s_1}b_1 = c_1\overline{t_1}$.

Case I. $w = a_1\overline{s_1}b_1 = a_2\overline{s_2}b_2$: we will show that $a_2s_2b_2 \equiv a_1s_1b_1 \bmod(S; w)$. There are three possibilities:

(i) If the subwords $\overline{s_1}$ and $\overline{s_2}$ have empty intersection in $a_1\overline{s_1}b_1$, then we may assume that $a_1s_1b_1 = as_1b\overline{s_2}c$ and $a_2s_2b_2 = a\overline{s_1}bs_2c$, where $a, b, c \in X^*$. Thus

$$a_2s_2b_2 - a_1s_1b_1 = -a(s_1 - \overline{s_1})bs_2c + as_1b(s_2 - \overline{s_2})c,$$

which implies $a_2s_2b_2 \equiv a_1s_1b_1 \bmod (S; w)$.

(ii) If $\overline{s_1} = u_1u_2, \overline{s_2} = u_2u_3$ for some $u_2 \neq 1$, then $a_2 = a_1u_1, b_1 = u_3b_2$, and

$$\begin{aligned} a_2s_2b_2 - a_1s_1b_1 &= a_1u_1s_2b_2 - a_1s_1u_3b_2 \\ &= -a_1(s_1u_3 - u_1s_2)b_2 = a_1(s_1, s_2)_{u_1u_2u_3}b_2. \end{aligned}$$

Since $\overline{(s_1, s_2)_{u_1u_2u_3}} \ll u_1u_2u_3$ and S is closed under the composition, we obtain $a_2s_2b_2 \equiv a_1s_1b_1 \bmod (S; w)$.

(iii) If $\overline{s_1} = u_1\overline{s_2}u_2$, then $a_2 = a_1u_1, b_2 = u_2b_1$, and

$$\begin{aligned} a_2s_2b_2 - a_1s_1b_1 &= a_1u_1s_2u_2b_1 - a_1s_1b_1 \\ &= a_1(u_1s_2u_2 - s_1)b_2 = a_1(s_2, s_1)_{\overline{s_1}}b_2. \end{aligned}$$

Since $\overline{(s_1, s_2)_{\overline{s_2}}} \ll \overline{s_2}$ and S is closed under the composition, we get $a_2s_2b_2 \equiv a_1s_1b_1 \bmod (S; w)$.

Case II. $w = c_1\overline{t_1} = c_2\overline{t_2}$: we will show that $c_2t_2 \equiv c_1t_1 \bmod (\emptyset, T; w)$. We may assume that $\overline{t_2} = u\overline{t_1}$, $u \in X^*$. Thus

$$c_2t_2 - c_1t_1 = c_2(t_2 - ut_1) = c_2(t_2t_1)_{\overline{t_2}}.$$

Since $(t_2, t_1)_{\overline{t_2}} \ll \overline{t_2}$ and T is closed under the right-justified composition with respect to S, $c_2t_2 \equiv c_1t_1 \bmod (S, T; w)$.

Case III. $w = a_1\overline{s_1}b_1 = c_1t_1$: we will show that

$$a_1s_1b_1 \equiv c_1t_1 \mod (S, T; w).$$

There are three possibilities:

(i) If the subword $\overline{s_1}$ and $\overline{t_1}$ have empty intersection in w, then as we have seen in **Case I** (i), we have $a_1s_1b_1 \equiv c_1t_1 \bmod (S, T; w)$.

(ii) If $\overline{s_1} = sb$, $\overline{t_1} = bb_1$ for some $b \neq 1$, then

$$a_1 s_1 b_1 - c_1 t_1 = a_1(s_1 b_1 - s t_1) = a_1(s_1, t_1)_w,$$

and we get $a_1 s_1 b_1 \equiv c_1 t_1 \mod (S, T; w)$.

(iii) If $\overline{t_1} = s\overline{s_1}b_1$, then

$$a_1 s_1 b_1 - c_1 t_1 = c_1(s s_1 b_1 - t_1) = c_1(s_1, t_1)_w,$$

which implies $a_1 s_1 b_1 \equiv c_1 t_1 \mod (S, T; w)$.

