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GLOBAL SHAPE OF FREE BOUNDARY SATISFYING BERNOULLI TYPE BOUNDARY CONDITION

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ABSTRACT. We study a free boundary problem satisfying Bernoulli type boundary condition along which the gradient of a piecewise harmonic solution jumps zero to a given constant value. In such problem, the free boundary splits the domain into two regions, the zero set and the harmonic region. Our main interest is to identify the global shape and the location of the zero set. In this paper, we find the lower and the upper bound of the zero set. In a convex domain, easier estimation of the upper bound and faster disk test technique are given to find a rough shape of the zero set. Also a simple proof on the convexity of zero set is given for a connected zero set in a convex domain.

1. Introduction

Free boundary value problems occur in various physical and engineering systems. Last few decades, significant progresses have been made in the area of local regularity of a free boundary by many researchers such as H. Alt, L. Caffarelli, A. Friedman [2, 3, 4, 5, 7, 16]. In practice, however, we met the problem of identifying the shape and the location of a free-boundary. Much less result has been known about the global

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geometry of the zero set partially due to the strong non-linearity of the free boundary.

A free boundary problem satisfying Bernoulli type boundary condition is our main subject in this paper. Suppose Ω is a bounded domain in $\mathbf{R}^2$ with $\mathcal{C}^2$ boundary $\partial\Omega$ and c, μ are given positive constants. Consider the following minimizing problem:

$$(1)\quad \mathcal{M}_\mu^\Omega \;\left|\; \begin{array}{ll} \text{Minimize} & J_\mu(w) := \int_\Omega |\nabla w|^2 - \mu^2|\{x \in \Omega : w(x) = 0\}| \\ \text{within the class} & \mathcal{K} = \{u \in H^1(\Omega) \;:\; u|_{\partial\Omega} = c\} \end{array}\right. .$$

A minimizer w_μ of the problem $\mathcal{M}_\mu^\Omega$ satisfies the following equations: (See [8] by Friedman and Liu)

$$(2)\qquad w \geq 0 \quad \text{in} \quad \Omega \quad \text{with} \quad w|_{\partial\Omega} = c$$

$$(3)\qquad \Delta w = 0 \quad \text{in} \quad \{w > 0\}$$

$$(4)\qquad |\nabla w^+|^2 = \mu^2 \quad \text{on} \quad \partial G_\mu^\Omega, \quad G_\mu^\Omega := \{x \in \Omega : w_\mu(x) = 0\}$$

where ∇w^+ denotes the limits of ∇w from outside of zero set G_μ. $\mathcal{M}_\mu^\Omega$ will be denoted by $\mathcal{M}_\mu$ and similarly G_μ^Ω by G_μ when the referring domain is fixed and clear from the text for simplicity of the notation.

We are interested in identifying the shape of the free boundaries ∂G_μ^Ω for a given domain Ω. Because of non-uniqueness of the free boundary and extremely sensitive dependency to the global geometry of Ω, it is not easy to identify the explicit shape of the free boundary. So, our aim is to extract some reduced information of the free boundary such as size, rough location, and upper/lower bound of the zero set G_μ.

In our recent papers [10, 14], we investigated the global shape of two-phase free boundary and some techniques can be similarly applied and extended to our one-phase problem. In Section 2, we give such extensions. For example, Lemma 2.1 shows that the Lebesque measure of G_μ strictly increases for μ bigger than a critical value μ_0^Ω. A practical method to guess approximate shape of two-phase free boundary, so called test-disk technique, can also be applied in Lemma 2.4. And using this new technique we have the following monotonicity properties in Theorem 2.5. If B_m and B_M are two disks with $B_m \subset \Omega \subset B_M$,

$$G_\mu^{B_m} \subset G_\mu^\Omega \subset G_\mu^{B_M}$$

where $G_\mu^{B_m}$ and $G_\mu^{B_M}$ be the zero set, which can be evaluated explicitly, corresponding to the problem $\mathcal{M}_\mu^{B_m}$ and $\mathcal{M}_\mu^{B_M}$, respectively. Also the zero set G_μ of $\mathcal{M}_\mu^\Omega$ is characterized by two shrunk sub-domains Ω_{ρ_1}, Ω_{ρ_2} in Corollary 2.7

$$\Omega_{\rho_1} \subset G_\mu \subset \Omega_{\rho_2}$$

where $\Omega_\rho := \{x \in \Omega : \text{distance}(x, \partial\Omega) \geq \rho\}$ and ρ_1, ρ_2 can be explicitly computed .

