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REFINEMENT PERMUTATIONS OF PRIME POWER ORDER

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ABSTRACT. For a permutation μ in S_b, the limit algebra A_μ of the stationary system given by μ is isomorphic to a refinement limit algebra if and only if its exponent set $E(\mu)$ is the set $\{0\}$. In the current paper, we prove a sufficient condition under which $E(\mu) = \{0\}$ when the order of μ is a power of p, where p is a prime number dividing b.

1. Introduction

For a positive integer b, let $[b]$ denote the set $\{0, 1, 2, \ldots, b-1\}$ and let $\mu \in S_b$ be a permutation on $[b]$, of order d. Let $\phi_d : \mathbf{Z} \longrightarrow [d]$ be the canonical surjection. Recall that the *exponent set* of μ is defined as follows:

$$E_1(\mu) = [d] \;=\; \{0, 1, 2, \ldots, d-1\},$$

and

$$E_{j+1}(\mu) = \{\phi_d(x - \mu^t(x) + bt) \mid x \in [b], t \in E_j(\mu)\},$$

for $j = 1, 2, \cdots$. It follows that $E_1(\mu) \supseteq E_2(\mu) \supseteq E_3(\mu) \supseteq \;\ldots$, and that, once two successive $E_j(\mu)$ are equal, all subsequent ones are also equal. Since $E_1(\mu)$ contains only d elements, stabilization occurs no later than at $E_d(\mu)$. We write $E(\mu) = E_d(\mu)$ and call it the *exponent set* of μ. Obviously, $0 \in E_j(\mu)$, for all j, and thus $0 \in E(\mu)$. We say μ has a *trivial* exponent set if $E(\mu) = \{0\}$.

Let us recollect some terminologies related to a homogeneous direct system of matrix algebras. Define U_μ to be the permutation unitary matrix whose i, j-entry is 1 if and only if $\mu(j) = i$. Then the unitary matrix

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U_μ defines a *homogeneous* embedding $\nu_\mu : T_n \longrightarrow T_{nb}$, by the formula $\nu_\mu(a_{ij}) = (a_{ij}U_\mu^{j-i})$. Such an embedding gives rise to a direct system, called a *stationary homogeneous system*, of upper triangular subalgebras of full matrix algebras in which each embedding is the homogeneous embedding ν_μ induced by a fixed permutation μ in S_b:

$$T_b \xrightarrow{\nu_\mu} T_{b^2} \xrightarrow{\nu_\mu} T_{b^3} \xrightarrow{\nu_\mu} \cdots \longrightarrow A_\mu. \tag{1}$$

If μ is the identity permutation in S_b, then ν_μ is the *refinement embedding*, and the limit algebra A_μ in (1) is called the *refinement algebra*, which is denoted by A_o. Not rarely a permutation μ distinct from the identity can give the limit algebra A_μ which is (*isomorphic* to) the refinement algebra. Such a permutation is sometimes called a *refinement permutation.*

A natural question arises here: For a fixed base b, which permutations in S_b are the refinement permutations?

Hopenwasser and Peters ([2]) showed the exponent set $E(\mu)$ gives a complete characterization of those permutations for which A_μ is a refinement algebra. Here are a few results of Hopenwasser and Peters, which we utilize in the later sections.

LEMMA 1.1 ([2]). *Let μ be a permutation in S_b with order d. Suppose $E(\mu) = \{0\}$. Then d divides a power of b.*

THEOREM 1.2 ([2]). *Let $\mu \in S_b$ and let A_μ be the limit algebra of the stationary system of nest embeddings associated with μ. A_μ is (isomorphic to) a refinement algebra A_μ if and only if $E(\mu) = \{0\}$.*

However, for a given permutation μ, to determine whether A_μ is a refinement algebra, it is an interesting approach to examine the permutation μ itself without a series of tedious computations of the exponent set $E(\mu)$. In the current paper, our special interest goes to the permutations of order a power of a prime number. Theorem 3.1 gives the sufficient conditions for a permutation of prime power order to have a trivial exponent set. The converses for a few specific cases are given in Theorem 3.7 and Theorem 3.10.

2. Trivial Exponent Sets

Let μ be a permutation in S_b. By R_μ, we denote the set of the members of $[b]$ which are not fixed by μ. That is,
$R_\mu = \{x \in [b] \mid x - \mu(x) \neq 0\}$.