Therefore, p can be written as $p = \sum \alpha_i' a_i' s_i' b_i' + \sum \beta_j' c_j' t_j'$, where $a_i'\overline{s_i'}b_i' \ll w$ for all i and $c_j'\overline{t_j'} \ll w$ for all j. Choose the maximal word w_1 in the ordering $\ll$ among $\{a_i'\overline{s_i'}b_i', c_j'\overline{t_j'}\}$. If $\overline{p} = w_1$, then we are done. If this is not the case, repeat the above process. Since X is indexed by the set of positive integers, this process must terminate in finite steps, which completes the proof. □

THEOREM 3.2. *Let (S, T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S, T). Suppose that (S, T) is a Gröbner-Shirshov pair for the A-module M. Then the set of (S, T)-reduced words forms a linear basis of M.*

Proof. Suppose $\sum \alpha_i u_i = 0$ in M, where $\alpha_i \in \mathbb{C}$ and u_i are distinct (S, T)-reduced words. Then, by Theorem 3.1, $\overline{\sum \alpha_i u_i}$ is (S, T)-reducible. Since each u_i is (S, T)-reduced, we must have $\alpha_i = 0$ for all i. Thus the set of (S, T)-reduced words is linearly independent.

Now we will show that any word $u \in \mathcal{A}_X$ can be written as

$$u = \sum \alpha_i u_i + \sum \beta_j a_j s_j b_j + \sum \gamma_k c_k t_k,$$

where u_i is an (S, T)-reduced word, $\alpha_i, \beta_j, \gamma_k \in \mathbb{C}$, $a_j, b_j, c_k \in X^*$, $s_j \in S$, $t_k \in T$, $a_j\overline{s_j}b_j \leqslant u$ and $c_k\overline{t_k} \leqslant u$ for all i, j, k. If u is (S, T)-reduced, then there is nothing to prove. If $u = a\overline{s}b$ with $s \in S$, then $u - asb = \sum \eta_i v_i$, $v_i \ll u$. If $u = c\overline{t}$ with $t \in T$, then $u - ct = \sum \eta_i' w_i$, $w_i \ll u$. We now apply the induction to complete the proof. □

COROLLARY 3.3. *Let (S, T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S, T). If M is finite dimensional and the number of (S, T)-reduced words is equal to the dimension of M, then (S, T) is a Gröbner-Shirshov pair for the A-module M.*

Proof. In the proof of the above theorem, we showed that the A-module M is linearly spanned by the set of (S,T)-reduced words. Since the number of (S,T)-reduced words is equal to the dimension of M, it forms a linear basis of M. Suppose (S,T) is not a Gröbner-Shirshov pair of M. Then there is a nontrivial composition among the elements in S and T, which can be written as a linear combination of (S,T)-reduced words. Since any composition should vanish in M, we get a nontrivial linear dependence relation among (S,T)-reduced words in M, which is a contradiction. □

REMARK. The above statement can be generalized to the graded A-modules with finite dimensional homogeneous subspaces by a straightforward modification of our argument.

Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S,T). We will show that how one can *complete* the pair (S,T) to get a Gröbner-Shirshov pair for the A-module M.

For any subset R of $\mathcal{A}_X$, we define

$$\widehat{R} = \{p/\alpha \mid \alpha \in \mathbb{C} \text{ is the leading coefficient of } p \in R\}.$$

Let $S^{(0)} = \widehat{S}$. For $i \geq 0$, set

$$\begin{aligned} S_{(i)} &= \{(f,g)_w \not\equiv 0 \bmod (S^{(i)};w) \mid f,g \in S^{(i)}\}, \\ S^{(i+1)} &= S^{(i)} \cup \widehat{S}_{(i)}. \end{aligned}$$

Then, clearly, the set $\mathcal{S} = \bigcup_{i\geq 0} S^{(i)}$ is closed under the composition. Note that the algebra A is defined also by $\mathcal{S}$.