Due to the extreme sensitivity of the zero set G_μ to the domain Ω, there are many counter intuitive examples in non-convex domain. However, if we restrict ourselves in the case of convex domain Ω, it is possible to draw more results on the zero set G_μ, which is our main interest in Section 3. Although Corollary 2.7 provides upper bound of G_μ, Theorem 3.1 provides a simpler way to get a rough estimation on the upper bound of the zero set G_μ

$$G_\mu \subset \Omega_{1/\mu}.$$

without solving sequence of solutions of Laplace equations. Theorem 3.4 gives properties of a convex extension G^* of the zero set G as follows:

$$\text{either} \quad G = G^* \quad \text{or} \quad \frac{|\partial G|}{|G|} < \frac{|\partial G^*|}{|G^*|},$$

therefore, connectedness of G implies convexity of G. Finally, Theorem 3.5 extends the test disk technique in the case of convex Ω and reduces the number of trying test-disks.

2. One phase free boundary

Let w_μ be any minimizer among many possible minimizers of $\mathcal{M}_\mu$ and G_μ be the corresponding zero set. It is easy to see that $G_0 = \emptyset$, $G_\infty = \Omega$, and

$$J_\mu(w_\mu) \leq J_\mu(w \equiv 1) = 0.$$

LEMMA 2.1. *If $\mu_1 < \mu_2$, then*

$$|G_{\mu_1}| \leq \frac{J_{\mu_1}(\mu_1) - J(\mu_2)}{\mu_2^2 - \mu_1^2} \leq |G_{\mu_2}| \tag{5}$$

where $J(\mu) := J_\mu(w_\mu)$ is independent of the choice of w_μ. Moreover, there is a positive constant $\mu_0 := \mu_0^\Omega > 2\sqrt{\frac{\pi}{|\Omega|}}$ so that

$$G_\mu = \begin{cases} G_\mu = \emptyset \text{ if } \mu < \mu_0 \\ G_\mu \neq \emptyset \text{ if } \mu > \mu_0. \end{cases} \tag{6}$$

Proof. Since w_{μ_1} and w_{μ_2} are minimizers of $\mathcal{M}_{\mu_1}$ and $\mathcal{M}_{\mu_2}$, respectively,

$$J_{\mu_1}(w_{\mu_1}) \leq J_{\mu_1}(w_{\mu_2}) = J_{\mu_2}(w_{\mu_2}) + (\mu_2^2 - \mu_1^2)|G_{\mu_2}|, \tag{7}$$

$$J_{\mu_2}(w_{\mu_2}) \leq J_{\mu_2}(w_{\mu_1}) = J_{\mu_1}(w_{\mu_1}) - (\mu_2^2 - \mu_1^2)|G_{\mu_1}|. \tag{8}$$

These inequalities give the lower and the upper bounds of $J\mu_1(w_{\mu_1}) - J_{\mu_2}(w_{\mu_2})$ in (5) which states that G_μ is non-decreasing with respect to μ. Although $|G_\mu|$ depends on choice of minimizer, $|G_\mu|$ is non-decreasing and may have jumps only on countably many points. Therefore,

$$J(\mu) := J_\mu(w_\mu) = -\int_0^\mu 2\lambda |G_\lambda| d\lambda \tag{9}$$

is well-defined and independent of minimizer. Also, $J(\mu)$ is Lipschitz continuous and non-increasing with respect to μ, $J(\mu) = 0 \to -\infty$ as $\mu = 0 \to \infty$.

