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# ON THE EXISTENCE OF AN INVARIANT PROBABILITY AND THE FUNCTIONAL CENTRAL LIMIT THEOREM OF A CLASS OF NONLINEAR AUTOREGRESSIVE PROCESSES

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ABSTRACT. Existence of a unique invariant probability is considered for a class of Markov processes which may not be irreducible and a functional central limit theorem for a class of nonlinear irreducible uniformly ergodic processes is derived as well.

## 1. Introduction

Let $(S,\rho)$ be a metric space, $\Gamma$ a set of measurable maps on $S$ into itself, $\mathcal{T}$ a $\sigma$-field on $\Gamma$ such that the map $(\gamma, x) \to \gamma(x)$ is measurable on $(\Gamma \times S, \mathcal{T} \otimes \mathcal{B}(S))$ into $(S, \mathcal{B}(S))$, where $\mathcal{B}(S)$ denotes the Borel $\sigma$-field of $S$.

Let $\mathbf{P}$ be a probability measure on $(\Gamma, \mathcal{T})$. On some probability space $(\Omega, \mathcal{F}, Q)$ is given a sequence of i.i.d. random maps $\alpha_1, \alpha_2, ...$ with common distribution $\mathbf{P}$. For a given random variable $X_0$, independent of $\alpha_n$, define $X_1 = \alpha_1 X_0, ..., X_n = \alpha_n X_{n-1} = \alpha_n \cdots \alpha_1 X_0 \cdots$. Then, in view of the independence of $\alpha_n$, $X_n$ is a Markov process on $S$.

Let $p(x, dy)$ denote the transition probability given by

$$(1.1) \qquad p(x,B) = \mathbf{P}(\{\gamma \in \Gamma : \gamma(x) \in B\}), \quad x \in S, \quad B \in \mathcal{B}(S).$$

We often write $X_n^{(x)}$ for $X_n$ in case $X_0 = x$. Denote by $\mathbf{P}^n$ the product measure $\mathbf{P} \times \cdots \times \mathbf{P}$ on $(\Gamma^n, \mathcal{T}^{\otimes n})$. A probability measure $\pi$ on $(S, \mathcal{B}(S))$ is said to be invariant for $p$ if $\pi(B) = \int p(x,B)\pi(dx), B \in \mathcal{B}(S)$.

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We shall write $p^{(n)}(x, dy)$ for the $n$-step transition probability with $p^{(1)} = p$. Then $p^{(n)}(x, dy)$ is the distribution of $X_n(x)$.

In this article $S$ is a closed subset of $\mathbb{R}^1$. For $\Gamma$ one takes a set of measurable monotone (non-increasing or non-decreasing) functions on $S$ into itself. We say that $p$ is $\varphi$-irreducible with respect to a nontrivial measure $\varphi$ on $S$ if $\varphi(B) > 0$ implies that for each $x$ there exists $n$ such that $p^{(n)}(x, B) > 0$.
The transition probability $p$ may not be $\varphi$-irreducible for any nonzero $\sigma$-finite measure $\varphi$. One of the two main interests in the paper is to look at one such class of processes $X_n$ to get some conditions under which there exist unique invariant probabilities $\pi$. And the other is, for an example of such process, which is known as a nonlinear 1st-order autoregressive process, under irreducibility, to identify broad classes of functions $\psi$ in $L^2(S, \pi)$ for which the functional central limit theorem (FCLT) holds, i.e., the sequence of stochastic processes.

(1.2)
$$n^{-1/2}\left[\sum_{j=0}^{[nt]}(\psi(X_j) - \int \psi d\pi) + (nt - [nt])(\psi(X_{[nt]+1}) - \int \psi d\pi\right](t \geq 0)$$

converges weakly to a Brownian motion under the initial distribution $\pi$. Even though the main results in the article are stated and proved for the one-dimensional case, it will be readily turned out that one can extend the results to $\mathbb{R}^k (k \geq 2)$ with more relaxed conditions on $S$.

## 2. Existence of a unique invariant probability

Define the transition operator $T$ on the linear space $B(S)$ of all real-valued bounded measurable functions on $S$ by

$$(Th)(x) = \int h(y)p(x, dy), h \in B(S). \tag{2.1}$$

We shall say that $p(x, dy)$ has the (weak) Feller-property if $(Th)$ is continuous whenever $h \in B(S)$.

