한국소음진동공학회지 제 10 권 제 6 호, pp. 1067~1074, 2000.

〈논 문〉

Free Vibrations and Buckling Loads of Columns with Multiple Elastic Springs

여러 개의 스프링으로 탄성지지된 기둥의 자유진동 및 좌굴하중

Byoung Koo Lee, Li Guangfan, Sang Jin Oh and Tae Ki Lee

이 병 구* · 李 光 范** · 오 상 진*** · 이태기****

(Received June 5, 2000 : Accepted October 9, 2000)

Key Words: Buckling Load(좌굴하중), Column(기둥), Elastic Foundation(탄성지반), Elastic Spring(탄성스프링), Free Vibration(자유진동), Mode Shape(진동형), Natural Frequency(고유진동수)

ABSTRACT

Numerical methods for calculating both the natural frequencies and buckling loads of columns with the multiple elastic springs are developed. In order to derive the governing equations of such columns, each elastic spring is modeled as a discrete elastic foundation with the finite longitudinal length. By using this model, the differential equations governing both the free vibrations and buckled shapes, respectively, of such columns are derived. These differential equations are solved numerically. The Runge- Kutta method is used to integrate the differential equations, and the determinant search method combined with Regula-Falsi method is used to determine the eingenvalues, namely natural frequencies and buckling loads. In the numerical examples, the clamped-clamped, clamped-hinged, hinged-clamped and hinged-hinged end constraints are considered. Extensive numerical results including the frequency parameters, mode shapes of free vibrations and buckling load parameters are presented in the non-dimensional forms.

요 약

이 논문은 여러 개의 스프링으로 탄성지지된 기둥의 고유진동수와 좌굴하중 산정에 관한 연구이다. 하나의 스프링을 폭이 매우 좁은 탄성지반으로 모형화하고, 이 모형을 이용하여 기둥의 자유진동과 좌굴된 기둥의 탄성곡선을 지배하는 미분방정식을 유도하였다. 이 미분방정식들을 Runge-Kutta법과 행열값 탐사법을 이용하여 미분방정식의 고유치 즉 고유진동수와 좌굴하중을 산정하였다. 수치해석 예에서는 고정-고정, 고정-회전, 회전-고정 및 회전-회전의 단부조건을 고려하였다. 수치해석의 결과로 기둥 변수들과 고유진동수 및 좌굴하중 사이의 관계를 고찰하고, 스프링으로 지지된 기둥과 지지되지 않은 기둥의 진동형을 비교하였다.

* Member, Dept. of Civil Engineering, Wonkwang University

** Professor, Dept. of Civil Engineering, Yanbian University, Yanji 13300, China

*** Member, Dept. of Civil Engineering, Provincial College of Damyang

**** Dae Woo Motor Sales/Engineering & Construction Corporation

1. Introduction

Columns are widely used in structural engineering fields since these units are basic structural forms. The static and dynamic problems, especially problems of both the free vibrations and buckling loads, of columns which are subjected to either axial static loads or dynamic loads have been investigated by many researchers during the past few decades. The following references and their citations include the governing differential equations and significant historical literature on this subject. Timoshenko and Gere[1], and Timoshenko, *et al.*[2] reported the natural frequencies and buckling loads of the beam-column, respectively. Abbas and Thomas[3] used the finite element method to predict the dynamic stability of Timoshenko beams on elastic foundations. Bokaian[4] studied the natural frequencies of beams under compressive axial loads. Pavlovic and Wylie[5] reported the natural frequencies of beams on the non-homogeneous foundations. Valsangkar and Pradhanang[6] reported the natural frequencies of partially supported piles. Lee and Oh[7] reported both the natural frequencies and buckling loads of the beam-column on elastic foundations. Lee, *et al.*[8] studied the free vibrations of tapered piles partially embedded in elastic foundations. Lee and Oh[9] reported the elastica and buckling load of simple tapered columns with constant volume.

The columns laterally supported by the multiple elastic springs are often used to control the free vibrations and to increase the capability of the axial compressive loads. In these cases, the problems of both the free vibrations and buckling loads have to be fully understood since the column designs are conducted under the comprehensive understanding the behaviors of such columns. From this viewpoint, the problems of both the free vibrations and buckling loads of columns laterally restrained by the multiple elastic springs are very attractive in the structural engineering fields.