Let $\mu_0 = \inf\{\mu \geq 0 : J(\mu) < 0\}$, then $G_\mu \neq \emptyset$ for $\mu > \mu_0$ and $J(\mu_0) = 0$ for $\mu < \mu_0$. Since $J_{\mu_0}(u \equiv c) = 0$, $w_{\mu_0} \equiv c$ is a minimizer of $\mathcal{M}_\mu$ with $G_{\mu_0} = \emptyset$. So it follows from (5) that $G_\mu = \emptyset$ for $\mu < \mu_0$. In order to estimate the lower bound of μ_0, recall the definition of minimizer J in (1) and the fact that $|\nabla w_\mu^+| = \mu$ on ∂G_μ (4),

$$J_\mu(w_\mu) = \int_{\partial G_\mu} |\nabla w_\mu| - \mu^2 |G_\mu| = \mu(|\partial G_\mu| - \mu|G_\mu|) \leq 0. \tag{10}$$

Therefore, if $G_\mu \neq \emptyset$, $\frac{|\partial G_\mu|}{|G_\mu|} \leq \mu$. A disk has the minimum perimeter to area ratio $2\sqrt{\frac{\pi}{area}}$ among all closed region in $\mathbf{R}^2$ with fixed area. Therefore, if $G_\mu \neq \emptyset$,

$$\mu \geq \frac{|\partial G_\mu|}{|G_\mu|} \geq 2\sqrt{\frac{\pi}{|G_\mu|}} > 2\sqrt{\frac{\pi}{|\Omega|}}. \tag{11}$$

This ends the proof of (6). □

LEMMA 2.2. *Let Ω be a disk B_a with radius a. Then*

$$G_\mu = \begin{cases} \emptyset, & \mu < \mu_0^{B_a} \\ B_\eta, & \mu > \mu_0^{B_a} \end{cases} \tag{12}$$

where the radius $\eta = \eta_\mu^a$ of zero set B_η satisfies $(\eta \log \frac{a}{\eta})^{-1} = \mu > \mu_0^{B_a} := \frac{2\sqrt{e}}{a}$.

Proof. As in [10], we can derive $w_\mu(x)$ is radial, that is, $w_\mu(x) = w_\mu(|x|)$. Hence G_μ is a disk or a empty set. Suppose that G_μ is a disk of radius $\eta > 0$. It follows from direct computation that

$$w_\mu(x) = \frac{\log |x|/\eta}{\log a/\eta} \quad \text{if } \eta < |x| < a \tag{13}$$

and the radius η is choose to minimize the energy functional (1),

$$J(\mu) = \inf_{0<s<a} \pi\left(\frac{2}{\log a/s} - \mu^2 s^2\right) = \pi\left(\frac{2}{\log a/\eta} - \mu^2\eta^2\right) \leq 0.$$

For sufficiently large μ, $J(\mu)$ has two critical values when

$$\eta \log a/\eta = \frac{1}{\mu} \tag{14}$$

and the larger radius η gives the minimum value

$$J(\mu) = \frac{\pi}{\log a/\eta}\left(2 - \frac{1}{\log a/\eta}\right) \leq 0. \tag{15}$$

Since $J(\mu) < 0$ implies $\eta > \frac{a}{\sqrt{e}}$, the lower bound of μ to make non-empty zero set is given as follows

$$\mu = (\eta \log a/\eta)^{-1} > 2\frac{\sqrt{e}}{a} = \mu_0^{B_a}. \tag{16}$$

This completes the proof. □

REMARK. When Ω is a disk of radius a and $\mu_0 = 2\frac{\sqrt{e}}{a}$, both $w \equiv 1$ and $w = 2(\log \frac{r}{a} + \frac{1}{2})\chi_{\{r > \frac{a}{\sqrt{e}}\}}$ have the same minimum $J(\mu_0) = 0$ for the problem $\mathcal{M}_{\mu_0}$. This tells us that there are two different minimizers when $\mu = \mu_0$.

Since a minimizer w_μ satisfies the Laplace equation (3) in $\Omega \setminus G_\mu$, it gives important information to investigate the harmonic function defined in $\Omega \setminus D$. Solving this forward problem with given zero set D is much easier than finding the zero set G_μ of the inverse problem $\mathcal{M}_\mu$. Our goal is to estimate G_μ by solving a sequence of the forward problems numerically.