We make the following assumptions:

($A_1$) There exists $x_0$ and a positive integer $n_0$ such that

$$Q(X_{n_0}(x) \leq x_0 \quad \forall x) > 0, \quad Q(X_{n_0}(x) \geq x_0 \quad \forall x) > 0.$$

($A_2$) p has the Feller-property.

Let $S \subset \mathbb{R}^1$, being the state space of the process, be a closed interval (not necessarily bounded) and let $\Gamma$ be a set of continuous monotone maps $\gamma$ on $S$ into $S$.
Write

$$\begin{aligned} \mathcal{A}_1 &= \{(-\infty, x] \cap S : x \geqslant x_0\}, \\ \mathcal{A}_2 &= \{(-\infty, x] \cap S : x < x_0\}, \\ \mathcal{A}_3 &= \{[x, \infty) \cap S : x \leq x_0\}, \\ \mathcal{A}_4 &= \{[x, \infty) \cap S : x > x_0\}, \\ \mathcal{A} &= \textstyle\bigcup_{i=1}^4 \mathcal{A}_i. \end{aligned}$$

Then it is simple to check that $\gamma^{-1}\mathcal{A} \subset \mathcal{A} \quad \forall \gamma \in \Gamma$, outside of a set of $\mathbf{P}$-probability zero. On the space $\mathcal{P}(S)$, define the distance $d$ by

$$d(\mu, \nu) = \sup\{|\mu(A) - \nu(A)| : A \in \mathcal{A}\}, \ \mu, \ \nu \in \mathcal{P}(S) \tag{2.2}$$

The following result may be found in Bhattacharya and Lee (1988) and its correction notes.

LEMMA 2.1. *The space $\mathcal{P}(S)$ is complete under the distance $d$ defined by* (2.2).

It is easily noticed that the lemma still works for our case also. One of the results to be noted in this article is

THEOREM 2.1. *If ($A_1$), ($A_2$) hold, then there exists a unique invariant probability $\pi$ for $p(x, dy)$.*

For the proof, in despite of differences in situations, one may follow the similar ways found in Bhattacharya and Lee (1988), and here we present a sketch of the proof, including key steps.

First, we note that, by (2.2), the map $\mu \to \mu \circ \gamma^{-1}$ is a contraction. Define the (adjoint) operator $T^*$ acting on $\mathcal{P}(S)$ by

$$(T^*\mu)(B) = \int p(x, B)\mu(dx).$$

It follows that $\mu \to T^*\mu$ is a contraction in metric $d$. Write

$$\begin{aligned} \Gamma_1 &= \{(\gamma_1, \cdots, \gamma_{n_0}) \in \Gamma^{n_0}; \gamma_{n_0} \cdots \gamma_1(S) \subset (-\infty, x_0]\} \\ \Gamma_2 &= \{(\gamma_1, \cdots, \gamma_{n_0}) \in \Gamma^{n_0}; \gamma_{n_0} \cdots \gamma_1(S) \subset [x_0, \infty)\} \end{aligned}$$

Then

$$(\mu \circ (\gamma_{n_0}, \cdots, \gamma_1)^{-1})(A) - (\nu \circ (\gamma_{n_0}, \cdots, \gamma_1)^{-1})(A) = 0$$

$$\begin{cases} \text{if} \quad \gamma \in \Gamma_1 \quad \text{and} \quad A \in \mathcal{A}_1 \\ \text{if} \quad \gamma \in \Gamma_2 \quad \text{and} \quad A \in \mathcal{A}_2 \\ \text{if} \quad \gamma \in \Gamma_1 \quad \text{and} \quad A \in \mathcal{A}_4 \\ \text{or} \\ \text{if} \quad \gamma \in \Gamma_2 \quad \text{and} \quad A \in \mathcal{A}_3 \quad \text{where} \quad \gamma = (\gamma_1, \cdots, \gamma_{n_0}). \end{cases}$$

