〈논 문〉

Vibration and Stability of Tapered Timoshenko Beams on Two-Parameter Elastic Foundations

두 파라미터 탄성기초를 갖는 테이퍼진 티모센코 보의 진동 및 안정성

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ABSTRACT

The paper describes the vibration and stability of tapered beams on two-parameter elastic foundations. The two-parameter elastic foundations are constructed by distributed Winkler springs and a shearing layer as often used in soil models. The shear deformation and the rotatory inertia of a beam are taken into account. Governing equations are derived from energy expressions using Hamilton's principle. The associated eigenvalue problems are solved to obtain the free vibration frequencies or the buckling loads. Numerical results for the vibration of a beam with an axial force are presented and compared when other solutions are available. Vibration frequencies, mode shapes, and critical forces of a tapered Timoshenko beam on elastic foundations under an axial force are investigated for various thickness ratios, shear foundation parameters, Winkler foundation parameters and boundary conditions.

요 약

본 논문은 이중 탄성기초 위에 놓인 테이퍼진 티모센코 보의 진동과 동적 안정성에 대한 연구로써, 이중 탄성기초는 지반모델에서 흔히 이용되는 분포 Winkler 스프링들과 전단기초층으로 구성된다. 보의 전단변형과 회전관성이 고려되고, 지배방정식은 Hamilton원리를 이용한 에너지 표현식에 의해 유도된다. 고유진동수와 좌굴하중을 구하기 위해 관계되는 고유치 문제를 풀며, 축력을 받는 보의 진동에 대한 수치해석결과들이 제시되고 다른 방법을 사용한 유용한 해의 결과들과 비교된다. 축력을 받고 탄성기초 위에 놓인 테이퍼진 티모센코 보의 고유진동수, 모드 형상, 그리고 임계하중 값들이 다양한 테이퍼 두께의 비, 전단기초 파라미터, Winkler 기초파라미터, 경계조건의 변화에 대해 조사된다.

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1. Introduction

The vibration and stability of beam structures on elastic foundations have long been studied in the field of structural dynamics in both mechanical and civil engineering. In the previous studies[1~3], the soil is often modeled as a linear elastic spring system, so-called the Winkler model, that takes the force only in the vertical direction. From an engineering point of view, the soil can be modeled as a continuous layer of independent linear springs by assuming that the soil is an elastic foundation. It is, however, known that the linear Winkler elastic foundation model does not represent the real system with enough accuracy, although it has the advantage of providing the closed-form solution.

For a better accuracy, the two-parameter foundation, so-called the Pasternak foundation, is employed by adding a shearing layer to the Winkler foundation. In recent years, many investigators have studied the vibration and stability of a beam structure on this type of foundation. Zhaohua and Cook[4] conducted the vibration study on a beam element on two-parameter elastic foundations using the finite element method based on the cubic hermitian shape function. Eisenberger and Clastornik[5] investigated vibrations and buckling of a beam on a variable Winkler elastic foundation. Franciosi and Masi[6] applied the finite element method using matrix displacement approach to study free vibrations of foundation beams on two-parameter elastic soil. De Rosa[7] obtained free vibrational frequencies, mode shapes and critical axial loads of two-parameter foundation beams for various boundary conditions and axial forces. Rossi and his coresearchers[8] investigated the vibration characteristics of a non-uniform cantilevered Timoshenko beam carrying a concentrated mass. They presented the exact solutions for a stepped beam while the approximate solutions for a tapered beam using the finite element method .

In addition to these studies, the vibration studies have also performed on a tapered non-uniform beam or a structure on two-parameter elastic foundations including the rotatory inertia and the shear deformation effect of a beam. Cheng and Pantelides[9] analyzed vibration characteristics of Timoshenko beam-columns on elastic media using the dynamic stiffness method, and Capron and Williams[10] obtained exact dynamic stiffness for an axially loaded uniform Timoshenko member embedded in an elastic medium.

Yokoyama[11] investigated natural frequencies and vibrational mode shapes of a Timoshenko beam with an axial force on two-parameter elastic foundations by employing the finite element method. He studied either a tapered beam without an elastic foundation or a uniform structure on the two-parameter elastic foundation or on the Winkler foundation. Recently, Hou and his collaborators[12] analyzed vibration characteristics of a non-uniform Timoshenko beam on two-parameter foundations for various shear foundation parameters and Winkler foundation parameters. Their study, however, does not include the stability problem of the system under an axial force.