DEFINITION 2.3. Let h_D^Ω be the solution of the following Dirichlet problem in $\Omega \setminus D$,

$$\Delta h_D^\Omega = 0 \quad \text{in} \quad \Omega \setminus D$$
$$h_R^\Omega|_{\partial D} = 0, \quad h_R^\Omega|_{\partial\Omega} = 1.$$

The harmonic function h_D^Ω is simply written as h_D if the domain Ω is implicitly known in the context and h_D sometimes denotes a function extended to the whole domain Ω with $h_D(x)|_D \equiv 0$.

Suppose D_1 and D_2 are subdomains of Ω with $D_1 \subset D_2$, then it follows from the maximum principles and the Hopf's lemma that

$$|\nabla h_{D_2}| \geq |\nabla h_{D_1}| \quad \text{on } \partial\Omega \tag{17}$$

$$|\nabla h_{D_2}| \leq |\nabla h_{D_1}| \quad \text{on } \partial D_1 \cap \partial D_2 \tag{18}$$

The following lemma states a test disk technique which checks if a test disk is a subset of G_μ. A similar result was introduced first in our previous paper [14] which dealt with two-phase free boundary value problem, however, we include the sketch of the proof again, for the sake of clarity.

LEMMA 2.4 (Test Disk Technique). *Suppose an open disk B in Ω and the corresponding harmonic function h_B satisfy the following radius and gradient conditions*

$$\frac{|\partial B|}{|B|} \leq \mu \quad \text{and} \quad |\nabla h_B| \leq \mu \text{ on } \partial B. \tag{19}$$

Then, the test ball B is a subset of G_μ,

$$B \subset G_\mu. \tag{20}$$

Proof. Let $G^* := B \cup G$ and assume $E = B \setminus G \neq \emptyset$. Since $|\nabla h_G| = \mu$ on ∂G and $|\nabla h_B| \leq \mu$ on ∂B, $|\nabla h_{G^*}| < \mu$ on ∂G^* by (18). A simple computation like (10) gives

$$\begin{aligned} J(h_{G^*}) &< \mu|\partial G^*| - \mu^2|G^*| \\ &= \mu|\partial G| - \mu^2|G| + \mu I \end{aligned}$$

where $I = |\partial B \setminus \bar{G}| - |\partial G \cap B| - \mu|E|$. Without loss of generality, we may assume that $\partial B \setminus \bar{G}$ has only one connected component, otherwise we can sum up the total I by adding I for each connected arc segments. Let Γ_1 be a connected arc in $\partial B \setminus \bar{G}$ and E_1 be the corresponding connected component of E near Γ_1. It is possible to prove that

$$I_1 := |\Gamma_1| - |\partial E_1 \cap B| - \mu|E_1| < 0$$

using polar coordinate on ∂E_1. We omit detailed proof since a similar computation has been given in our previous paper [14]. The fact that $I < I_1 < 0$ under the assumption $E \neq \emptyset$ draws a contradiction to h_G is a minimizer of $\mathcal{M}_\mu^\Omega$ since $J(h_{G^*}) < J(h_G) + \mu I$. Therefore, $E = \emptyset$ and $B \subset G_\mu$. $\square$

REMARK. The zero set $G_\mu = B_\eta$ of $\mathcal{M}_\mu^{B_a}$, $\mu \geq \mu_0$ satisfies the radius condition (19) in the test disk technique

$$\frac{|\partial B|}{|B|} = \frac{2}{\eta} = 2\mu \log\frac{a}{\eta} \leq \mu, \text{ for } \mu \geq \mu_0. \tag{21}$$

For $\mu = \mu_0$, B_η is the only disk satisfying the radius condition (19). As μ becomes larger, $|G_\mu|$ is getting bigger and smaller test disks can be used, which makes the technique more easily applicable.