The main purpose of the present paper is to compute both the natural frequencies and buckling loads of columns with the multiple elastic springs. In free vibration analysis, the effect of axial load is included. In this study, each elastic spring of the column is modeled as a discrete elastic foundation whose longitudinal length is very short. On the basis of this model, the differential equation governing the free vibrations of columns with the elastic springs is derived. Also the differential equation governing the buckled shape of such columns is derived using the differential equation of free vibration and the relationship between the natural frequencies and axial compressive loads. These differential equations are solved numerically for calculating the natural frequencies with the corresponding mode shapes and the buckling loads of such columns. In the numerical examples, four end constraints of clamped-clamped, clamped-hinged, hinged-clamped and hinged-hinged ends are considered. The effects of various non-dimensional system parameters on the natural frequencies and the buckling loads are reported in tables and figures. Also the typical mode shapes of free vibrations are reported.

2. Mathematical model

Fig. 1 shows the typical mode shape of uniform column with span length l, which is laterally restrained by the multiple elastic springs. Both ends are supported by either clamped or hinged end. Each spring constant and its distance from the left end are depicted as S_j and L_j, respectively. Here the subscript j is the spring number of the intermediate multiple elastic springs so that the subscript n is the total number of springs. Also the column is subjected to an axial load P in which the compressive load is positive. As shown in this figure, the rectangular coordinates (x, w) are depicted for defining the dynamic vertical displacement $W = W(x, t)$, in which w is the amplitude of W.

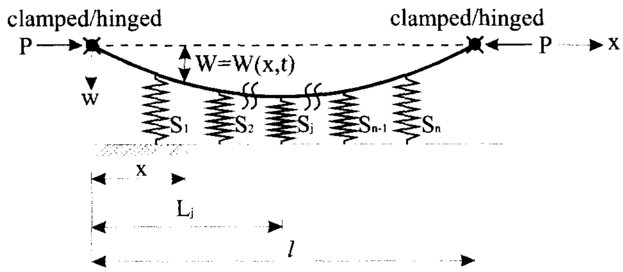

Fig. 1 Typical vibration mode shape of column with elastic springs

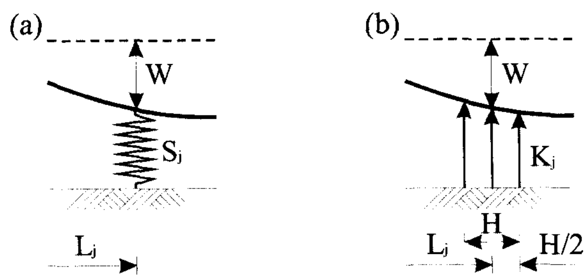

Fig. 2 Column supported by (a) elastic spring and (b) elastic foundation with finite length

In this study, each spring depicted in Fig. 2(a) is modeled as the discrete elastic foundation whose longitudinal length is very short as shown in Fig. 2(b). The foundation modulus and its finite length are depicted as K_j and H, respectively. Also the dynamic vertical displacement of column is expressed as W as shown already in Fig. 1.

Now the partial differential equation[2] governing the free vibration of column on the elastic foundation is introduced as follows, in order to apply the column with elastic springs to the column on the discrete elastic foundation.

$$EI\frac{\partial^4 W}{\partial x^4}+m\frac{\partial^2 W}{\partial t^2}+P\frac{\partial^2 W}{\partial x^2}+KW=0 \quad (1)$$

where EI, m and t are the flexural rigidity, mass per unit column length and time, respectively.

The column is assumed to be in harmonic motion

$$W=W(x,t)=w\sin(\omega_i t) \quad (2)$$

where ω_i is the ith angular frequency.

In Fig. 2, the internal force of jth spring and the restoring force of jth foundation due to the dynamic displacement W at L_j are $S_j w$ and $K_j Hw$, respectively. Since each spring is modeled as the discrete foundation with the finite longitudinal length H, the spring force and the restoring force of foundation are equal to each other. When the former is equivalent to the latter, it gives the following Eq. (3).

$$S_j w=K_j Hw \quad (3)$$

Rewriting the Eq. (3) gives

$$K_j=S_j/H=\alpha S_j/l \quad (4)$$

in which α is the ratio of span length l to finite foundation length H, defined as

$$\alpha=l/H \quad (5)$$

Substituting Eq. (4) into Eq. (1) gives the following equation.

$$EI\frac{\partial^4 W}{\partial x^4}+m\frac{\partial^2 W}{\partial t^2}+P\frac{\partial^2 W}{\partial x^2}+\alpha\frac{S_j}{l}W=0,$$
$$S_j=0 \text{ not for } L_j-H/2\le x\le L_j+H/2 \quad (6)$$

It is noted that in Eq. (6), the spring constant S_j not for $L_j-H/2\le x\le L_j+H/2$ is zero since the column is restrained by elastic foundations with length H for only $L_j-H/2\le x\le L_j+H/2$ as shown in Fig. 2(b).