The objective of the study is to investigate the vibration and the stability of a tapered Timoshenko beam on two-parameter elastic foundations under an axial force. The study was conducted by finding vibration frequencies and critical forces of the system for various thickness ratios, shear foundation parameters, and Winkler foundation parameters.

2. Theoretical Analyses

2.1 Mathematical Model and Governing Equations

Figure 1 depicts the mathematical model of a beam with a constant width on the two-parameter elastic foundation under an axial force. In Fig. 1, P is the constant axial force, L is the entire length of a beam, k_w is the Winkler foundation modulus, k_s is the shear foundation modulus, x and u are the axial and the horizontal coordinates, respectively, h_1 and h_2 are the thicknesses of a tapered beam, and $y(x,t)$ is the vertical deflection as a function of position x and time t.

The governing equation for the model in Fig. 1 can be derived from energy expression. The elastic strain energy of the system U, kinetic energy T, the work done W by the axial force P can be expressed as follows.

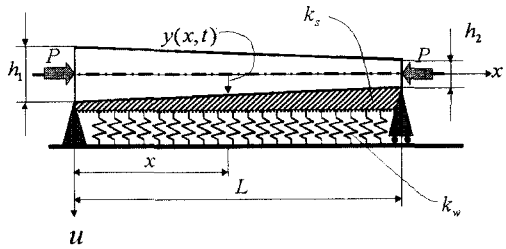

Fig. 1 A tapered Timoshenko beam resting on two-layer elastic foundations under a axial force

$$U=\frac{1}{2}\int_0^L EI(x)\phi_x^2 dx+\frac{1}{2}\int_0^L kA(x)G((y_x-\phi)^2)dx$$
$$+\frac{1}{2}\int_0^L k_w y^2 dx+\frac{1}{2}\int_0^L k_s y_x^2 dx \tag{1}$$

$$T=\frac{1}{2}\int_0^L \rho A(x)y_t^2 dx+\frac{1}{2}\int_0^L \rho I(x)\phi_t^2 dx \tag{2}$$

$$W=\frac{1}{2}\int_0^L P y_x^2 dx \tag{3}$$

In equations (1) through (3), E is the Young's modulus, $A(x)$ is the cross-sectional area which varies with the axial length, $I(x)$ is the second area of moment of inertia of the cross section, ρ is the density of a beam, ϕ is the bending slope of the beam, G is the shear modulus, subscripts x and t are the derivatives with respect to position and time, respectively.

Substituting equations (1) through (3) into Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T-U+W)dt=0 \tag{4}$$

and rearranging gives

$$\int_0^L\Bigg\{\rho A_1\left(D+(1-D)\left(1-\frac{x}{L}\right)\right)y_{tt}\delta y$$
$$+\rho I_1\left(D+(1-D)\left(1-\frac{x}{L}\right)\right)^3\phi_{tt}\delta\phi$$
$$+EI_1\left(D+(1-D)\left(1-\frac{x}{L}\right)\right)^3\phi_x\delta\phi_x$$
$$+kA_1G\left(D+(1-D)(1-\frac{x}{L})\right)(y_x-\phi)\delta(y_x-\phi)$$
$$+k_w y\delta y+k_s y_x\delta y_x-Py_x\delta y_x\Bigg\}dx=0 \tag{5}$$

where, D is h_2/h_1 which presents the thickness ratio, A_1 and I_1 are the cross-sectional area and the second area moment of inertia of the cross section on the thicker side, respectively.

2.2 Finite Element Analyses

In order to apply the finite element method to equation (4), the beam is divided into N elements of uniform length l as shown in Fig. 2. In the figure, numbers in circles represent the element number, l is the length of an element.