It is natural to ask if there is a simple relationship between Ω and G_μ^Ω for fixed μ. For $\Omega_1 \subset \Omega_2$, it is easy to see that

$$J_\mu^{\Omega_1}(w_\mu^{\Omega_1}) = J_\mu^{\Omega_2}(\bar{w}_\mu^{\Omega_1}) \geq J_\mu^{\Omega_2}(w_\mu^{\Omega_2}) \tag{22}$$

where $\bar{w}_\mu^{\Omega_1}$ is an extension of $w_\mu^{\Omega_1}$ and defined to be 1 in $\Omega_2 \setminus \Omega_1$. Since $J_\mu^\Omega(w_\mu^\Omega) = 0$ for $\mu = \mu_0^\Omega$,

$$\mu_0^{\Omega_1} \geq \mu_0^{\Omega_2}. \tag{23}$$

One might ask if a statement that $G_\mu^{\Omega_1} \subset G_\mu^{\Omega_2}$ is true or not. The next theorem gives a partial result about this conjecture.

THEOREM 2.5 (Disk Inclusion). *Suppose Ω is bounded by two disks B_m, B_M of radius m, M which may not be concentric,*

$$B_m \subset \Omega \subset B_M. \tag{24}$$

Then, the zero set G_μ of the minimization problem $\mathcal{M}_\mu^\Omega$ for $\mu \geq \mu_0^\Omega$ is bounded by two zero sets of B_m and B_M,

$$G_\mu^{B_m} \subset G_\mu \subset G_\mu^{B_M} \tag{25}$$

where $G_\mu^{B_m}$, $G_\mu^{B_M}$ are given explicitly in (12).

Proof. By maximum principle for two harmonic functions in $B_m \setminus \bar{G}_\mu^{B_m}$,

$$h^\Omega_{G_\mu^{B_m}} \leq h^{B_m}_{G_\mu^{B_m}} \quad \text{in } B_m \setminus \bar{G}_\mu^{B_m}.$$

Therefore,

$$|\nabla h^\Omega_{G_\mu^{B_m}}| \leq |\nabla h^{B_m}_{G_\mu^{B_m}}| = \mu \quad \text{on } \partial G_\mu^{B_m}.$$

Since the zero set $G_\mu^{B_m}$ satisfies the radius condition (21), it follows from the test disk technique that $G_\mu^{B_m} \subset G_\mu$.

Now we will prove $G_\mu \subset G_\mu^{B_M}$. Since $G_\infty^{B_M} = B_M \supset \Omega$ and $G_\lambda^{B_M}$ is continuously increasing with respect to $\lambda > \mu_0^{B_M}$,

$$\mu^* = \inf\{\lambda : G_\mu^\Omega \subset G_\lambda^{B_M}\} \tag{26}$$

is well-defined. Suppose $\mu^* = \mu_0^{B_M}$, then $\mu \geq \mu_0^\Omega \geq \mu^*$ which proves $G_\mu^\Omega \subset G_{\mu^*}^{B_M} \subset G_\mu^{B_M}$. Now, we need to check the case of $\mu^* > \mu_0^{B_M}$. In such case, $\partial G_{\mu^*}^{B_M}$ touches ∂G_μ^Ω. Let P be a contact point and let us compare the minimizers $w_{\mu^*}^{B_M}$ and w_μ^Ω in $\Omega \setminus G_{\mu^*}^{B_M}$.

$$0 = w_{\mu^*}^{B_M} \leq w_\mu^\Omega \qquad \partial G_{\mu^*}^{B_M} \tag{27}$$

$$w_{\mu^*}^{B_M} \leq w_\mu^\Omega = 1 \quad \partial\Omega. \tag{28}$$

Therefore, $w_{\mu^*}^{B_M} \leq w_\mu^\Omega$ by maximum principle, and Hopf's lemma gives

$$|\nabla w_{\mu^*}^{B_M}(P)| = \mu^* \leq |\nabla w_\mu^\Omega(P)| = \mu. \tag{29}$$

And it proves that $G_\mu^\Omega \subset G_{\mu^*}^{B_M} \subset G_\mu^{B_M}$. □

To improve the above theorem a little bit, let us define C^2-character and a concept of shrunk domain Ω_ρ for a given domain Ω.

