Propagation Speed of Torsional Waves in a Circular Rod with Harmonically Varying Material Properties

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Abstract

The paper describes a theoretical study on the speed of torsional elastic waves propagating in a circular rod whose material properties vary periodically as harmonic functions of the axial coordinate. An approximate solution for the phase speed has been obtained by using the perturbation technique for sinusoidal modulation of small amplitude. This solution shows that the wave speed in the nonuniform rod is dependent on the wave frequency as well as the periodic variation of the material properties. It implies that the torsional waves considered in this paper are dispersive even in the fundamental mode.

I. Introduction

Torsional elastic waves propagating in a waveguide have been adopted to develop on-line sensors measuring fluid viscosity[1]. The measurement is based on the phenomenon that the interaction between the torsional wave and the adjacent fluid affects the propagation characteristics of the torsional wave such as the speed and attenuation. The developed sensor includes an elastic waveguide with a circular cross-section.

One method for increasing the sensor's sensitivity is to reduce the mass moment of inertia of the waveguide. Another method is to increase the surface area per unit length of the waveguide contacting with the fluid. The latter can be accomplished by corrugating or modulating the surface of the waveguide, and the wave propagation in such a corrugated waveguide has been studied[2,3]. The former can be easily accomplished by taking a hollow cylinder rather than a solid one. It may, however, be necessary to consider the integrity of the hollow cylinder. In order to enhance the stiffness of the cylinder, it is reasovable to use a bamboo-shaped cylinder as shown in Fig. 1(a). Then, it is needed to understand wave propagation in such a waveguide.

The bamboo-shaped cylinder can be modelled for analysis as a circular rod with harmonically varying material properties as shown in Fig. 1(b). It is understood that the simplification of the model neglects the wave reflection which may occur at the material discontinuity. Wave propagation in periodic structures has been studied since Brillouin summarized some interesting topics in his book[4], and recently it has been applied to understand the mechanical phenomenon in a flat-panel speaker[5].

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In this paper, the propagation of the fundamental mode of torsional elastic waves in a circular rod, whose shear modulus and mass density vary periodically as harmonic functions of the axial coordinate, is theoretically studied. The analysis consists of a perturbation expansion in the amplitude of the material property variation. The approximate solution of the phase speed is obtained using the perturbation technique for sinusoidal modulation of small amplitude.

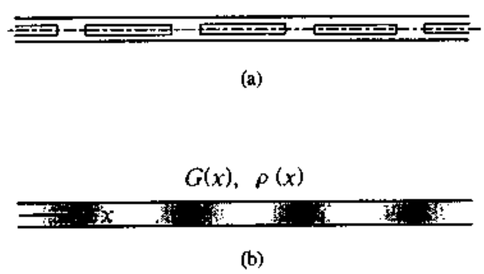


Figure 1. Cross-sectional view of the periodically nonuniform circular rod;
(a) bamboo-shaped cylinder,
(b) a rod modol with harmonically varying material properties.

II. Problem Formulation

Fig. 1(b) shows a waveguide where torsional elastic waves propagate. The waveguide consists of an elastic rod which has a circular cross-section and material properties, such as the shear modulus $G(z)$ and mass density $\rho(z)$, varying periodically as a function of the axial coordinate z. For the fundamental mode of torsional waves, the differential equation of motion describing the dependence of the angular displacement $\theta(z, t)$ on the axial coordinate z and time t is given as follows[6].

$$\frac{\partial}{\partial z}\left[G(z)\frac{\partial\theta(z,t)}{\partial z}\right] = \rho(z)\frac{\partial^2\theta(z,t)}{\partial t^2} \quad (1)$$

If the motion at each location is assumed to be time-harmonic, the variable $\theta(z,t)$ can be separated as follows.

$$\theta(z,t) = \Theta(z)\exp(-j\omega t) \quad (2)$$

Inserting Eq. (2) into Eq. (1) results in the following governing equation.

$$\frac{d}{dz}\left[G(z)\frac{d\Theta(z)}{dz}\right] = -\omega^2\rho(z)\Theta(z) \quad (3)$$

The material properties, which vary periodically as harmonic functions of the axial coordinate, are expressed as

$$G(z) = G_0(1+\varepsilon\sin\alpha z) \quad (4)$$

$$\rho(z) = \rho_0(1+\varepsilon\gamma\sin\alpha z) \quad (5)$$

where ε is the variation amplitude of the shear modulus relative to the mean value G_0, and $\varepsilon\gamma$ is the variation amplitude of the mass density relative to the mean value ρ_0. Here α is the variation number of the material properties per unit length.