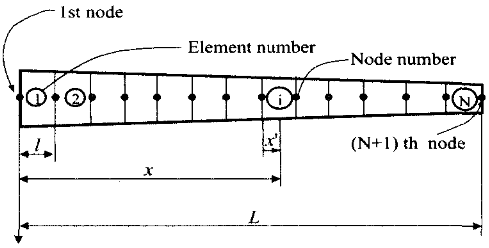

Fig. 2 A finite element model

The following nondimensional terms and local coordinates are introduced for the calculation purpose.

$$\xi=\frac{x'}{l},\quad \eta=\frac{y}{l},\quad x'=x-(i-1)l \tag{6}$$

The dimensionless displacement function for the i th element is assumed as

$$\eta(\xi)^{(i)}=a_0+a_1\xi+a_2\xi^2+a_3\xi^3$$
$$\phi(\xi)^{(i)}=b_0+b_1\xi+b_2\xi^2+b_3\xi^3 \tag{7}$$

Finding the coefficient of equation (7) to satisfy the compatibility condition between finite elements and rearranging give

$$\eta(\xi)^{(i)}=\{N_w\}^T\{H\}^{(i)},\quad \phi(\xi)^{(i)}=\{N_w\}^T\{\Phi\}^{(i)} \tag{8}$$

where the superscript T means transpose, $\{H\}^{(i)}$ and $\{\Phi\}^{(i)}$ are the generalized element nodal vectors, and $\{N_w\}$ is the shape function vector as follows.

$$\{N_w\}^T=\{1-3\xi^2+2\xi^3,\ \xi-2\xi^2+\xi^3,\ 3\xi^2-2\xi^3,-\xi^2+\xi^3\} \tag{9}$$

Equation (5) can be rearranged in the following discretized equation by using the relationship among equations (6) through (9) gives

$$\sum_{i=1}^{N}\int_0^1\Bigg[\frac{\Omega^2}{N^4}\left\{\left(1+\frac{FB}{N}\right)+\left(\frac{B}{N}\right)\xi\right\}\eta^{(i)}(\xi)\delta\eta^{(i)}(\xi)$$
$$+\frac{\Omega^2R}{N^2}\Bigg\{\left(\frac{F^3}{N^3}\right)\xi^3+\left(\frac{3F^2B}{N^3}+\frac{3F^3}{N^2}\right)\xi^2$$
$$+\left(\frac{3F^3B^2}{N^3}+\frac{6F^2B}{N^2}+\frac{3F}{N}\right)\xi$$
$$+\left(\frac{F^3B^3}{N^3}+\frac{3F^2B^2}{N^2}+\frac{3FB}{N}+1\right)\Bigg\}\phi^{(i)}(\xi)\delta\phi^{(i)}(\xi)$$
$$+\Bigg\{\left(\frac{F^3}{N^3}\right)\xi^3+\left(\frac{3F^2B}{N^3}+\frac{3F^2}{N^2}\right)\xi^2$$

$$+\left(\frac{3F^3B^2}{N^3}+\frac{6F^2B}{N^2}+\frac{3F}{N}\right)\xi \quad (10)$$
$$+\left(\frac{F^3B^3}{N^3}+\frac{3F^2B^2}{N^2}+\frac{3FB}{N}+1\right)\Big\}\phi_\xi^{(i)}(\xi)\,\delta\phi_\xi^{(i)}(\xi)$$
$$+\frac{S}{N^2}\left\{\left(1+\frac{FB}{N}\right)+\left(\frac{F}{N}\right)\xi\right\}(\eta_\xi^{(i)}-\phi^{(i)})\delta(\eta_\xi^{(i)}-\phi^{(i)})\Big]$$
$$d\xi=0$$

where, $F=D-1$, $B=i-1$.

Combining N elements of the entire beam gives the following eigenvalue equation in matrix form.

$$[\,[K]-\Omega^2[M]\,]\{q\}=\{0\} \quad (11)$$

where, $[K]$ and $[M]$ are total stiffness matrix and total mass matrix, respectively.

Meanwhile, the following dimensionless parameters are used to get the numerical solution in the study.

$$\Omega^2=\frac{\rho A_1L^4}{EI_1}\omega^2,\ K_s=\frac{k_sL^2}{EI_1},\ K_w=\frac{k_wL^4}{EI_1},$$
$$Q=\frac{PL^2}{EI_1},\ D=\frac{h_2}{h_1},\ R=\frac{I_1}{A_1L^2},\ S=\frac{kA_1GL^2}{EI_1} \quad (12)$$

where Ω is the frequency parameter, K_s is the shear foundation parameter, K_w is the Winkler foundation parameter, Q is the axial force parameter, and D is the thickness ratio of a tapered beam.