To facilitate the numerical studies and to obtain the most general results for this class of problems, the following non-dimensional system variables are defined. The first is the frequency parameter,

$$c_i=\omega_i l^2(m/EI)^{1/2} \quad (7)$$

which is written in terms of ith angular frequency ω_i, $i=1,2,3,\cdots$. The load parameter is

$$p=Pl^2/(\pi^2 EI) \quad (8)$$

The spring parameter is

$$s_j=S_j l^3/EI,\quad j=1,2,3,\cdots,n \quad (9)$$

The coordinates x and w are normalized by the span length l, or

$$\xi=x/l \quad (10)$$

$$\eta=w/l \quad (11)$$

Also, the position parameter of each spring is defined as

$$\lambda_j=L_j/l,\quad j=1,2,3,\cdots,n \quad (12)$$

When each term of $\partial^2 W/\partial x^2$, $\partial^4 W/\partial x^4$ and $\partial^2 W/\partial t^2$ obtained by differentiating the Eq. (2) is substituted into Eq. (6) and the non-dimensional forms defined in Eq. (5) and Eqs. (7)~(12) are used, then the result is

$$\frac{d^4\eta}{d\xi^4}=-\pi^2 p\frac{d^2\eta}{d\xi^2}+(c_i^2-\alpha s_j)\eta,$$
$$s_j=0 \text{ not for } \lambda_j-1/(2\alpha)\le\xi\le\lambda_j+1/(2\alpha) \quad (13)$$

which is the non-dimensional differential equation governing the free vibration of column with the intermediate multiple elastic springs.

Now consider the buckling load problems. The axial compressive loads of the columns are increased gradually and then the natural frequencies are decreased. When the compressive load finally coincides with the buckling load, the column is buckled and the respective angular frequency becomes zero. Thus, substituting $c_i=0$ and $p=b_i$ into Eq. (13) gives the following equation which governs the buckled shape of column with the intermediate multiple elastic springs.

$$\frac{d^4\eta}{d\xi^4} = -\pi^2 b_i \frac{d^2\eta}{d\xi^2} - \alpha s_j \eta,$$
$$s_j = 0 \ \text{ not for } \ \lambda_j - 1/(2\alpha) \le \xi \le \lambda_j + 1/(2\alpha) \tag{14}$$

where b_i is the buckling load parameter defined as

$$b_i = B_i l^2/(\pi^2 EI),\ \ i = 1, 2, 3, \cdots \tag{15}$$

in which B_i, $i=1, 2, 3, \cdots$ is the ith buckling load.

The boundary conditions for hinged end ($x=0$ or $x=l$) are

$$\eta = 0 \ \text{ at } \ \xi = 0 \text{ or } 1 \tag{16}$$

$$\frac{d^2\eta}{d\xi^2} = 0 \ \text{ at } \ \xi = 0 \text{ or } 1 \tag{17}$$

which assure that the vertical displacement and bending moment are zero at the hinged end, respectively.

The boundary conditions for clamped end ($x=0$ or $x=l$) are

$$\eta = 0 \ \text{ at } \ \xi = 0 \text{ or } 1 \tag{18}$$

$$\frac{d\eta}{d\xi} = 0 \ \text{ at } \ \xi = 0 \text{ or } 1 \tag{19}$$

which imply that the vertical displacement and rotation are zero at the clamped end, respectively.

3. Numerical methods

Based on the above analysis, the algorithms were developed to calculate both the natural frequencies with the corresponding mode shapes and the buckling loads of the columns. The algorithms similar to those described by Lee and Oh[7], Lee, *et al.*[9] and Oh, *et al.*[10] were used to solve the differential equations (13) and (14). The Runge-Kutta method[11] and the determinant search method[12] combined with Regula-Falsi method[11] were used to integrate differential equations and to determine the eigenvalues, namely the frequency parameter c_i or the buckling load parameter b_i, respectively. For the sake of completeness, these algorithms are summarized as follows. The first is for the free vibration problems.