Since G and ρ are functions of z as in Eqs. (4) and (5), an exact solution of Eq. (3) is not available, and therefore an approximate solution is obtained as follows.

III. Perturbation Solution

Since an exact solution to the problem formulated above is unlikely, an approximate perturbation analysis is adopted. The angular displacement amplitude $\Theta(z)$ is expanded into a power series of ε.

$$\Theta(z,\varepsilon) = \Theta_0(z)+\varepsilon\Theta_1(z)+\varepsilon^2\Theta_2(z)+\cdots \quad (6)$$

According to the perturbation technique well described by Nayfeh[7], an infinite sets of differential equations are obtained from the coefficients of like powers in ε by substituting Eqs. (4)-(6) into (3) and they are solved to yield Θ_0, Θ_1, Θ_2, $\cdots$.

3.1. Solution of $O(\varepsilon^0)$

The terms of $O(\varepsilon^0)$ results in the leading order governing equation as follows.

$$\frac{d^2\Theta_0(z)}{dz^2}+\beta_0^2\Theta_0(z) = 0 \quad (7)$$

where

$$\beta_0^2 = \frac{\omega^2\rho_0}{G_0} \quad (8)$$

This problem admits the well-known solution[8],

$$\Theta_0(z) = A_0e^{jkz} \quad (9)$$

where $j = (-1)^{1/2}$, A_0 is the wave amplitude depending on the initial conditions, and k is the wavenumber. Since the wavenumber can be varied due to the nonuniformity of the material properties, k has the following form of series in terms of the perturbation ε.

$$k = k_0+\varepsilon k_1+\varepsilon^2k_2+\cdots \quad (10)$$

Substituting Eq. (10) into Eq. (9) and then Eq. (9) into Eq. (7) yields an algebraic equation, and a root corresponding to the wave propagating in the positive direction is obtained as follows.

$$k_0 = \beta_0 \quad (11)$$

Thus the leading order solution is

$$\Theta_0(z) = A_0e^{j(\beta_0+\varepsilon k_1+\cdots)z} \quad (12)$$

3.2. Solution of $O(\varepsilon^1)$

The terms of $O(\varepsilon^1)$ result in the first order governing equation as follows.

$$\frac{d^2\Theta_1(z)}{dz^2}+\beta_0^2\Theta_1(z) = -\sin\alpha z\frac{d^2\Theta_0(z)}{dz^2}-\alpha\cos\alpha z\frac{d\Theta_0(z)}{dz} - \beta_0^2\gamma\sin\alpha z\Theta_0(z) \quad (13)$$

Substituting the leading order solution $\Theta_0(z)$ into the right-hand-side of Eq. (13) and changing the harmonic functions into exponential functions result in the following equation.

$$\frac{d^2\Theta_1(z)}{dz^2} + \beta_0^2\Theta_1(z)$$
$$= \frac{j}{2}[-(\beta_0+\epsilon k_1+\cdots)^2 - \alpha(\beta_0+\epsilon k_1+\cdots)+\beta_0^2\gamma] A_0 e^{j[(\beta_0+\alpha)+\epsilon k_1+\cdots]z}$$
$$-\frac{j}{2}[-(\beta_0+\epsilon k_1+\cdots)^2 + \alpha(\beta_0+\epsilon k_1+\cdots)+\beta_0^2\gamma] A_0 e^{j[(\beta_0-\alpha)+\epsilon k_1+\cdots]z} \tag{14}$$

The general solution of Eq. (14) has the following form.

$$\Theta_1(z) = B_1 e^{j(\beta_0+\epsilon k_1+\cdots)z} + B_2 e^{j[(\beta_0+\alpha)+\epsilon k_1+\cdots]z} + B_3 e^{j[(\beta_0-\alpha)+\epsilon k_1+\cdots]z} \tag{15}$$

By the way, the exponents of Eqs. (12) and (15) include the power series of ϵ, and they should have been considered when Eq .(6) was substituted into Eq. (3). Therefore, Eqs. (12) and (15) are inserted in Eq. (6) and then Eqs. (4)-(6) are substituted into Eq. (3). The equations of the like orders of ϵ are solved again. The leading order solution is the same as before, and Eqs. (11) and (12) are still valid. The following term is added to the right-hand side of Eq. (14): $2\beta_0 k_1 A_0 \exp[j(\beta_0+\epsilon k_1+\cdots)z]$