3. Numerical Analysis Results and Discussion

A beam is divided into 20 finite elements for the numerical analysis in the study. The accuracy of numerical solutions is tested by comparing with the results for the uniform beam under various boundary conditions presented in References[9,11]. A good agreement was obtained as shown in Table 1.

Table 1 presents the first three lowest natural frequencies when dimensionless parameters of K_w/π^4, Q_{cr}/π^2, and K_s/π^2 are constant for two different boundary conditions, hinged-hinged and hinged-clamped.

As can be seen in the table, the maximum error is less than 0.2% when the comparison is made with the exact results presented in Reference[9].

The beam can be considered as an Euler-Bernoulli beam when the shear deformation parameter of the beam, S, in the governing equation goes to the infinite and the rotatory inertia parameter of the beam, R, approaches to zero.

For the numerical analyses in the study, $S=10^6$ and $R=0.0$ are used for the Euler-Bernoulli beam. $S=26.6667$ and $R=0.01$ are chosen for the Timoshenko beam as used in reference (11). The shear coefficient of the beam, k, and Poisson's ratio, ν, are set to 2/3 and 0.25, respectively.

Figure 3 through Fig. 5 present the critical axial force for various thickness ratios and foundation parameters for both a tapered Euler-Bernoulli beam and a tapered

Table 1 Comparison present results with other available results for the first lowest three natural frequencies

Parameters	Method	Exact Ref.(9)	8 Element Ref.(11)	16 Element Ref.(11)	20 Element Present	Error (%)
Both hinged	$K_w/\pi^4=0.0$	8.21	8.23	8.22	8.21469	-0.06
	$K_s/\pi^2=0.0$	24.23	24.56	24.31	24.2281	0.01
	$Q/\pi^2=0.0$	41.54	43.22	41.96	41.5417	0.00
	$K_w/\pi^4=0.6$	8.21	8.22	8.22	8.21469	-0.06
	$K_s/\pi^2=0.0$	20.59	20.90	20.67	20.5896	0.00
	$Q/\pi^2=0.6$	35.86	37.42	36.25	35.8568	0.01
Hinged clamped	$K_w/\pi^4=0.0$	10.63	10.66	10.63	10.6266	0.03
	$K_s/\pi^2=0.0$	25.62	26.01	25.71	25.6160	0.02
	$Q/\pi^2=0.0$	42.03	43.75	42.46	42.0314	0.00
	$K_w/\pi^4=0.6$	10.46	10.51	10.49	10.4806	-0.20
	$K_s/\pi^2=0.0$	22.20	22.57	22.30	22.2068	-0.03
	$Q/\pi^2=0.6$	36.50	38.11	36.90	36.5041	-0.01

※ Error(%) $=\left(\frac{exact(Ref[9])-present}{exact(Ref[9])}\times 100\right)$

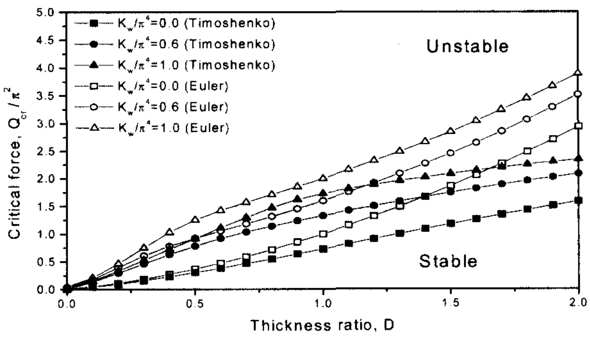

Fig. 3 Critical force depending on thickness ratio and Winkler foundation parameter for both-hinged boundary condition

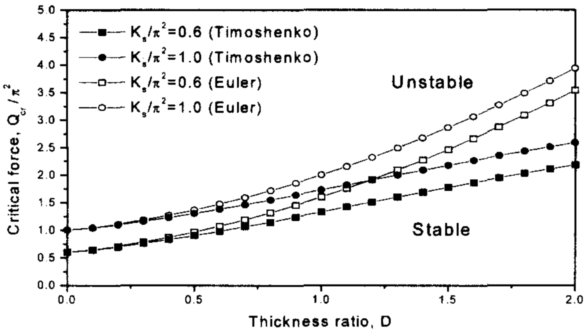

Fig. 4 Critical force depending on thickness ratio and shear foundation parameter for both-hinged boundary condition