Let $T^{(0)} = \widehat{T}$. For $i \geq 0$, set

$$\begin{aligned} T_{(i)} &= \{(f,g)_w \not\equiv 0 \bmod (\mathcal{S},T^{(i)};w) \mid f,g \in T^{(i)}, (f,g)_w \text{ is right-justified}\}, \\ T^{(i+1)} &= T^{(i)} \cup \widehat{T}_{(i)}. \end{aligned}$$

Then the set $T^c = \bigcup_{i\geq 0} T^{(i)}$ is closed under the right-justified composition with respect to $\mathcal{S}$.

We now consider the compositions between $\mathcal{S}$ and T^c. Let $X^{(0)} = T^c$. For $i \geq 0$, set

$$X_{(i)} = \{(f,g)_w \not\equiv 0 \bmod (\mathcal{S}, X^{(i)}; w) \mid f \in \mathcal{S}, g \in X^{(i)}\},$$
$$X^{(i+1)} = (X^{(i)} \cup \widehat{X}_{(i)})^c.$$

Let $\mathcal{T} = \bigcup_{i \geq 0} X^{(i)}$. Then the A-module M is defined also by the pair $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{S}, \mathcal{T})$ is a Gröbner-Shirshov pair of M.

We summarize the above discussion in the following theorem.

THEOREM 3.4. *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the left A-module defined by (S,T). Then the pair (S,T) can be completed to a Gröbner-Shirshov pair $(\mathcal{S}, \mathcal{T})$ for the A-module M.*

4. Kac-Moody algebras

In this section, we show how to apply our Gröbner-Shirshov basis theory for representations to solve the reduction problem for integrable highest weight modules over symmetrizable Kac-Moody algebras.

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite index set and let $A = (a_{ij})_{i,j \in \Omega}$ be a symmetrizable generalized Cartan Matrix of lank l. Fix a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of A in the sense of [8]. Extend the set $\Pi^\vee$ to obtain a basis

$$H = \{h_1, \cdots, h_n, h_{n+1}, \cdots, h_{2n-l}\}$$

for $\mathfrak{h}$. Let $E = \{e_i\}_{i \in \Omega}, F = \{f_i\}_{i \in \Omega}$, and $X = E \cup H \cup F$. We define a linear ordering on X by setting

$$e_i \succ h_j \succ f_k \quad \text{for all } i, j, k \in \Omega,$$
$$e_i \succ e_j, \quad h_i \succ h_j, \quad f_i \succ f_j \quad \text{if } i > j.$$

Then we have the lexicographic ordering and the length-lexicographic ordering on X^* as in Section 2. We denote the left adjoint action of a Lie algebra by ad and the right adjoint action by $\widetilde{\mathrm{ad}}$.

The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ *associated with* A is defined to be the Lie algebra with generators X and the following defining relations:

$$
(1)\quad
\begin{array}{rl}
W: & [h_i h_j] \quad (i > j), \\
 & [e_i f_j] - \delta_{ij} h_i, \quad [e_i h_j] + \alpha_i(h_j) e_i, \quad [h_i f_j] + \alpha_j(h_i) f_j, \\
S_+: & (\mathrm{ad} e_i)^{1-a_{ij}} e_j \quad (i > j), \\
 & e_i(\widetilde{\mathrm{ad}} e_j)^{1-a_{ji}} \quad (i > j), \\
S_-: & (\mathrm{ad} f_i)^{1-a_{ij}} f_j \quad (i > j), \\
 & f_i(\widetilde{\mathrm{ad}} f_j)^{1-a_{ji}} \quad (i > j).
\end{array}
$$

Let $S = S_+ \cup W \cup S_-$. We denote by $\mathfrak{g}_+$ (resp. $\mathfrak{h}$ and $\mathfrak{g}_-$) the subalgebra of $\mathfrak{g}$ generated by E (resp. H and F). If we consider the set of relations S as a subset of $\mathcal{A}_X$ with $[x,y] = xy - yx$, we can easily see that the universal enveloping algebra $U = U(\mathfrak{g})$ of $\mathfrak{g}$ is the algebra defined by S in $\mathcal{A}_X$. We denote by U_+ (resp. U_0 and U_-) the $\mathbb{C}$-subalgebra of U with 1 generated by e_i's (resp. h_i's and f_i's).