The terms of $O(\epsilon^1)$ yield three exponential functions, the coefficients of which compose the following equations.

$$(-\beta_0^2+k_0^2)B_1 = 2\beta_0 k_1 A_0 \tag{16}$$

$$[(\beta_0+\alpha)^2-\beta_0^2]B_2 = \frac{jA_0}{2}\beta_0(\beta_0+\alpha-\gamma\beta_0) \tag{17}$$

$$[(\beta_0-\alpha)^2-\beta_0^2]B_3 = -\frac{jA_0}{2}\beta_0(\beta_0-\alpha-\gamma\beta_0) \tag{18}$$

Eq. (16) yields

$$k_1 = 0 \tag{19}$$

and Eqs. (17) and (18) yield

$$B_2 = \frac{jA_0}{2}\frac{\frac{\beta_0}{\alpha}\left[\left(\frac{\beta_0}{\alpha}+1\right)-\gamma\frac{\beta_0}{\alpha}\right]}{2\frac{\beta_0}{\alpha}+1} \tag{20}$$

$$B_3 = \frac{jA_0}{2}\frac{\frac{\beta_0}{\alpha}\left[\left(\frac{\beta_0}{\alpha}-1\right)-\gamma\frac{\beta_0}{\alpha}\right]}{2\frac{\beta_0}{\alpha}-1} \tag{21}$$

For uniqueness, the homogeneous solution shown in the first term of Eq. (15) can be regarded as being accounted for in Eq. (12) by letting

$$B_1 = 0 \tag{22}$$

When β^0 approaches $\alpha/2$, the denominator of Eq. (21) becomes zero and B_3 is not defined. This situation corresponds to resonance, and another mathematical scheme[5] should be adopted to obtain the solution, if needed.

Substituting Eqs. (20)-(22) into Eq. (15) determines the first order solution, which provides the correction of the wave amplitude according to the perturbation of the material properties. The correction of the wave speed has not been achieved because $k_1 = 0$, and thus the next order solution is needed.

3.3. Solution of $O(\epsilon^2)$

The terms of $O(\epsilon^2)$ results in the second order governing equation as follows.

$$\frac{d^2\Theta_2(z)}{dz^2} + \beta_0^2\Theta_2(z) = RHS \tag{23}$$

The right-hand-side of Eq. (23) consists of five terms of exponential functions. Accordingly, the general solution of Θ_2 has the following form.

$$\Theta_2(z) = C_1 e^{j(\beta_0+\epsilon^2 k_2+\cdots)z} + C_2 e^{j[(\beta_0+\alpha)+\epsilon^2 k_2+\cdots]z} + C_3 e^{j[(\beta_0-\alpha)+\epsilon^2 k_2+\cdots]z} + C_4 e^{j[(\beta_0+2\alpha)+\epsilon^2 k_2+\cdots]z} + C_3 e^{j[(\beta_0-2\alpha)+\epsilon^2 k_2+\cdots]z} \tag{24}$$

Inserting Eq. (24) into Eq. (23) and collecting the coefficients of the first exponential function yield the following equation.

$$-2\beta_0 k_2 A_0 - \frac{j}{2}(\beta_0+\alpha)^2 B_2 + \frac{j}{2}(\beta_0-\alpha)^2 B_3 + \frac{j}{2}\alpha(\beta_0+\alpha)B_2 + \frac{j}{2}\alpha(\beta_0-\alpha)B_3 + \frac{j}{2}\gamma\beta_0^2 B_2 - \frac{j}{2}\gamma\beta_0^2 B_3 = 0 \tag{25}$$

The expressions for B_2 and B_3 in Eqs. (20) and (21) are inserted in Eq. (25), and k_2 is obtained as follows.

$$\frac{k_2}{k_0} = \frac{1}{8}\left\{\frac{\left[(1-\gamma)\frac{\beta_0}{\alpha}+1\right]\left(\frac{\beta_0}{\alpha}+1-\gamma\right)}{2\frac{\beta_0}{\alpha}+1} - \frac{\left[(1-\gamma)\frac{\beta_0}{\alpha}-1\right]\left(\frac{\beta_0}{\alpha}-1-\gamma\right)}{2\frac{\beta_0}{\alpha}-1}\right\} \tag{26}$$

This is the correction of the wave speed.

The correction of the wave amplitude can also be obtained, but it is not of interest in this paper. When β_0 approaches $\alpha/2$, one denominator in Eq. (26) becomes zero and k_2 is not defined. This situation corresponds to the resonance, and another mathematical scheme[5] should be adopted to obtain the solution, if needed.