Therefore, for $A \in \mathcal{A}_1 \cup \mathcal{A}_4$,

$$|T^{*n_0}(A) - T^{*n_0}\nu(A)|$$

$$(2.3) \leq \int_{\Gamma^{n_0}} |\mu((\gamma_{n_0} \cdots \gamma_1)^{-1}A) - \nu((\gamma_{n_0} \cdots \gamma_1)^{-1}A)| \, \mathbf{P}^{n_0}(d\gamma_1 \cdots d\gamma_{n_0})$$

$$\leq \int_{\Gamma^{n_0} \setminus \Gamma_1} |\mu((\gamma_{n_0} \cdots \gamma_1)^{-1}A) - \nu((\gamma_{n_0} \cdots \gamma_1)^{-1}A)| \, \mathbf{P}^{n_0}(d\gamma_1 \cdots d\gamma_{n_0})$$

$$\leq (1 - P^{n_0}(\Gamma_1))d(\mu, \nu).$$

For $A \in \mathcal{A}_2 \cup \mathcal{A}_3$, similarly,

$$|T^{*n_0}(A) - T^{*n_0}\nu(A)| \leq (1 - \mathbf{P}^{n_0}(\Gamma_2))d(\mu, \nu).$$

Combining (2.3), (2.4),

$$(2.4) \qquad d(T^{*n_0}\mu, T^{*n_0}\nu) \leq \max\Big\{1 - P^{n_0}(\Gamma_1), 1 - P^{n_0}(\Gamma_2)\Big\} d(\mu, \nu).$$

(2.5) and the fact that $\mu \longrightarrow T^*\mu$ is a contraction in metric $d$ together imply

$$d(T^{*n}\mu, T^{*n}\nu) \leq \delta^{[n/n_0]} d(\mu, \nu), \quad \forall n = 1, 2, ...,$$

where $[n/n_0]$ is the integer part of $n/n_0$, and $\delta = \max\{1 - P^{n_0}(\Gamma_1),\ 1 - P^{n_0}(\Gamma_2)\}$, which is less than 1.
For $n' > n$, one has

$$(2.5) \qquad d\big(p^{(n)}(x, dy), p^{(n')}(x, dy)\big) = d(T^{*n}\mu, T^{*n}\nu) \leq \delta^{[n/n_0]}$$

with $\mu = \delta_x$ (point mass at $x$) and $\nu = T^{*(n'-n)}\delta_x$.

Hence, $p^{(n)}(x, dy)$ is a Cauchy sequence in the metric $d$. Let $\pi$ be its limit, which exists by Lemma 2.1.
Then $\pi$ is uniform in $x$ and letting $n' \longrightarrow \infty$ in (2.6), we see that

$$\sup_{x \in S} d\big(p^{(n)}(x, dy), \pi(dy)\big) \leq \delta^{[n/n_0]} \longrightarrow 0 \text{ as n} \longrightarrow \infty.$$

Therefore, $p^{(n)}(x, dy)$ converges weakly to the same probability measure $\pi(dy)$ on $S$ for every $x \in S$. Since $p$ has the Feller property, it is trivial to show that $\pi$ is invariant.

If $\pi'$ is another invariant probability, then $T^{*n}\pi' = \pi'$, which implies

$$d(\pi', \pi) = d(T^{*n}\pi', \pi) \to 0 \text{ as } n \to \infty.$$

Therefore, $d(\pi', \pi) = 0$, so that $\pi' = \pi$.

## 3. Some results on the functional central limit theorem under irreducibility

As an example of the process $X_n$ considered in the previous sections, which is generated by successive iterations of an i.i.d. sequence of random maps, on a probability space, consider the following process

$$X_{n+1} = f(X_n) + \varepsilon_{n+1} \quad (n \geqslant 0), \tag{3.1}$$

which is generated by the random maps $\alpha_n$ defined by $x \to \alpha_n x = f(x) + \varepsilon_n \quad (n \geqslant 1)$, where $f$ is real-valued Borel measurable and continuous monotone on S, $\{\varepsilon_n : n \geqslant 1\}$ is a sequence of i.i.d. random variables and $X_0$ is arbitrarily prescribable real-valued random variable independent of $\{\varepsilon_n : n \geqslant 1\}$.