(1) Specify the column geometry (end constraint, load parameter p, and each position parameter λ_j and spring parameter s_j).

(2) Consider the fourth order system, Eq. (13), as two initial value problems whose initial value are the homogeneous boundary conditions at $\xi=0$ in accordance with the end constraint as chosen in step (1). Then assume a trial frequency parameter c_i, namely an eigenvalue, in which the first trial value is zero.

(3) Using the Runge-Kutta method, integrate the Eq. (13) from $\xi=0$ to 1. Perform two separate integrations, one for each of two chosen boundary conditions.

(4) From the Runge-Kutta solutions, evaluate at $\xi=1$ the determinant D of the coefficient matrix for the chosen set of two homogeneous boundary conditions. If $D=0$, then the trial value of c_i is an eigenvalue. If $D\neq 0$, then increment c_i and go to step (3).

(5) Repeat steps (3) and (4) and note the sign of D in each iteration. If D changes sign between two consecutive trials, then the eigenvalue lies between these last two trial values of c_i.

(6) Use Regula-Falsi method to compute the advanced trial c_i based on its two previous values.

(7) Terminate calculations and print the value of c_i and the corresponding mode shape $\eta=\eta(\xi)$ when the appropriate convergence criteria are met.

The second is for the buckling load problems. Also, the buckling load parameters b_i were calculated in a straightforward way described in free vibration problems. Specify the column geometry, of course not p in step (1). And assume the trial value b_i instead of c_i in step (2). Remaining numerical procedures are the same as the above procedures, and of course the characteristic value of Eq. (14) is the buckling load parameter b_i.

Based on these algorithms, two FORTRAN computer programs were written to solve the frequency parameters c_i and the buckling load parameters b_i, respectively. In these FORTRAN codes, the clamped-clamped, clamped-hinged, hinged-clamped and hinged-hinged ends are considered as the end constraints of columns.

4. Numerical results and discussions

Before discussing the numerical examples, the convergence analysis was firstly conducted, in which the suitable convergence of solutions was obtained for the value of $\alpha=100$. The convergence criterion was that both the c_i and b_i solutions obtained with a more crude value of $\alpha=20$ agreed with those obtained with $\alpha=100$ to within three significant figures. The numerical results are now discussed.

The first series of numerical studies are served as validations of the analysis for both the free vibrations and buckling loads presented herein. For comparison purposes, finite element solutions based on the SAP 90 were used to compute the first four frequency parameters c_i and the results are compared in Table 1. In this study, the end constraints of hinged and clamped ends are depicted as 'h' and 'c', respectively. For example, 'h-c' means the end constraint of hinged-clamped ends.

Also comparisons are made between the first four buckling parameters b_i computed using the present analysis and those given by Timoshenko and Gere(1) in Table 2. These two Tables show that both the natural frequencies and buckling loads predicted herein agree closely with those of the references.

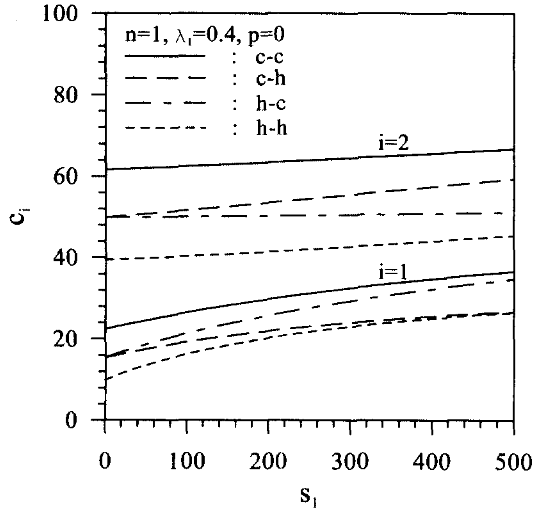

Fig. 3 c_i versus s_1 curves

Table 1 Comparison of frequency parameters c_i between this study and finite element method SAP 90 for $p=0$

ξ_j, s_j	End constraint	Data source	Frequency parameter, c_i			
			i=1	i=2	i=3	i=4
$\xi_1=0.5$ $s_1=50.$	h-h	This study	14.0	39.5	89.4	158.
		SAP 90	14.0	39.5	89.3	158.
	c-c	This study	25.0	61.7	121.	200.
		SAP 90	25.0	61.7	121.	200.
$\xi_1=0.2$ $s_1=50.$ $\xi_2=0.7$ $s_2=100.$	h-c	This study	19.0	53.1	105.	178.
		SAP 90	19.0	53.1	105.	178.
	c-h	This study	20.6	52.0	105.	179.
		SAP 90	20.6	51.9	105.	178.