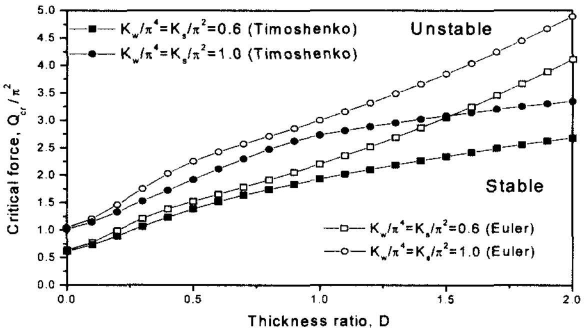

Fig. 5 Critical force depending on thickness ratio and two-layer foundation parameter for both-hinged boundary condition

Timoshenko beam. They are hinged at both ends. Figure 3 is for beams on the Winkler foundation. Figure 4 is for beams on the shear foundation. Figure 5 is for beams on both the Winkler foundation and the shear foundation. As shown in these figures, the critical axial force increases as either the foundation parameter or the thickness ratio increases. The critical force of the Timoshenko beam is less than that of the Euler-Bernoulli beam for the same thickness ratio and foundation parameter.

Figure 6 shows the critical force for several rotatory inertia parameters. As can be seen in the figure, the rotatory inertia parameter, R, has no effect on the critical force for all thickness ratios. This is because the rotatory inertia is believed to have no effect on the critical axial force when the axial force is purely conservative.

Figure 7 shows the critical force for several shear deformation parameters. The critical axial force increases as the shear deformation parameter increases for all thickness ratios as shown in the figure. However, the increment of the shear deformation parameter, S, has little effect on the critical force for $S \geq 10^3$.

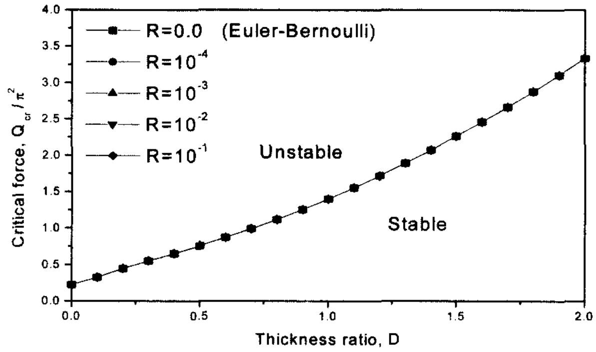

Fig. 6 Critical force depending on rotary inertia parameter and thickness ratios of the beams ($S = 10^6$)

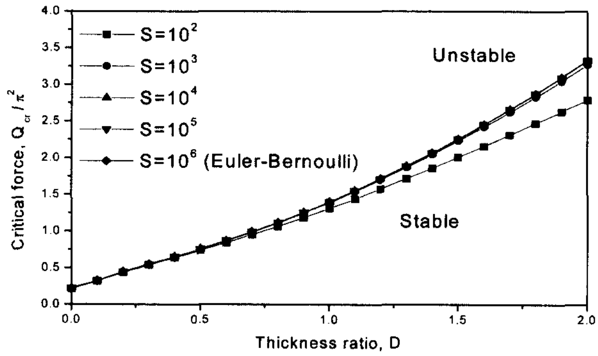

Fig. 7 Critical force depending on shear deformation parameter and thickness ratios of the beams ($R = 0.0$)

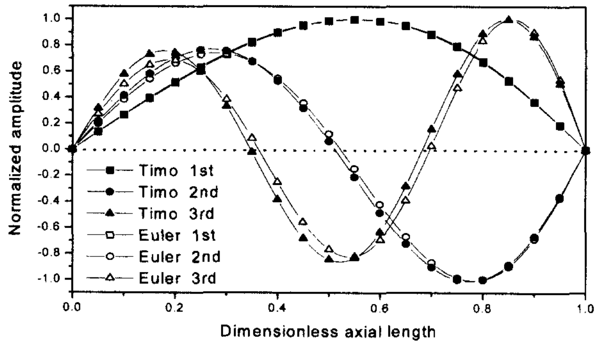

Fig. 8 The first three lowest mode shapes depending on Winkler foundation parameter for thickness ratios (D = 0.5, K_w/π^4 = 0.2, K_s/π^2 = 0.0)