LEMMA 2.1. *Let $\mu \in S_b$ and $\sigma \in S_c$ be the permutations of order d such that $R_\mu = R_\sigma$, $\mu(x) = \sigma(x)$ for $x \in R_\mu$. Then $\phi_d(b) = \phi_d(c)$ implies $E(\mu) = E(\sigma)$.*

PROOF. Suppose that $\phi_d(b) = \phi_d(c)$. It is obvious that $E_1(\mu) = [d] = E_1(\sigma)$. Proceeding by induction, we assume that $E_m(\mu) = E_m(\sigma)$ holds for some $m \geq 1$. We have

$$\begin{aligned} E_{m+1}(\mu) &= \{\phi_d(x - \mu^n(x) + bn) \mid x \in [b], n \in E_m(\mu)\} \\ &= \{\phi_d(x - \mu^n(x)) + \phi_d(b)\phi_d(n) \mid x \in [b], n \in E_m(\mu)\} \\ &= \{\phi_d(x - \mu^n(x)) + \phi_d(b)\phi_d(n) \mid x \in R_\mu, n \in E_m(\mu)\} \\ &\quad \cup \ \{\phi_d(x - \mu^n(x)) + \phi_d(b)\phi_d(n) \mid x \notin R_\mu, n \in E_m(\mu)\} \\ &= \{\phi_d(x - \mu^n(x)) + \phi_d(b)\phi_d(n) \mid x \in R_\mu, n \in E_m(\mu)\} \\ &\quad \cup \ \{\phi_d(b)\phi_d(n) \mid n \in E_m(\mu)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} E_{m+1}(\sigma) &= \{\phi_d(x - \sigma^n(x) + cn) \mid x \in [c], n \in E_m(\sigma)\} \\ &= \{\phi_d(x - \sigma^n(x)) + \phi_d(c)\phi_d(n) \mid x \in R_\sigma, n \in E_m(\sigma)\} \\ &\quad \cup \ \{\phi_d(c)\phi_d(n) \mid n \in E_m(\sigma)\}. \end{aligned}$$

Since $\mu(x) = \sigma(x)$ for $x \in R_\mu = R_\sigma$, it follows that $\mu^n(x) = \sigma^n(x)$ for any nonnegative integer n. Thus, by equating every pair of corresponding objects, we have $E_{m+1}(\mu) = E_{m+1}(\sigma)$. So, by induction, we have $E_m(\mu) = E_m(\sigma)$ for every $m \geq 1$. In particular, $E(\mu) = E(\sigma)$. □

With Lemma 2.1, in a practical computation of $E(\mu)$ for a permutation μ of order d in S_b, we may take b as small as possible, provided that $\mu \in S_b$ and the number $\phi_d(b)$ remains unchanged.

LEMMA 2.2. *Let μ be a permutation of order d in S_b, where d divides a power of b. If $\phi_d(x - \mu(x)) = 0$ for all $x \in R_\mu$, then $E(\mu) = \{0\}$.*

PROOF. Assume $\phi_d(x-\mu(x))=0$ for all $x \in R_\mu$. Let d divide b^k for some $k \geq 1$. Then

$$\begin{aligned} E_1(\mu) &= [d], \\ E_2(\mu) &= \{\phi_d(x-\mu^n(x)+bn) \mid x \in [b], n \in E_1(\mu)\}. \end{aligned}$$

For any $x \in R_\mu$ and for any $n \in E_1(\mu)$, either $\mu^n(x) \in R_\mu$ or x is a fixed point of μ^n. Thus $\phi_d(x-\mu^n(x))=0$. Therefore

$$E_2(\mu) = \{\phi_d(bn) \mid n \in E_1(\mu)\}.$$

$$\begin{aligned} E_3(\mu) &= \{\phi_d(x-\mu^n(x)+bn) \mid x \in [b], n \in E_2(\mu)\} \\ &= \{\phi_d(bn) \mid n \in E_2(\mu)\} \\ &= \{\phi_d(b\phi_d(bn)) \mid n \in E_1(\mu)\} \\ &= \{\phi_d(b^2n) \mid n \in E_1(\mu)\}. \end{aligned}$$

Repeating, we have

$$E_{k+1}(\mu) = \{\phi_d(b^kn) \mid n \in E_1(\mu) = [d]\}.$$

Since d divides b^k, $E_{k+1}(\mu) = \{0\}$. Therefore $E(\mu) = \{0\}$. □

If x is a fixed point of μ, then it is always true that $\phi_d(x-\mu(x))=0$. So, the condition $\phi_d(x-\mu(x))=0$ holds for all $x \in R_\mu$ if and only if it holds for all $x \in [b]$. We will mention R_μ instead of $[b]$ to emphasize the permutation μ itself. Also note that, for a cycle μ, $\phi_d(x-\mu(x))=0$ for all $x \in R_\mu$ if and only if $\phi_d(x-y)=0$ for any two elements x and y of R_μ.