DEFINITION 2.6. Ω is called C^2-domain with C^2-character L if

$$radius(D_x) \geq L \text{ for all } x \in \partial\Omega \tag{30}$$

where D_x is an inscribed disk in Ω contacting $\partial\Omega$ at x. We also define a shrunk domain

$$\Omega_\rho := \{x \in \Omega : \text{distance}(x, \partial\Omega) \geq \rho\}. \tag{31}$$

COROLLARY 2.7. *Let Ω be a domain with C^2-character L. For $\mu > \mu_0^{B_L} = \frac{2\sqrt{e}}{L}$, the zero set G_μ of $\mathcal{M}_\mu^\Omega$ is characterized by two shrunk subdomains Ω_{ρ_1}, Ω_{ρ_2} as follows:*

$$\Omega_{\rho_1} \subset G_\mu \subset \Omega_{\rho_2} \tag{32}$$

where $\rho_1 = L - \eta_\mu^L = L - radius(G_\mu^{B_L})$ and $\rho_2 = \inf\{\rho > 0 : \inf_{\partial\Omega_\rho} |\nabla h_{\Omega_\rho}| \leq \mu\}$.

Proof. From Theorem 2.5, the zero set G_μ must contain a concentric subdisk with radius η_μ^L of the inscribing tangent disk D_x with radius L at $x \in \partial\Omega$. If we can apply these along the boundary $\partial\Omega$, we could conclude that the set $\Omega_{L-\eta_\mu^L} \setminus \Omega_{L+\eta_\mu^L}$ is contained in the set G_μ. Hence, the maximum principle and the test disk technique proves that $\Omega_{L-\eta_\mu^L} \subset G_\mu$.

Now we prove the upper bound. Let P is a point on ∂G nearest to the boundary of the domain $\partial\Omega$ so that $\rho = dist(P, \partial\Omega) = dist(\partial G, \partial\Omega)$. Then $|\nabla h(P)| = \mu$. Note $|\nabla h_\rho(P)| \le \mu$ by maximum principle.

$$\inf_{\partial\Omega_\rho} |\nabla w_\rho| \le \mu \quad \Rightarrow \quad \rho > \rho_2. \tag{33}$$

Therefore, $G \subset \Omega_\rho \subset \Omega_{\rho_2}$. □

3. Convex Domains

Our main concern in this section is the shape of the zero set G_μ under the assumption that the given domain Ω is convex. In [9], Henrot and Shahgholian proved that the zero set G_μ in two dimensional domain Ω is convex provided it is connected with finite perimeter, however, the connectivity of G_μ is not known yet. When the domain Ω is steiner symmetric, we can use Serrin's moving plane method to get some symmetry for the zero set G_μ. For example, if Ω is steiner symmetric with respect to x- and y-axis, so does the zero set G_μ, therefore, G_μ is convex and connected. However, as far as we know, the convexity of G_μ for general convex domain $\Omega \subset R^n (n \ge 2)$is still missing. And in many practical situation, it is sometimes more interesting to estimate approximate shape of G_μ instead of convexity. The purpose of this section is to get some partial information for approximate shape and location of the zero set G_μ and we shall assume $\mu > \mu_0$, that is, G_μ is a non-empty set throughout this section.

Although Corollary 2.7 provides upper bound of G_μ using the harmonic function h_{Ω_ρ}, it requires to solve the Laplace equation in $\Omega \setminus \Omega_\rho$. The following theorem provides a rough estimation on the upper bound of the zero set G_μ without any further computation.

THEOREM 3.1. *For $\mu > \mu_0^\Omega$, the zero set G_μ of $\mathcal{M}_\mu^\Omega$ is subset of $\Omega_{1/\mu}$,*

$$G_\mu \subset \Omega_{1/\mu}. \tag{34}$$

Proof. Let ρ be the largest positive number with $G_\mu \subset \Omega_\rho$. Pick a point $x_0 \in \partial\Omega_\rho \cap \partial G$ and define a strip

$$\mathcal{S} := \{x \,|\, 0 < \langle x - x_0, \nu(x_0)\rangle < \rho\}$$

where $\nu(x_0)$ is the outward unit normal vector at $x_0 \in \partial\Omega$ and $\langle A, B\rangle$ is the inner product of the vectors A and B. Since Ω and Ω_ρ are convex,