Table 2 Comparison of buckling load parameters b_i between this study and reference(1) for $n=1$ and hinged-hinged ends.

ξ_1, s_1	Data source	Buckling load parameter, b_i			
		i=1	i=2	i=3	i=4
$\xi_1=0.5$ $s_1=50.$	This study	2.008	4.000	9.130	16.00
	reference (1)	2.008	4.000	9.130	16.00
$\xi_1=0.4$ $s_1=100.$	This study	2.615	4.368	9.106	16.14
	reference (1)	2.613	4.370	9.104	16.14

The effects of various non-dimensional system parameters on c_i and b_i are shown in Fig. 3 through 10. First, the free vibration problems are illustrated. Curves of the c_i versus s_1 with $n=1$, $\lambda_1=0.4$ and $p=0$ are given in Fig. 3, in which two lowest values of c_i are presented. The trend is expected: the c_i values increase as s_1 increases, and as the end constraint increases from hinged-hinged to hinged-clamped to clamped-hinged to clamped-clamped, each value of c_i increases. However, it is true that the fact is reversed in case of the first mode, between the hinged-clamped and clamped-hinged ends.

The curves of the $c_i\,(i=1,2)$ versus λ_1 with $n=1$, $s_1=50$ for $p=0$ are shown in Fig. 4. It is noted that for $i=1$, the c_1 values of hinged-hinged and clamped-clamped ends become maximum at $\lambda_1=0.5$, respectively; and the c_1 value for hinged-clamped case become maximum at $\lambda_1=0.42$ where the spring's position is located eccentrically to the hinged end. Also, the curves of c_i of hinged-clamped and clamped-hinged ends are symmetric with each other about $\lambda_1=0.5$. For $i=2$, the c_2 values of hinged-hinged and clamped-clamped ends for $\lambda_1=0.5$ are equal to those of columns without the elastic springs.

Shown in Fig. 5 are the variation of first frequency parameter c_1 with the number of springs n equally spaced for $p=0$. The c_1 values increase as the number of springs increases, as it is expected. It is noted that the c_1 values of clamped-hinged and hinged-clamped ends are equal to each other since the elastic springs are equally spaced in accordance with the number of springs.