The converse of Lemma 2.2 need not be true in general:

Let $\mu = (0\ 2\ 4\ 6)(1\ 3\ 5\ 7)$ be a permutation in S_8. Then $E(\mu)=\{0\}$, but $\phi_4(x-\mu(x)) \neq 0$ for any $x \in R_\mu$.

THEOREM 2.3 ([3]). *Let p be a prime number, and μ be a permutation of order p in S_b, where p divides b. $E(\mu)=\{0\}$ if and only if $\phi_p(x-\mu(x))=0$ for all $x \in R_\mu$.*

PROOF. By Lemma 2.2, it suffices to prove that if $E(\mu) = \{0\}$, then $\phi_p(x - \mu(x)) = 0$ for all $x \in R_\mu$. Suppose that there exists $y_0 \in R_\mu$ such that $\phi_p(y_0 - \mu(y_0)) \neq 0$. Obviously, $E_1(\mu) = [p] \neq \{0\}$. Assume $E_m(\mu) \neq \{0\}$ for some $m \geq 1$. Then there exists a nonzero $t \in E_m(\mu)$. Since p is a prime number and $E_m(\mu) \subseteq [p]$, t is relatively prime to p. Therefore $R_{\mu^t} = R_\mu$, and thus there exists $x_0 \in R_\mu$ such that $\phi_p(x_0 - \mu^t(x_0)) \neq 0$. Indeed, if $\phi_p(x - \mu^t(x)) = 0$ for all $x \in R_{\mu^t}$, then, since t is relatively prime to p, $\phi_p(x - y) = 0$ for all $x, y \in R_{\mu^t}$. Regarding y_0 and $\mu(y_0)$ as elements of R_{μ^t}, we have $\phi_p(y_0 - \mu(y_0)) = 0$ contradictory to the choice of y_0. Since $\phi_p(x_0 - \mu^t(x_0)) \in E_{m+1}(\mu)$, we have $E_{m+1}(\mu) \neq \{0\}$. Thus, by induction, $E_m(\mu) \neq \{0\}$ for all m. In particular, $E(\mu) \neq \{0\}$. □

THEOREM 2.4 ([3]). *Let $\pi = \mu\sigma$, where μ and σ are disjoint permutations in S_b.*

If $A_\pi \cong A_o$, then $A_\mu \cong A_o$ and $A_\sigma \cong A_o$.

The converse fails: Let $\mu = (0\ 3\ 6)$ and $\sigma = (1\ 5)$ in S_{12}. Then $E(\mu) = E(\sigma) = \{0\}$, by Theorem 2.3. But $E(\mu\sigma) = \{0, 2, 3, 4\}$.

COROLLARY 2.5. *Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a product of disjoint permutations in S_b.*

If $A_\pi \cong A_o$, then $A_{\pi_i} \cong A_o$ for each i.

Here is a partial converse of Theorem 2.4.

THEOREM 2.6 ([3]). *Let μ, σ be disjoint cycles in S_b of order p, q, respectively, where p and q are distinct prime numbers both dividing b. Let $\pi = \mu\sigma$. Suppose $E(\mu) = \{0\}$ and $E(\sigma) = \{0\}$. Then $E(\pi) = \{0\}$ if and only if at least one of the following conditions holds:*

$$(i)\ \phi_q(x - \mu(x)) = 0, \ \text{for all } x \in R_\mu,$$
$$(ii)\ \phi_p(x - \sigma(x)) = 0, \ \text{for all } x \in R_\sigma.$$

If A_μ is a refinement algebra, Lemma 2.1 can be much enhanced:

LEMMA 2.7. *Let $\mu \in S_b$ and $\sigma \in S_c$ be permutations of order d dividing both a power of b and a power of c, $R_\mu = R_\sigma$ and $\mu(x) = \sigma(x)$ for $x \in R_\mu$. Assume both b and c have the same supernatural number. Then $E(\mu) = \{0\}$ if and only if $E(\sigma) = \{0\}$.*