Let

$$
P = \{\lambda \in \mathfrak{h}^* | \lambda(h_i) \in \mathbb{Z}, i \in \Omega\},
$$
$$
P_+ = \{\lambda \in P | \lambda(h_i) \geq 0, i \in \Omega\}.
$$

The set P is called the *weight lattice* and an element $\lambda \in P_+$ is called a *dominant integral weight.* Let

$$
T_{Ver} = \{e_i, h_i - \lambda(h_i) | i \in \Omega\},
$$

and for $\lambda \in P_+$, set

$$
T_\lambda = \{f_i^{\lambda(h_i)+1} | i \in \Omega\} \quad \text{and} \quad T = T_{Ver} \cup T_\lambda.
$$

Let $V(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P_+$. Then any highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P_+$ and highest weight vector v_λ is isomorphic to $V(\lambda)$ if and only if $f_i^{\lambda(h_i)+1} v_\lambda = 0$ for all $i \in \Omega$ ([8]). Using the language of Gröbner-Shirshov basis theory, this fact can be rephrased as follows:

PROPOSITION 4.1. *The irreducible highest weight* $\mathfrak{g}$*-module* $V(\lambda)$ *with highest weight* $\lambda \in P_+$ *is isomorphic to the* $U(\mathfrak{g})$*-module defined by the pair* (S, T)*. Moreover, due to the triangular decomposition of* $U(\mathfrak{g})$*,* $V(\lambda)$ *can be regarded as the* U_-*-module defined by the pair* (S_-, T_λ) *in* $\mathcal{A}_F$*.*

As we have seen in Theorem 3.4, the pair (S_-, T_λ) can be completed to a Gröbner-Shirshov pair $(\mathcal{S}, \mathcal{T})$ of the integrable highest weight $U(\mathfrak{g})$-module $V(\lambda)$. Consider first the set S_- of Serre relations. To obtain a Gröbner-Shirshov basis $\mathcal{S}$ for the algebra U_-, one can take a direct approach of computing all the possible non-trivial compositions in S_- as was described in Theorem 3.4. An alternative approach is to compute the *Lie compositions* and use the main results of [3]: a Gröbner-Shirshov basis for the Lie algebra $\mathfrak{g}$ is also a Gröbner-Shirshov basis for its universal enveloping algebra $U(\mathfrak{g})$.

For the finite dimensional simple Lie algebras, the Gröbner-Shirshov bases were completely determined in [4, 5, 6]. For the affine Kac-Moody algebras, the Gröbner-Shirshov bases were constructed only for the simplest case – for the affine Kac-Moody algebra of type $A_n^{(1)}$ [11].

Now assume that we have a Gröbner-Shirshov basis $\mathcal{S}$ for $U(\mathfrak{g})$. Note that the set T_λ is already closed under the right-justified composition with respect to $\mathcal{S}$. Therefore, to construct a Gröbner-Shirshov pair $(\mathcal{S}, \mathcal{T})$ for the integrable highest weight $U(\mathfrak{g})$-module $V(\lambda)$, we have only to carry out the last step of the algorithm given in Theorem 3.4, and this process works for any symmetrizable Kac-Moody algebras. However, when the character of $V(\lambda)$ is known, it would be more practical to compute sufficiently many relations in $V(\lambda)$ and make use of Corollary 3.3. This is the approach we take in the next section to construct Gröbner-Shirshov pairs for the simple Lie algebra sl_3.

5. Irreducible modules of sl_3

In this section, we will give an explicit construction of Gröbner-Shirshov pairs and monomial bases for finite dimensional irreducible representations of the simple Lie algebra sl_3. We will also show that each of these bases is in 1-1 correspondence with the set of semistandard Young tableaux of a given shape.