$\Omega \cap \mathcal{S} \neq \emptyset$ and $\Omega_\rho \cap \mathcal{S} = \emptyset$. Let $h(x)$ be a linear(harmonic) function defined in $\Omega \cap \mathcal{S}$

$$h(x) := \frac{1}{\rho}\langle x - x_0, \nu(x_0)\rangle.$$

Let us compare $h(x)$ with the minimizer $w = w_\mu^\Omega$ of $\mathcal{M}_\mu^\Omega$. Since $h \leq w$ on $\partial(\mathcal{S}\cap\Omega)$, $h < w$ in $\mathcal{S}\cap\Omega$ by the maximum principle for the harmonic functions w and h. Then it follows from Hopf boundary point lemma that

$$\frac{1}{\rho} = |\nabla h(x_0)| < |\nabla w(x_0)| = \mu.$$

This completes the proof. □

The following lemma gives us a useful tool to estimate the gradient of a harmonic function. Let us consider a domain defined by two level curves of a harmonic function. In such case, it is possible to estimate the gradient of the harmonic function at the critical points of a level curve using a similar strip introduced in the previous theorem. This idea comes from the result of Henrot and Shahgholian [9].

LEMMA 3.2. *For $\phi \in C^2[-1,1]$ with $\phi(1) = \phi(-1) = 0$, let D be a simply connected bounded domain defined by*

$$D := \{(x_1, x_2) : \ -1 < x_1 < 1, 0 < x_2 < \phi(x_1)\}.$$

Suppose that a harmonic function u in D has (i) *positive constant boundary value along $\{(x_1, \phi(x_1)) : -1 < |x_1| < 1\}$ and* (ii) *its minimal value at $x^- := (x_1^-, 0)$. Then,*

$$|\nabla u(x^-)| \geq |\nabla u(x^*)|$$

where $x^* = (x_1^*, \phi(x_1^*))$ and x_1^* is any critical points of ϕ.

Proof. We may assume $|\nabla u(x^*)| = 1$ without loss of generality. For each $0 < s < 1$, define $v_s(x_1, x_2) = sx_2 - u(x_1, x_2)$ in D. Then v has a maximum at either x^- or x^* in the region $D \cap \{x_2 < \phi(x_1)\}$. Suppose v has a maximum at x^*, then

$$\frac{\partial v}{\partial x_2}(x^*) = s - \frac{\partial u}{\partial x_2}(x^*) = s - 1 < 0$$

which is a contradiction. Hence v must have a maximum at x^-. By Hopf lemma,

$$0 > \frac{\partial v}{\partial x_2}(x^-) = s - \frac{\partial u}{\partial x_2}(x^-).$$

Since the above inequality is true for all $0 < s < 1$, $|\nabla u(x^-)| \geq 1$. This completes the proof. □

COROLLARY 3.3. *Let $F_1, F_2, \cdots, F_N$ be subsets in a convex domain Ω and*

$$|\nabla h_{F_i}| \leq \mu \ \text{ on } \ \partial F_i, \quad \text{for all } \ i = 1, \cdots N.$$

Then the gradient of the harmonic function for the convex hull F of $F_1, F_2, \cdots, F_N$ satisfies

$$|\nabla h_F| \leq \mu \text{ in } \Omega \setminus F.$$

Proof. It was known from J. Lewis [13] that a level curve of h_F is convex. Moreover, it follows from the maximum principle that $|\nabla h_F| \leq \mu$ on $\partial F \cap (\partial F_1 \cup \cdots \cup \partial F_N)$. Thus,

$$|\nabla h_F| \leq \mu \ \text{ in } \ \Omega \setminus \bar{F}$$

using Lemma 3.2. □

Before further discussion on the properties of zero set G, let us introduce a new concept which extends the idea of convex hull. Convex extension of a set is an extension of the original set with disjoint union of the convex hulls of its component or some components. For example, two possible convex extensions of a set with two components are the union of the convex hulls of each components and the convex hull of both components.