PROOF. By Power ([10]), a refinement algebra in S_b is isomorphic to a refinement algebra in S_c if and only if both b and c have the same supernatural number. Thus if this refinement algebra is arising from a permutation, then its exponent set must be $\{0\}$. $\square$

3. Refinement Permutations of Order p^n

In the previous section, we have introduced some known results on the refinement permutations. Let μ be a refinement permutation in S_b. If the order of μ has the prime factorization $p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$, then, by Lemma 1.1, each p_i necessarily divides b. The simplest form of such order is p^n, where p is a prime number which divides b. We now prove the main result of the current section:

THEOREM 3.1. *Let μ be a permutation of order p^n in S_b, where p is a prime number which divides b. If $\phi_{p^j}(x-\mu^{p^{j-1}}(x))=0$ for every $x\in R_\mu$, and for each $j=1,2,\ldots,n$, then $E(\mu)=\{0\}$.*

PROOF. We may assume, by Lemma 2.7, that p^n divide b so that we can drop the term $+\ bt$ in the computation of the exponent sets. To find out the exponent set of μ, we start with

$$\begin{aligned} E_1(\mu) &= [p^n], \\ E_2(\mu) &= \{\phi_{p^n}(x-\mu^t(x)) \mid x\in[b], t\in E_1(\mu)\}. \end{aligned}$$

Since $\phi_p(x-\mu(x))=0$ for all $x\in R_\mu$ with $j=1$, we have $\phi_p(x-y)=0$ for any pair $x,y\in R_\mu$. Hence for each $t\in E_1(\mu)$,

$$\phi_p\phi_{p^n}(x-\mu^t(x))=\phi_{p^n}\phi_p(x-\mu^t(x))=\phi_{p^n}(0)=0.$$

Thus

$$E_2(\mu)\subseteq p\mathbf{Z}\cap[p^n]=\{0,\ p,\ 2p,\ \cdots,\ (p^{n-1}-1)p\}.$$

Next, assume inductively that

$$E_m(\mu)\subseteq p^{m-1}\mathbf{Z}\cap[p^n]=\{0,\ p^{m-1},\ 2p^{m-1},\ \cdots,\ (p^{n-m+1}-1)p^{m-1}\},$$

for $m \leq n$. Then

$$\begin{aligned}E_{m+1}(\mu) &= \{\phi_{p^n}(x - \mu^t(x)) \mid x \in [b], t \in E_m(\mu)\} \\ &\subseteq \{\phi_{p^n}(x - \mu^t(x)) \mid x \in [b], t = kp^{m-1}, k = 0, 1, \cdots, p^{n-m+1} - 1\}.\end{aligned}$$

Let $x \in [b]$, $t = kp^{m-1}$ with $k = 0, 1, \cdots, p^{n-m+1} - 1$. Since $\phi_{p^m}(x - \mu^{p^{m-1}}(x)) = 0$, we have

$$\begin{aligned}&\phi_{p^m}(x - \mu^{kp^{m-1}}(x)) \\ &= \phi_{p^m}(x - \mu^{p^{m-1}}(x) + \mu^{p^{m-1}}(x) - \mu^{2p^{m-1}}(x) + \cdots - \mu^{kp^{m-1}}(x)) \\ &= \phi_{p^m}(\sum_{j=0}^{k-1} \phi_{p^m}(\mu^{jp^{m-1}}(x) - \mu^{(j+1)p^{m-1}}(x))) \\ &= \phi_{p^m}(\sum_{j=0}^{k-1} \phi_{p^m}(\mu^{jp^{m-1}}(x) - \mu^{p^{m-1}}(\mu^{jp^{m-1}}(x)))) \\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\phi_{p^m}(\phi_{p^n}(x - \mu^t(x))) &= \phi_{p^n}(\phi_{p^m}(x - \mu^t(x))) \\ &= \phi_{p^n}(\phi_{p^m}(x - \mu^{kp^{m-1}}(x))) \\ &= \phi_{p^n}(0) \\ &= 0.\end{aligned}$$

It follows that

$$\begin{aligned}E_{m+1}(\mu) &\subseteq p^m\mathbf{Z} \cap E_m(\mu) \\ &\subseteq p^m\mathbf{Z} \cap p^{m-1}\mathbf{Z} \cap [p^n] \\ &= p^m\mathbf{Z} \cap [p^n].\end{aligned}$$