Recall that the simple Lie algebra sl_3 is a Kac-Moody algebra associated with the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Hence the algebra U_- is the associative algebra defined by the set $S_- = \{[f_2[f_2f_1]], [[f_2f_1]f_1]\}$ of the Serre relations in $\mathcal{A}_F$, where $F = \{f_1, f_2\}$. It can be easily verified that S_- is already closed under the composition (see, for example, [4]). Thus the algebra U_- has a monomial basis consisting of S_--reduced words $f_1^a(f_2f_1)^bf_2^c$ in F^* $(a, b, c \geq 0)$.

LEMMA 5.1. *The relations*

$$f_2f_1^k - kf_1^{k-1}[f_2f_1] - f_1^kf_2 \quad (k \geq 2)$$

hold in U_-. In other words, they belong to the two-sided ideal generated by S_- in $\mathcal{A}_F$.

Proof. If $k = 2$, then it is just $[[f_2f_1]f_1] = 0$, which is given by S_-. Assume that the relation holds for some fixed k. Since $[f_2f_1]f_1 = f_1[f_2f_1]$, we obtain

$$\begin{aligned} f_2f_1^{k+1} &= kf_1^{k-1}[f_2f_1]f_1 + f_1^kf_2f_1 \\ &= (k+1)f_1^k[f_2f_1] + f_1^{k+1}f_2 \end{aligned}$$

as desired. □

Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight for sl_3, where Λ_i are the fundamental weights $(i = 1, 2)$, and let $V(\lambda)$ be the irreducible highest weight module over sl_3 with highest weight λ. Then, as a U_--module, $V(\lambda)$ is defined by the pair (S_-, T_λ), where $T_\lambda = \{f_1^{m+1}, f_2^{n+1}\}$. From now on, we will say that *a relation $R = 0$ holds in* $V(\lambda)$ whenever R is contained in the left ideal of U_- generated by T_λ.

LEMMA 5.2. *The following relations hold in $V(\lambda)$:*
(a) $f_1^{m+c+1}f_2^c = 0 \quad (c \geq 0)$,
(b) $f_1^{m+c}[f_2f_1]f_2^c + \dfrac{1}{m+c+1}f_1^{m+c+1}f_2^{c+1} = 0 \quad (c \geq 0)$.

Proof. If $c = 0$, then it is just f_1^{m+1} belonging to T_λ. Assume that we have the first relation for some fixed c. Multiplying f_2 from the left, Lemma 5.1 yields

$$f_2f_1^{m+c+1}f_2^c = (m+c+1)f_1^{m+c}[f_2f_1]f_2^c + f_1^{m+c+1}f_2^{c+1},$$

which gives the second relation for c. Now, multiplying $[f_2f_1]$ to the left of the first relation and using the second relation obtained in the above, we get

$$[f_2f_1]f_1^{m+c+1}f_2^c = f_1^{m+c+1}[f_2f_1]f_2^c = -\frac{1}{m+c+1}f_1^{m+c+2}f_2^{c+1}.$$

Thus we get the first relation for $c + 1$. By multiplying f_2 from the left again, Lemma 5.1 yields the second relation for $c + 1$. Hence by induction, we obtain the desired relations. □

LEMMA 5.3. *The relations*

$$\sum_{r=0}^{b} \frac{(m-b+c+1)!}{(m-b+c+r+1)!}\binom{b}{r} f_1^{m-b+c+r+1}[f_2f_1]^{b-r} f_2^{c+r} = 0$$

hold in $V(\lambda)$ *for* $c \geq 0$ *and* $0 \leq b \leq m+c+1$.