Figure 8 through Figure 10 present the first three normalized natural modes of both an Euler-Bernoulli beam and a Timoshenko beam on the Winkler foundation for three different Winkler foundation parameters when the

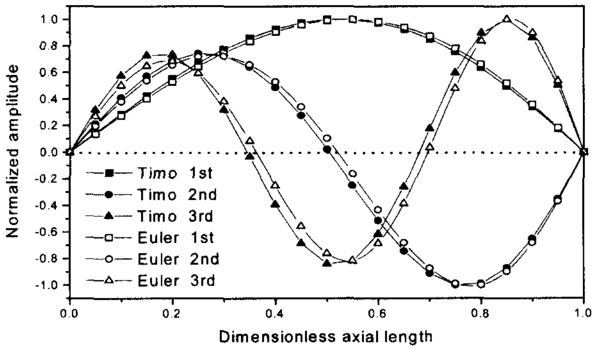

Fig. 9 The first three lowest mode shapes depending on Winkler foundation parameter for thickness ratios ($D = 0.5,\ K_w/\pi^4 = 1.0,\ K_s/\pi^2 = 0.0$)

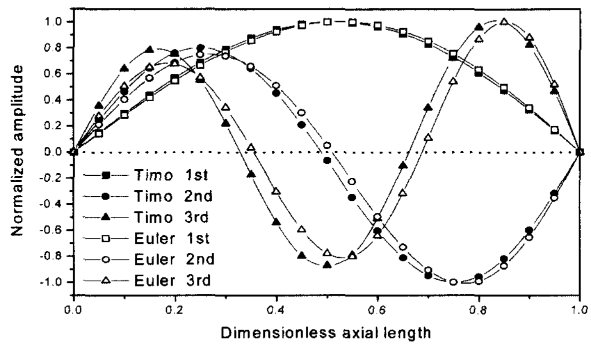

Fig. 12 The first three lowest mode shapes depending on shear foundation parameter for thickness ratios($D=0.5,\ K_w/\pi^4=0.0,\ K_s/\pi^2=1.0$)

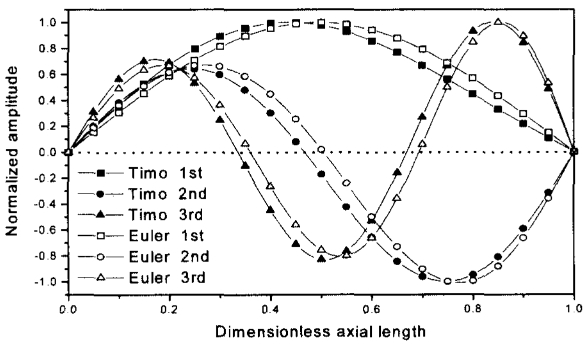

Fig. 10 The first three lowest mode shapes depending on Winkler foundation parameter for thickness ratios ($D=0.5,\ K_w/\pi^4=5.0,\ K_s/\pi^2=0.0$)

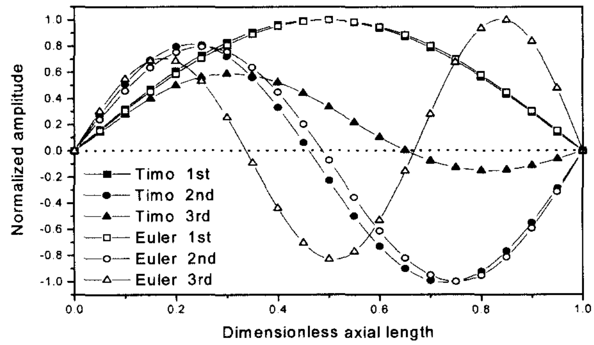

Fig. 13 The first three lowest mode shapes depending on shear foundation parameter for thickness ratios($D=0.5,\ K_w/\pi^4=0.0,\ K_s/\pi^2=5.0$)