With the assumptions $(A_1)$ and $(A_2)$ hold, as though there exists a unique invariant for the process, the transition probability $p$ may not be $\varphi$-irreducible for any nonzero $\sigma$-finite measure $\varphi$, not even strongly mixing. Indeed, the tail $\sigma$-field may not be nontrivial. Here, the irreducibility plays a role, under which it enables us to see these properties hold. A $\varphi$-irreducible aperiodic Markov process with transition probability $p(x, dy)$ is said to be *geometrically (Harris) ergodic* if there exists a probability measure $\pi$ such that

$$||p^{(n)}(x, dy) - \pi(dy)|| \to 0 \text{ exponentially fast as n} \to \infty, \quad \forall x \in \mathbb{R}^1. \tag{3.2}$$

Here $||\cdot||$ denotes the variation norm on the Banach space of finite signed measure on $(\mathbb{R}^1, \mathcal{B}(\mathbb{R}^1))$.

Recently there have been considerable works on kth-order $(k \geq 1)$ nonlinear autoregressive models, most of which provide some verifiable criteria for geometric ergodicity (see, e.g., Chan and Tong (1985), $Tj\phi$stheim (1990), Bhattacharya and Lee (1995a), (1995b), Lee (1998)).

If (3.2) holds, then $\pi$ is necessarily the unique invariant probability for $p(x, dy)$, and the process having $\pi$ as the initial distribution is stationary.

We assume that the process (3.1) is uniformly (geometrically) ergodic, that is, $\sup_{x \in S} ||p^{(n)}(x, dy) - \pi(dy)||$ goes to zero exponentially fast as $n \to \infty$, and that $X_0$ has the unique invariant $\pi$ as its distribution. Note that, by contractivity, the convergence automatically has geometric rate. A set $B \in \mathcal{B}(S)$ is said to be *small* (with respect to $\varphi$) if $\varphi(B) > 0$, and for every $A \in \mathcal{B}(S)$ with $\varphi(A) > 0$ there exists $j \geq 1$ such that

$$\inf_{x \in B} \sum_{n=1}^{j} p^{(n)}(x, A) > 0.$$

It also turns out that uniform ergodicity can be characterized as the smallness of the state space $S$, that is, a $\varphi$-irreducible aperiodic Markov process with transition probability $p(x, dy)$ is uniformly ergodic if the state space $S$ is small.

Let $\psi$ belong to the range ($\subset L^2 \equiv L^2(S, \pi)$) of the operator $T - I$ on $L^2$, where $T$ is the operator defined by (2.1), and $I$ is the identity. Then the functional central limit theorem (FCLT), as stated in the following proposition, holds for (1.2) (Gordin and Lifšic, 1978).

PROPOSITION 3.1. *Assume $p(x, dy)$ admits an invariant probability $\pi$ and, under the initial distribution $\pi$, (3.1) is ergodic. Assume also that $\tilde{\psi} = \psi - \bar{\psi}$ is in the range of $I - T$. Then*

$$n^{-1/2} \left[ \sum_{j=0}^{[nt]} (\psi(X_j) - \bar{\psi}) + (nt - [nt])(\psi(X_{[nt]+1}) - \bar{\psi}) \right] \quad (t \geq 0) \tag{3.3}$$

*converges weakly to a Brownian motion with mean zero and variance parameter $||h||_2^2 - ||Th||_2^2$, where $(I - T)h = \tilde{\psi}$, $[nt]$ is the integer part of $nt$ and $\bar{\psi} = \int \psi d\pi$.*

Our main result is the following theorem.

THEOREM 3.1. *Assume that the process in (3.1) is uniformly (geometrically) ergodic and that $X_0$ has the unique invariant $\pi$ as its distribution. Then for every $\psi$ in $L^2(S, \pi)$ such that $\int \psi d\pi = 0$, $\psi$ belongs to the range of $I - T$, i.e., (3.3) holds for every $\psi$ in $L^2(S, \pi)$ such that $\int \psi d\pi = 0$.*

*Proof.* Given the one-sided (strictly stationary) process $\{X_n : n \geq 0\}$, we can always construct a two-sided $\{X_n : n = 0, \pm 1, \pm 2, \cdots\}$ with the same finite-dimensional distribution. We may call $\{X_n : n = 0, \pm 1, \pm 2, \cdots\}$ the doubly infinite extension of $\{X_n : n \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$.