Proof. Fix an arbitrary $c \geq 0$. If $b = 0$, then the above relation is just $f_1^{m+c+1}f_2^c = 0$, which is the first relation in Lemma 5.2. Assume that the relation holds for some fixed b. Multiplying by f_2 from the left and using $f_2[f_2f_1] = [f_2f_1]f_2$, we get

$$\begin{aligned}
&\sum_{r=0}^{b} \frac{(m-b+c+1)!}{(m-b+c+r+1)!}\binom{b}{r} f_2 f_1^{m-b+c+1+r}[f_2f_1]^{b-r} f_2^{r+c} \\
= \; &\sum_{r=0}^{b} \frac{(m+c-b+1)!}{(m-b+c+r)!}\binom{b}{r} f_1^{m+c-b+r}[f_2f_1]^{b-r+1} f_2^{r+c} \\
&+\sum_{r=0}^{b} \frac{(m-b+c+1)!}{(m-b+c+r+1)!}\binom{b}{r} f_1^{m-b+c+r+1}[f_2f_1]^{b-r} f_2^{r+c+1} \\
= \; &(m-b+c+1)f_1^{m-b+c}[f_2f_1]^{b+1}f_2^c + \frac{(m-b+c+1)!}{(m+c+1)!} f_1^{m+c+1} f_2^{b+c+1} \\
&+\sum_{r=1}^{b} \frac{(m-b+c+1)!}{(m-b+c+r+1)!}\binom{b+1}{r} f_1^{m-b+c+r}[f_2f_1]^{b-r+1} f_2^{c+r}.
\end{aligned}$$

Dividing out by the leading coefficient, we get

$$\sum_{r=0}^{b+1} \frac{(m-b+c)!}{(m-b+c+r)!}\binom{b+1}{r} f_1^{m-b+c+r}[f_2f_1]^{b-r+1} f_2^{r+c} = 0,$$

which is the desired relation for $b+1$. □

LEMMA 5.4. *The relations*

$$[f_2f_1]^{m+1}f_2^c + \sum_{r=1}^{m+1} \frac{c!}{(c+r)!}\binom{m+1}{r} f_1^r[f_2f_1]^{m-r+1} f_2^{c+r} = 0$$

hold in $V(\lambda)$ *for* $c \geq 0$.

Proof. If $c = 0$, it is the same as the case of $b = m+1$, $c = 0$ in Lemma 5.3. Assume that the relation holds for some fixed c. Multiplying by f_2, we have

$$\begin{aligned}
&[f_2f_1]^{m+1}f_2^{c+1}+\sum_{r=1}^{m+1}\frac{c!}{(c+r)!}\binom{m+1}{r}f_2f_1^r[f_2f_1]^{m-r+1}f_2^{c+r}\\
= \ &[f_2f_1]^{m+1}f_2^{c+1}+\sum_{r=1}^{m+1}\frac{c!}{(c+r)!}\binom{m+1}{r}rf_1^{r-1}[f_2f_1]^{m-r+2}f_2^{c+r}\\
&+\sum_{r=1}^{m+1}\frac{c!}{(c+r)!}\binom{m+1}{r}f_1^r[f_2f_1]^{m-r+1}f_2^{c+r+1}\\
= \ &\frac{c+m+2}{c+1}[f_2f_1]^{m+1}f_2^{c+1}+\frac{c!}{(c+m+1)!}f_1^{m+1}f_2^{c+m+2}\\
&+\sum_{r=1}^{m}A\frac{c!}{(c+r+1)!}f_1^r[f_2f_1]^{m-r+1}f_2^{c+r+1},
\end{aligned}$$

where $A=\binom{m+1}{r+1}(r+1)+\binom{m+1}{r}(c+r+1)$. Dividing out by the leading coefficient, we get the desired relation for $c+1$. $\square$

THEOREM 5.5. *The pair $(\mathcal{S},\mathcal{T}_\lambda)$ is a Gröbner-Shirshov pair for the irreducible highest weight module $V(\lambda)$ over the simple Lie algebra sl_3, where $\mathcal{S}=S_-$ and $\mathcal{T}_\lambda$ consists of the following elements:*

(a) f_2^{n+1},

(b) $[f_2f_1]^{m+1}f_2^c+\sum_{r=1}^{m+1}\frac{c!}{(c+r)!}\binom{m+1}{r}f_1^r[f_2f_1]^{m-r+1}f_2^{c+r}$

$$(1\le c\le n),$$

(c) $\sum_{r=0}^{b}\frac{(m-b+c+1)!}{(m-b+c+r+1)!}\binom{b}{r}f_1^{m-b+c+r+1}[f_2f_1]^{b-r}f_2^{c+r}$

$$(0\le b\le m,\ 0\le c\le n).$$

Hence the set of the monomials of the form

$$f_1^a(f_2f_1)^bf_2^c\quad(0\le c\le n,\ 0\le b\le m,\ 0\le a\le m-b+c)$$

forms a linear basis of $V(\lambda)$.