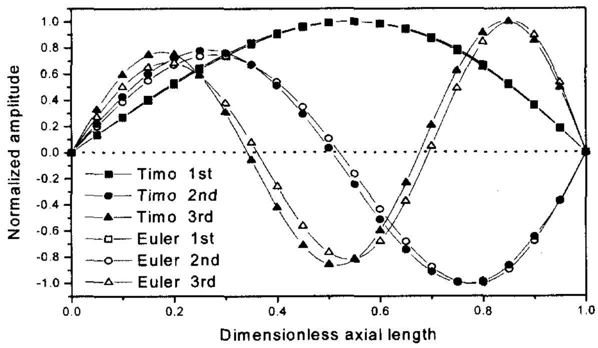

Fig. 11 The first three lowest mode shapes depending on shear foundation parameter for thickness ratios($D=0.5,\ K_w/\pi^4=0.0,\ K_s/\pi^2=0.2$)

thickness ratio, D, is 0.5. The difference in the mode shapes between an Euler-Bernoulli beam and a Timoshenko beam is getting bigger as the Winkler foundation parameter increases.

Figure 11 through Fig. 13 present the first three normalized natural modes of both an Euler-Bernoulli beam and a Timoshenko beam on the shear foundation for three different shear foundation parameters when the thickness ratio is 0.5. There is a big difference in the third mode between an Euler-Bernoulli beam and a Timoshenko beam when $K_s/\pi^2=5.0$ as shown in Fig. 13.

Figure 14 and Fig. 15 present the first three normalized natural modes of both an Euler-Bernoulli beam and a Timoshenko beam on the Winkler foundation when the thickness ratio, D, is 1.5. As shown in these figures, there is a significant difference in the third mode between an Euler-Bernoulli beam and a Timoshenko beam even when $K_w/\pi^4=1.0$. The thickness ratio and the foundation parameter affect more the mode shape of a Timoshenko beam than those of an Euler-Bernoulli beam.

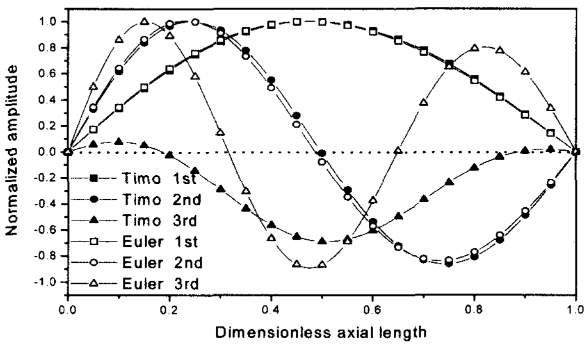

Fig. 14 The first three lowest mode shapes depending on Winkler foundation parameter for thickness ratios($D=1.5$, $K_w/\pi^4=1.0$, $K_s/\pi^2=0.0$)

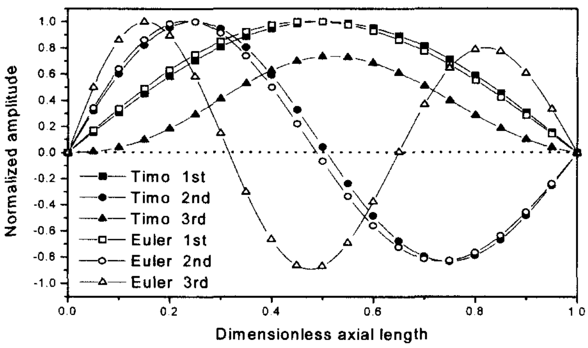

Fig. 15 The first three lowest mode shapes depending on Winkler foundation parameter for thickness ratios($D=1.5$, $K_w/\pi^4=5.0$, $K_s/\pi^2=0.0$)

Table 2 shows the first three lowest natural frequencies of a Timoshenko beam for various thickness ratios and boundary conditions. As shown in the table, the natural frequencies of the beam without an axial force and a foundation are the same as those of the beam on the shear foundation when the axial force parameter, Q/π^2, equals to the shear foundation parameter, K_s/π^2. The same phenomenon is observed for any combination of thickness ratios and boundary conditions. This is because the shear foundation is believed to produce the same stiffening effect as the tensile axial force produces and cancel out the effect of the compressive force applied.