THEOREM 3.4. *If G^* be a convex extension of the zero set G for $\mathcal{M}_\mu^\Omega$ in a convex domain Ω, then*

$$\text{either} \quad G^* = G \quad \text{or} \quad \frac{|\partial G|}{|G|} < \frac{|\partial G^*|}{|G^*|}. \tag{35}$$

Therefore, if G is connected which implies $\frac{|\partial G|}{|G|} \geq \frac{|\partial G^*|}{|G^*|}$, then $G^* = G$, that is, G is convex.

Suppose $G^* \neq G$ and $\frac{|\partial G^*|}{|G^*|} \leq \frac{|\partial G|}{|G|}$. Since $|\nabla h_{G^*}| \leq \mu$ by Corollary 3.3,

$$\begin{aligned} 0 \geq J(w) &= \int_\Omega |\nabla w|^2 - \mu^2 |G| = \mu |G| \left(\frac{|\partial G|}{|G|} - \mu \right) \\ &\geq \mu |G| \left(\frac{|\partial G^*|}{|G^*|} - \mu \right) > \mu |\partial G^*| - \mu^2 |G^*| \\ &\geq \int_\Omega |\nabla h_{G^*}|^2 - \mu^2 |G^*| = J(h_{G^*}) \end{aligned}$$

which contradict to the fact that w is a minimizer of $\mathcal{M}_\mu^\Omega$.

The test disk technique stated in Lemma 2.4 is useful tool to guess approximate shape of the zero set by applying disks on the domain. In the case where Ω is convex, the number of trying test-disks on the domain Ω can be reduced significantly by the following theorem.

THEOREM 3.5. *Let $B_1, B_2, ..., B_N$ be disks in a convex domain Ω satisfying two test conditions for $\mathcal{M}_\mu^\Omega$ in Theorem 2.4. Then the zero set contains the convex hull of the test disks,*

$$\textit{Convex hull}\left(\cup_{i=1}^N B_i\right) \subset G_\mu. \tag{36}$$

Proof. Let F be the convex hull of the test disks. Then $|\nabla h_F| \le \mu$ in $\Omega \setminus \bar{F}$ using Corollary 3.3. Let $E = G_\mu \cup F$ then

$$|\nabla h_E| \le \mu \text{ on } \partial E$$

since $|\nabla w| \le \mu$ on ∂G_μ and $|\nabla h_F| \le \mu$ on ∂F. Comparing two harmonic functions w and h_E, we get

$$\begin{aligned} J_\mu(w) &= \int_{\partial G_\mu} |\nabla w| - \mu^2|G_\mu| = \mu\left(|\partial G| - \mu|G|\right) \\ &\le J_\mu(h_E) \le \mu\left(|\partial E| - \mu|E|\right). \end{aligned}$$

Therefore,

$$I := |\partial E| - |\partial G_\mu| - \mu(|E| - |G_\mu|) \ge 0. \tag{37}$$

We want to prove that $G_\mu = E$. Suppose not. Let $P_1 \in \partial E \setminus \bar{G}_\mu$. It follows from the construction of F that there is a tangent disk B^{P_1} of ∂E at P_1 so that $B^{P_1} \in F$ and $\frac{|\partial B^{P_1}|}{|B^{P_1}|} \le \mu$. Let $E_1 = B^{P_1} \cup G_\mu$. From the proof of theorem 2.4, we obtain

$$|\partial E_1| - \mu|E_1| < |\partial G_\mu| - \mu|G_\mu|.$$

Next, pick $P_2 \in \partial E \setminus \bar{E}_1$ if exist and denote again by B^{P_2} the corresponding tangent disk at P_2. If $E_2 = B^{P_1} \cup G_\mu$, then, as before,

$$|\partial E_2| - \mu|E_2| < |\partial E_1| - \mu|E_1|.$$

We can choose a sequence in such a way that $E_1 \subset E_2 .... \subset E_k \to E$. (When $E_k = E$, stop the process.) Since $a_k = |\partial E_k| - \mu|E_k|$ is strictly decreasing sequence,

$$|\partial E| - \mu|E| < |\partial G_\mu| - \mu|G_\mu|.$$

which contradict to (37). This completes the proof. □

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