Proof. By Lemma 5.1 – Lemma 5.4, we see that the above relations hold in $V(\lambda)$. Note that the set of $(\mathcal{S},\mathcal{T}_\lambda)$-reduced words is given by:

$$f_1^a(f_2f_1)^bf_2^c\quad(0\le c\le n,\ 0\le b\le m,\ 0\le a\le m-b+c).$$

Hence the number of $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words is

$$\sum_{c=0}^{n} \sum_{b=0}^{m} (m-b+c+1) = \frac{1}{2}(m+1)(n+1)(m+n+2).$$

This is exactly the dimension of $V(\lambda)$. Hence by Corollary 3.3, the pair $(\mathcal{S}, \mathcal{T}_\lambda)$ is a Gröbner-Shirshov pair for $V(\lambda)$. $\square$

Let $\lambda = m\Lambda_1 + n\Lambda_2$ be a dominant integral weight for sl_3 and let $Y^\lambda = \{(1,i),(2,j) | 1 \le i \le m+n, 1 \le j \le n\}$ be the *Young frame* of shape λ. We define a *semistandard Young tableau of shape* λ to be a function τ of Y^λ into the set $\{1,2,3\}$ such that

$$\begin{aligned} &\tau(k,i) \le \tau(k,i+1) \quad \text{for } k = 1,2, \\ &\tau(1,j) < \tau(2,j) \quad \text{for all } j = 1, \cdots, n. \end{aligned}$$

As usual, we can present a semistandard Young tableau by an array of colored boxes. For example, the following are semistandard tableaux of shape $2\Lambda_1$, $3\Lambda_2$ and $2\Lambda_1 + \Lambda_2$, respectively.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$$

We would like to identify the monomial basis consisting of $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words with the set of semistandard Young tableaux of shape λ. Consider the empty word as the semistandard Young tableau τ^λ of shape λ defined by $\tau^\lambda(1,i) = 1$ and $\tau^\lambda(2,j) = 2$ for all i, j. To each $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced word $f_1^a(f_2f_1)^b f_2^c$ with $0 \le c \le n$, $0 \le b \le m$, $0 \le a \le m-b+c$, we associate the semistandard Young tableau τ as follows. Start with the tableau τ^λ and change its entries by the following rule:

(i) Let the word f_2 change the box $\boxed{2}$ to the box $\boxed{3}$, let the word f_2f_1 change $\boxed{1}$ to $\boxed{3}$ and let f_1 change $\boxed{1}$ to $\boxed{2}$.

(ii) Let the words f_2, f_2f_1 and f_1 in $f_1^a(f_2f_1)^b f_2^c$ act successively on τ^λ changing the boxes in τ^λ from the right.

For example, the word $f_1^2(f_2f_1)^2 f_2$ occurring in $V(3\Lambda_1 + 3\Lambda_2)$ corresponds to the semistandard Young tableau $\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array}$.

More explicitly, to each $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced word $f_1^a(f_2f_1)^b f_2^c$ with $0 \le c \le n$, $0 \le b \le m$, $0 \le a \le m-b+c$, we associate the semistandard

Young tableau τ defined by:

$$\tau(1,i) = \begin{cases} 1 & \text{if } 1 \le i \le m+n-1-b, \\ 2 & \text{if } m+n-a-b+1 \le i \le m+n-b, \\ 3 & \text{if } m+n-b+1 \le i \le m+n, \end{cases}$$
$$\tau(2,j) = \begin{cases} 2 & \text{if } 1 \le j \le n-c, \\ 3 & \text{if } n-c+1 \le j \le n. \end{cases}$$

Then it is now easy to verify that the above correspondence defines a bijection between the set of $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words and the set of semistandard Young tableaux of shape λ.