The natural frequencies of the beam without an axial force and a foundation are different from those of the beam on the Winkler foundation when the axial force parameter, Q/π^2, equals to the Winkler foundation parameter, K_w/π^4, except for the first natural frequency of an uniform beam (i.e. the thickness ratio, $D=1.0$) hinged at both ends.

Table 2 The first three lowest natural frequencies Ω_i for the change of thickness ratios and boundary conditions of Timoshenko beams ($S=26.6667$, $R=0.01$)

Thickness ratios	Parameters	Boundary conditions Both -Hinged	Clamped -Free	Hinged -Clamped	Both -Clamped
0.5	$K_w/\pi^4=0.0$	6.3588	3.8238	9.5319	11.4234
	$K_s/\pi^2=0.0$	20.8811	18.3171	22.9061	24.5733
	$Q/\pi^2=0.0$	37.6321	47.2650	38.5748	39.9218
	$K_w/\pi^4=0.1$	6.3331	4.4011	9.4014	11.3655
	$K_s/\pi^2=0.0$	19.9093	12.0718	21.9955	23.7495
	$Q/\pi^2=0.1$	36.2267	26.1076	37.2159	38.6240
	$K_w/\pi^4=0.0$	6.3588	3.8238	9.5319	11.4234
	$K_s/\pi^2=0.1$	20.8811	18.3171	22.9061	24.5733
	$Q/\pi^2=0.1$	37.6321	47.2650	38.5748	39.9218
1.0	$K_w/\pi^4=0.0$	8.2147	3.1887	10.6663	13.0748
	$K_s/\pi^2=0.0$	24.2281	13.8755	25.6160	26.6924
	$Q/\pi^2=0.0$	41.5417	29.9222	42.0314	42.6488
	$K_w/\pi^4=0.1$	8.2147	3.9288	10.6081	13.0615
	$K_s/\pi^2=0.0$	23.6615	13.3792	25.0802	26.1740
	$Q/\pi^2=0.1$	40.6514	29.1827	41.1659	41.7968
	$K_w/\pi^4=0.0$	8.2147	3.1887	10.6266	13.0748
	$K_s/\pi^2=0.1$	24.2281	13.8755	25.6160	26.6924
	$Q/\pi^2=0.1$	41.5417	29.9222	42.0314	42.6488
1.5	$K_w/\pi^4=0.0$	9.3842	3.0079	11.1552	13.9255
	$K_s/\pi^2=0.0$	26.2096	13.8566	27.1754	27.6602
	$Q/\pi^2=0.0$	39.6798	30.7216	40.4097	44.0435
	$K_w/\pi^4=0.1$	9.3815	3.6139	11.1526	13.9182
	$K_s/\pi^2=0.0$	25.7810	13.5647	26.7677	27.2647
	$Q/\pi^2=0.1$	39.6720	30.2524	40.3459	43.3734
	$K_w/\pi^4=0.0$	9.3842	3.0079	11.1552	13.9255
	$K_s/\pi^2=0.1$	26.2096	13.8566	27.1754	27.6602
	$Q/\pi^2=0.1$	39.6798	30.7216	40.4097	44.0435

4. Concluding Remarks

The following results were obtained from the study on the stability and the vibration of a tapered Timoshenko beam on two-parameter elastic foundations subjected to the axial force.

(1) The critical force increases as the thickness ratio of

the tapered beam and foundation parameters increase for all boundary conditions considered.

(2) The natural frequencies of the beam without an axial force and a foundation are the same as those of the beam on the shear foundation when the axial force parameter, Q/π^2, equals to the shear foundation parameter, K_s/π^2, for all boundary conditions considered.

(3) The natural frequencies of the beam without an axial force and a foundation are different from those of the beam on the Winkler foundation when the axial force parameter, Q/π^2, equals to the Winkler foundation parameter, K_w/π^4, except for the first natural frequency of a uniform beam hinged at both ends.

(4) For a fixed thickness ratio, the critical axial force increases as the shear deformation parameter, S, increases. However, the increment of S has little effect on the critical force for $S \geq 10^3$.

(5) The thickness ratio and the foundation parameter affect more the mode shape of a Timoshenko beam than those of an Euler-Bernoulli beam.

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