Thus $E_{m+1}(\mu) \subseteq p^m\mathbf{Z} \cap [p^n]$ for every $m \leq n$. In particular,

$$E_{n+1}(\mu) \subseteq p^n\mathbf{Z} \cap [p^n] = \{0\}.$$

Therefore $E(\mu) = \{0\}$. $\square$

EXAMPLE 3.2. Let $\mu = (0\ 3\ 6\ 9\ 12\ 15\ 18\ 21\ 24)$ be a permutation in S_{30}. The order of μ is $9 = p^n$ with $p = 3$, $n = 2$. Since $\mu^3 = (0\ 9\ 18)(3\ 12\ 21)(6\ 15\ 24)$, it is obvious that the condition in Theorem 3.1 holds for μ: $\phi_p(x - \mu(x)) = 0$ and $\phi_{p^2}(x - \mu^p(x)) = 0$, for any $x \in R_\mu$. Therefore the exponent set $E(\mu)$ must be $\{0\}$ by Theorem 3.1. Indeed we have

$$\begin{aligned} E_1(\mu) &= \{0,1,2,3,4,5,6,7,8\}, \\ E_2(\mu) &= \{0,3,6\}, \\ E_3(\mu) &= \{0\}. \end{aligned}$$

It seems that the condition in Theorem 3.1 is necessary for an exponent set to be trivial. Here are a few suggestive examples.

EXAMPLE 3.3. Let $\mu = (1\ 3\ 5\ 9)$ be a permutation of order p^n in S_{10} with $p = 2$, $n = 2$. Then the condition of Theorem 3.1 holds for $j = 1$ while it fails for $j = 2$: $\phi_4(3 - 9) \neq 0$. The exponent set is not trivial. In fact, $E(\mu) = \{0, 2\}$.

EXAMPLE 3.4. Let $\mu = (0\ 4\ 6\ 9\ 13\ 15\ 18\ 22\ 24)$ be a permutation in S_{27} of order 3^2. Then the condition of Theorem 3.1 holds for $j = 2$, since $\mu^3 = (0\ 9\ 18)(4\ 13\ 22)(6\ 15\ 24)$. But it fails for $j = 1$: $\phi_3(0 - 4) \neq 0$. We have $E(\mu) = \{0, 1, 3, 6, 8\}$.

EXAMPLE 3.5. Let $\mu = (0\ 2\ 4\ 6\ 8\ 10\ 12\ 18)$ be a cycle of order 2^3 in S_{20}. Then the condition of Theorem 3.1 holds for $j = 1$ and for $j = 2$ while the condition fails for $j = 3$. The exponent set $E(\mu)$ is $\{0, 4\}$. If $\sigma = (0\ 4\ 2\ 6\ 8\ 12\ 10\ 14)$, then the condition holds for $j = 1$ and for $j = 3$ while it fails for $j = 2$. We have $E(\sigma) = \{0, 2, 6\}$. Now if $\tau = (0\ 1\ 4\ 5\ 8\ 9\ 12\ 13)$, then the condition is false only for $j = 1$ while it is true for both $j = 2$ and $j = 3$. The exponent set is $E(\tau) = \{0, 1, 3, 4, 5, 7\}$.

The exponent sets are *symmetric* in the following sense.

LEMMA 3.6 ([3]). *Let μ be a permutation of order d in S_b. Then $t \in E_k(\mu)$ if and only if $d - t \in E_k(\mu)$, for each $k \geq 1$. In particular, $t \in E(\mu)$ if and only if $d - t \in E(\mu)$.*

PROOF. Since $E_1(\mu) = [d]$, if $t \in E_1(\mu)$, then $d - t \in E_1(\mu)$. We proceed by induction. Let $t \in E_{k+1}(\mu)$ for some $k \geq 1$. Then there exist $x \in [b]$ and $s \in E_k(\mu)$ such that

$$\phi_d(x - \mu^s(x) + bs) = t. \tag{2}$$

Put $y = \mu^s(x)$. Negating both sides of (2) and using the facts that $\mu^d = \mu^0$ and $\phi_d(bd) = 0$, we obtain:

$$\phi_d(y - \mu^{d-s}(y) + b(d - s)) = d - t. \tag{3}$$

Since $y \in [b]$ and $d - s \in E_k(\mu)$ by the induction hypothesis, (3) implies $d - t \in E_{k+1}(\mu)$. Thus we have proved that if $t \in E_k(\mu)$, then $d - t \in E_k(\mu)$ for every $k \geq 1$. In particular, if $t \in E(\mu)$, then $d - t \in E(\mu)$. Since $t = d - (d - t)$, the reverse implication follows. □