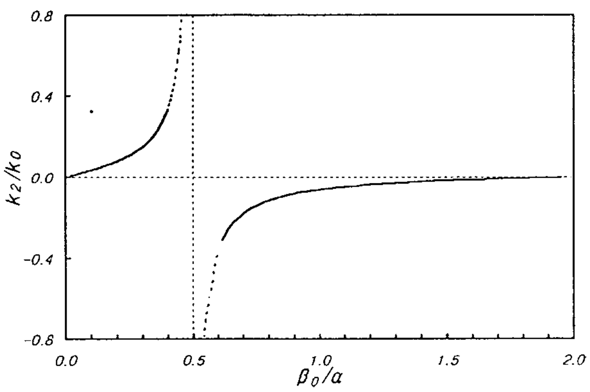
IV. Results and Discussion

Based on the information obtained so far, an approximation for the phase speed of the torsional wave propagating in a circular rod with periodically nonuniform material properties has been constructed. Wave speed has the relationship with wave frequency and wavenumber as $c = \omega/k$. Thus the wave speed c normalized by c_0 ($=\sqrt{G_0/\rho_0}$), which is the speed of the torsional wave in a uniform rod, is expressed as follows.

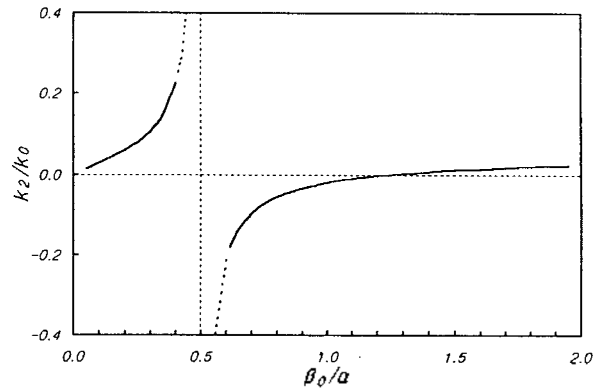
$$\frac{c}{c_0} = \frac{k_0}{k} = 1 - \varepsilon^2 \frac{k_2}{k_0} + O(\varepsilon^3) \qquad (27)$$

The normalized wave speed is obtained by inserting k_2/k_0 expressed in Eq. (26).

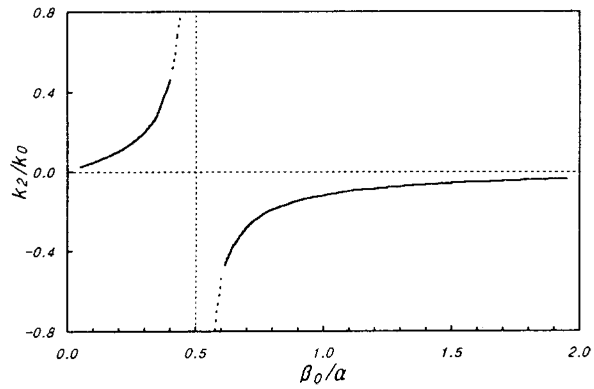
In order to evaluate the change of the wave speed due to the nonuniformity, the values of k_2/k_0 have been calculated for some cases. The values of k_2/k_0 are determined according to the values of γ and β_0/α. Figs. 2(a), 2(b), and 2(c) show the values of k_2/k_0 as functions of β_0/α for the cases of γ equal to 0.2, 0.5, and 0.8, respectively. The graph is not valid where $\beta_0/\alpha \approx 0.5$, because Eq. (26) is not defined when β_0 approaches $\alpha/2$.



(a) $\gamma = 0.2$



(b) $\gamma = 0.5$



(c) $\gamma = 0.8$

Figure 2. Wave speed corections k_2/k_0 depicted as a function of the wavenumber ratio β_0/α

It is shown that the value of k_2/k_0 is slightly different at different value of β_0/α, which is a function of frequency ω due to Eq. (8). This implies that the torsional waves considered in this paper are dispersive even in the fundamental mode, even though the fundamental mode of torsional waves propagating in a uniform circular rod is nondispersive[9].

V. Conclusion

An approximate theory has been derived to obtain the speed of torsional elastic waves propagating in a circular rod with harmonically varying material properties. The solution shows that the wave speed in the nonuniform rod is dependent on the wave frequency as well as the periodic variation of the material properties. It implies that the torsional waves considered in this paper are dispersive even in the fundamental mode.

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