For $a \leq b$, define $\mathcal{F}_a^b$ as the $\sigma$-field generated by the random variables $X_a, \cdots, X_b$: define $\mathcal{F}_{-\infty}^a$ as the $\sigma$-field generated by $\cdots, X_{a-1}, X_a$: and define $\mathcal{F}_a^\infty$ as the $\sigma$-field generated by $X_a, X_{a+1}, \cdots$.

We claim that for each $m$ $(-\infty < m < \infty)$ and for each $n$ $(n \geq 1)$, $A \in \mathcal{F}_{-\infty}^m$, $B \in \mathcal{F}_{m+n}^\infty$ together imply

$$|P(A \cap B) - P(A)P(B)| \leq \varphi(n)P(A), \tag{3.4}$$

where $\varphi(n)$ is a nonnegative function of positive integers.

Let $g(X_{m+n})$ denote $E(I_B | \mathcal{F}_{-\infty}^{m+n})$, where $I_B$ is the indicator function of $B$. Then

$$\begin{aligned} E[I_A E(g(X_{m+n})|\mathcal{F}_{-\infty}^m)] &= E[E(I_A g(X_{m+n})|\mathcal{F}_{-\infty}^m)] \\ &= E[I_A g(X_{m+n})] \\ &= E[I_A g(X_m)] \\ &= E[I_A I_B] \\ &= P(A \cap B). \end{aligned}$$

On the other hand,

$$P(A)P(B) = E\left[I_A \int g(y)\pi(dy)\right].$$

Therefore,

$$\begin{aligned} &|P(A \cap B) - P(A)P(B)| \\ &= \left| E\left[I_A E(g(X_{m+n})|\mathcal{F}_{-\infty}^m)\right] - E\left[I_A \int g(y)\pi(dy)\right] \right| \\ &= \left| E\left[I_A \left( \int g(y)p^{(n)}(x, dy) - \int g(y)\pi(dy) \right)\right] \right| \\ &\leq E[I_A ||g||_\infty ||p^{(n)}(x, dy) - \pi(dy)||]. \end{aligned} \tag{3.5}$$

Since the process $\{X_n : n \geq 0\}$ is uniformly ergodic, there exist constants $\varepsilon > 0, \quad 0 < \rho < 1$, such that

$$||p^{(n)}(x, dy) - \pi(dy)|| < \varepsilon\rho^n$$

for all sufficiently large $n$, uniformly for $x \in S$. Thus, taking $\varepsilon\rho^n$ as our $\varphi(n)$, we have shown that for every sufficiently large $n$, (3.4) holds, and from the inequality of (3.5), a constant function $\varphi(n)$ can be taken for the other finite many number of n's, justifying our claim. Let $\psi \in 1^{\perp}$, the set of all mean zero $L^2$-functions. Then

$$\begin{aligned} ||T^n\psi||_2^2 &\equiv E(\{E[\psi(X_n)|X_0]\}^2) \\ &= E(E\{E[\psi(X_n)|X_0]\psi(X_n)|X_0\}) \\ &= E(E(\psi(X_n)|X_0)\psi(X_n)) = E((T^n\psi)(X_0)\psi(X_n)) \\ &\leq \rho(n)||\psi||_2^2 \end{aligned} \tag{3.6}$$

(by definition of the maximal correlation coefficient $\rho(n)$).

Now consider the function $g(y) := -\sum_{n=0}^{\infty}(T^n\psi)(y)$. Note that

$$||g||_2 \leq \sum_{n=0}^{\infty} ||T^n\psi||_2 \leq ||\psi||_2 \sum_{n=0}^{\infty} \sqrt{\rho(n)} < \infty,$$

by the inequality (3.6). Hence $g \in L^2(S, \pi)$. But $(T - I)g = \psi$. Hence $\psi$ is in the range of $(T - I)$. □

REMARK 1. The functional law of the iterated logarithm (FLIL) holds under the additional assumption $\int |\psi|^{2+\delta} d\pi < \infty$ for some $\delta > 0$ (Bhattacharya (1982)).

REMARK 2. As references for earlier work on a class of nonlinear autoregressive processes, see Bhattacharya and Lee (1988), and Nummelin (1984), where they have provided some verifiable conditions under which the functional central limit theorem (FCLT) holds for a class of functions $\psi$ in $L^2(\mathbb{R}^1, \pi)$.

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