We now make use of Lemma 3.6 to prove the converse of Theorem 3.1 for $p = 2$ and $n = 2$:

THEOREM 3.7 ([6]). *Let μ be a permutation of order 4 in S_b, where b is an even integer. If $E(\mu) = \{0\}$, then both (i) $\phi_2(x - \mu(x)) = 0$ and (ii) $\phi_4(x - \mu^2(x)) = 0$ hold for all $x \in R_\mu$.*

PROOF. We may assume, by Lemma 2.7, that b is divisible by 4. Observe that every permutation of order 4 is the disjoint product of a cycle of order 4 and cycles of order 4 and/or transpositions. In case a transposition τ occurs in the product, the condition (*ii*) is always true for all $x \in R_\tau$ because the order of τ is 2. Thus, by Theorem 2.4, we can also assume that μ is a cycle of order 4.

Let $\mu = (m_1\ m_2\ m_3\ m_4)$ be a cycle of order 4, with each $m_i \in [b]$. Suppose that $\phi_2(x - \mu(x)) \neq 0$ for some $x \in R_\mu$. With the cyclicity of the cyclic notation, we may assume $x = m_1$ so that $\mu(x) = \mu(m_1) = m_2$, and thus $\phi_2(m_1 - m_2) \neq 0$ or $\phi_2(m_1 - m_2) = 1$. Thus we have either $\phi_4(m_1 - m_2) = 1$ or $\phi_4(m_1 - m_2) = 3$. By Lemma 3.6, the symmetry of the exponent sets, we observe that both 1 and 3 are contained in $E_2(\mu)$. Taking $x = m_1$ and $t = 1$, we see that $E_3(\mu)$ contains $\phi_4(x - \mu^t(x) = \phi_4(m_1 - m_2)$. That is, $E_3(\mu)$ contains both 1 and 3. Repeatedly taking $x = m_1$ and $t = 1$, we have that every $E_k(\mu)$ contains 1 and 3, which is impossible because $E(\mu) = \{0\}$. Therefore the condition (i) must hold for all $x \in R_\mu$.

Now suppose $\phi_4(m_1 - m_3) = \phi_4(m_1 - \mu^2(m_1)) \neq 0$. Then, since we already have $\phi_2(m_1 - m_3) = 0$, it must be $\phi_4(m_1 - m_3) = 2$. It follows that $2 \in E_2(\mu)$. Taking $x = m_1$ and $t = 2$, we see that $E_3(\mu)$ contains $\phi_4(x - \mu^t(x) = 2$. Repeating this, we have $2 \in E(\mu)$, another contradiction. Therefore the condition (ii) must hold for all $x \in R_\mu$. □

The condition (i) in Theorem 3.7 shows that the smallest possible value of b for S_b to contain a refinement permutation of order 4 is 8.

EXAMPLE 3.8. There are 44 refinement permutations of order 4 out of 43020 permutations in S_8, including
(0 2 4 6), (0 6 4 2), (1 3 5 7), (0 2 4 6)(1 7 5 3),
(0 2 4 6)(1 3), (1 7 5 3)(0 6), (0 2 4 6)(1 7)(3 5).

LEMMA 3.9. *For a positive integer n, let μ be a permutation of order p^n in S_b, and let there exist an element x of R_μ such that $\phi_p(x-\mu(x)) \neq 0$. If $t \in [p^n]\backslash p\mathbf{Z}$, then there exist $s \in [p^n]\backslash p\mathbf{Z}$ and $x_0 \in R_\mu$ such that $\phi_{p^n}(x_0 - \mu^t(x_0)) = s$.*