PROPOSITION 5.6. *The monomial basis of $V(\lambda)$ given by $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words is in* 1-1 *correspondence with the set of semistandard Young tableaux of shape λ.*

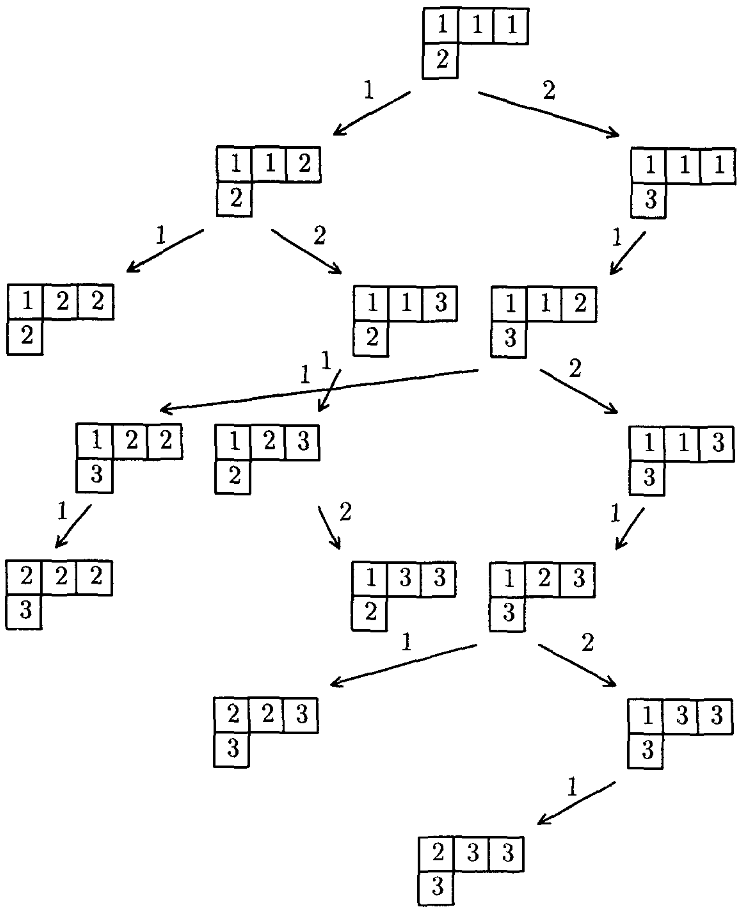
Furthermore, we can define a colored oriented graph structure on the monomial basis of $V(\lambda)$ consisting of $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words (and hence on the set of semistandard Young tableaux of shape λ): for each $i = 1, 2$, we define $w \xrightarrow{i} w'$ if and only if $w' = f_i w$. This graph is usually different from the *crystal graph* developed by Kashiwara ([9], [10]).

Our discussion will be illustrated in the following example.

EXAMPLE 5.7. Let $\lambda = 2\Lambda_1 + \Lambda_2$. In the following, we list all the $(\mathcal{S}, \mathcal{T}_\lambda)$-reduced words and the corresponding semistandard Young tableaux of shape λ.

1	f_1	f_2	f_1^2	$f_1 f_2$
$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&1\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&2\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&2\\\hline 3\\\cline{1-1}\end{array}$
$f_2 f_1$	$f_1^2 f_2$	$f_1 f_2 f_1$	$f_2 f_1 f_2$	$f_1^3 f_2$
$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&1&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 2&2&2\\\hline 3\\\cline{1-1}\end{array}$
$f_1 f_2 f_1 f_2$	$(f_2 f_1)^2$	$f_1^2 f_2 f_1 f_2$	$(f_2 f_1)^2 f_2$	$f_1 (f_2 f_1)^2 f_2$
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 2&2&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&3\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 2&3&3\\\hline 3\\\cline{1-1}\end{array}$

The colored oriented graph for $V(\lambda)$ is given in the following figure.



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