PROOF. Suppose to the contrary that $\phi_{p^n}(x - \mu^t(x)) = 0$ for every $x \in R_\mu$. Then, for $j = 1, 2, 3, \cdots$, $\phi_{p^n}(x - \mu^{jt}(x)) = \phi_{p^n}(x - \mu^t(x) + \mu^t(x) - \mu^{2t}(x) + - \cdots + \mu^{(j-1)t}(x) - \mu^{jt}(x)) = 0$. Since p^n and t are relatively prime, for any fixed x, $\{\mu^{jt}(x) \mid j = 0, 1, 2, \cdots\}$ exhausts all elements of R_μ. Thus $\phi_{p^n}(x - y) = 0$ for any pair of elements x and y of R_μ. In particular, $\phi_{p^n}(x - \mu(x)) = 0$ for every $x \in R_\mu$, hence $\phi_p(x - \mu(x)) = 0$ for every $x \in R_\mu$, which is a contradiction. Therefore there exists $x_0 \in R_\mu$ such that $\phi_{p^n}(x_0 - \mu^t(x_0)) \neq 0$.

Let $s = \phi_{p^n}(x_0 - \mu^t(x_0))$. It remains to show that x_0 can be chosen so that $\phi_p(s) \neq 0$. Suppose that $\phi_p(s) = \phi_p(\phi_{p^n}(x - \mu^t(x))) = 0$ for every $x \in R_\mu$. Then $\phi_{p^n}(\phi_p(x - \mu^t(x))) = 0$. Thus $\phi_p(x - \mu^t(x))$ is a multiple of p^n which is less than p. That is, $\phi_p(x - \mu^t(x)) = 0$ for every $x \in R_\mu$. Since t is relatively prime to p, we have $\phi_p(x - y) = 0$ for any pair of elements x and y of R_μ. Hence $\phi_p(x - \mu(x)) = 0$ for every $x \in R_\mu$, which is another contradiction. □

Now we prove a partial converse of Theorem 3.1.

THEOREM 3.10. *Let μ be a permutation of order p^n in S_b, where p is a prime number dividing b. If $E(\mu) = \{0\}$, then $\phi_p(x - \mu(x)) = 0$ for every $x \in R_\mu$.*

PROOF. Assume that p^n divides b. Suppose that $\phi_p(x_0 - \mu(x_0)) \neq 0$ for some $x_0 \in R_\mu$, that is, $\phi_p(x_0 - \mu(x_0)) \in \{1, 2, \cdots, p-1\}$. Then $\phi_{p^n}(x_0 - \mu(x_0)) \in [p^n]\backslash p\mathbf{Z}$. Let $t = \phi_{p^n}(x_0 - \mu(x_0))$. By Lemma 3.9, there exist $x_1 \in R_\mu$ and $s \in [p^n]\backslash p\mathbf{Z}$ such that $\phi_{p^n}(x_1 - \mu^t(x_1)) = s$. This means that if $0 \neq t \in E_k(\mu)$ for some $k \geq 1$, then $0 \neq s \in E_{k+1}(\mu)$. Since $s \in [p^n]\backslash p\mathbf{Z}$, Lemma 3.9 applies repeatedly. Thus $E_k(\mu) \neq \{0\}$ implies $E_{k+1}(\mu) \neq \{0\}$ for *every* $k \geq 1$. It follows that $E(\mu) \neq \{0\}$. □

References

[1] A. Hopenwasser, S. Power, *Classification of Limits of Triangular Matrix Algebras*, Proc. Edinburgh Math. Soc. **36** (1992), 107-121.

[2] A. Hopenwasser, J. Peters, *Full Nest Algebras*, Illinois J. of Math. **38** (1994), 501-520.

[3] D. Park, *Limit algebras of stationary systems*, Dissertation, Univ. of Alabama (1995).

[4] ———, *Refinement algebras arisen from stationary systems*, Youngnam Math. J. **2** (1995), 115-130.

[5] ———, *Exponent tables and limit algebras of stationary systems*, Korean J. Math. Sci. **3** (1996), 179-188.

[6] ———, *Permutations of prime square order with trivial exponent sets*, Korean J. Math. Sci. **5** (1998), 229-237.

[7] S. Power, *The Classification of tensor products of triangular operator algebras*, Proc. London Math. Soc. **61** (1990), 571-614.

[8] ———, *The Classification of triangular subalgebras of AF algebras*, Bull. London Math. Soc. **22** (1990), 269-272.

[9] ———, *Non-self-adjoint operator algebras and inverse systems of simplicial complexes*, J. Reine Angew. Math. **421** (1991), 43-61.

[10] ———, *Limit Algebras: An Introduction to Subalgebras of C^*−algebras*, Pitman Research Notes, Longman, 1992.

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