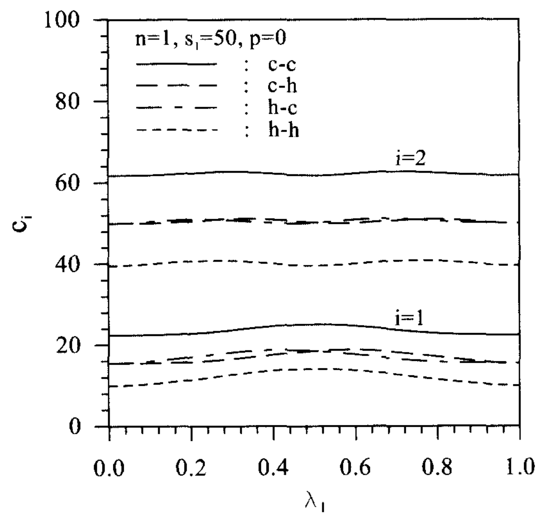

Fig. 4 c_i versus λ_1 curves

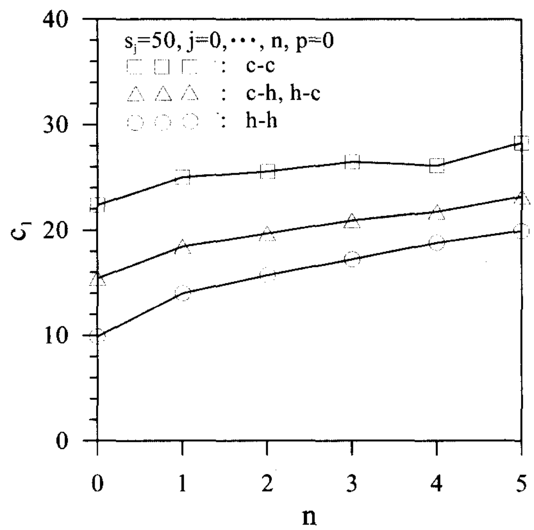

Fig. 5 Variation of c_1 with the number of elastic springs equally spaced

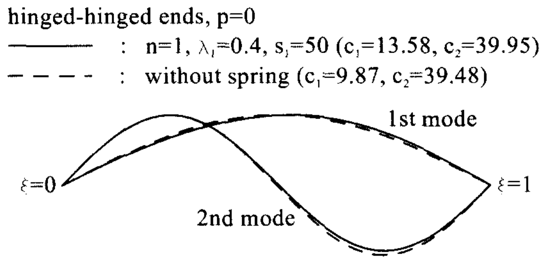

Fig. 6 Example of mode shapes for hinged-hinged ends

Shown in Fig. 6 are the first and second mode shapes of hinged-hinged ends. In this figure, the solid and dashed curves are the mode shapes with elastic spring for $n=1$, $\lambda_1=0.4$, $s_1=50$ and $p=0$, and without spring, respectively. This figure shows that the maximum amplitudes of first mode are accorded at $\xi=0.5$ for the column without the spring, and at 0.53 with the spring, respectively. Also comparing $c_1=9.87$ and 13.58 for these respective columns, it is found that the values of c_i are increased when the column is restrained by its elastic springs. In case of second modes, it is shown that also two mode shapes are much different from each other.

Fig. 7 shows the effects of axial loads on the natural

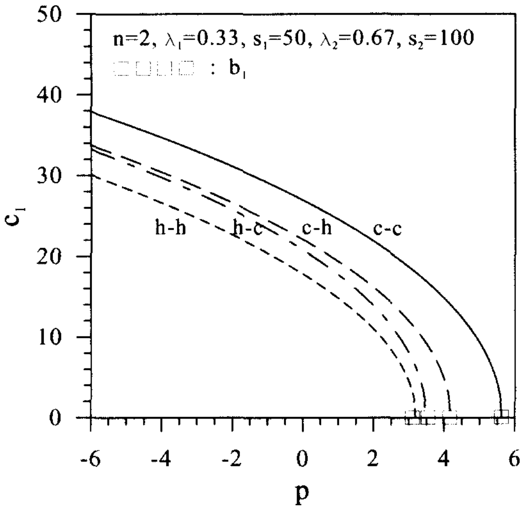

Fig. 7 c_1 versus p curves

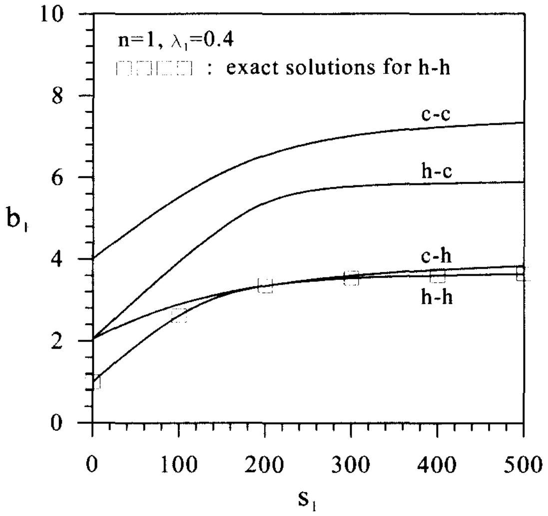

Fig. 8 b_1 versus s_1 curves

frequencies, in which only the first frequencies are presented. See the column geometry in this figure. The values of c_1 decrease as the axial compressive load increases, and the c_1 values increase as the tensile axial load increases. It is noted that the values of p on the horizontal p axis, marked by □, are the first buckling load parameters b_1 corresponding to each end constraints for a given set of column geometry.

Second, the buckling loads problems are illustrated. Fig. 8 shows the curves of b_1 versus s_1 for $\lambda_1=0.4$. Marked □ are the exact solutions by Timoshenko and Gere[1] which serve to validate the present theory. The values of b_1 increase as the spring parameter s_1 increases.

The curves of the b_1 versus λ_1 with $s_1=50$ are shown in Fig. 9. Also, marked □ are the exact solutions given by Timoshenko and Gere[1]. The trend is as expected as shown in the Fig. 4 in free vibration problems. It is noted that the b_1 values of hinged-hinged and clamped-clamped ends become maximum at $\lambda_1=0.5$, respectively, and the b_1 value of the hinged-clamped end become maximum at $\lambda_1=0.39$ where the spring's position is placed eccentrically to the hinged end.

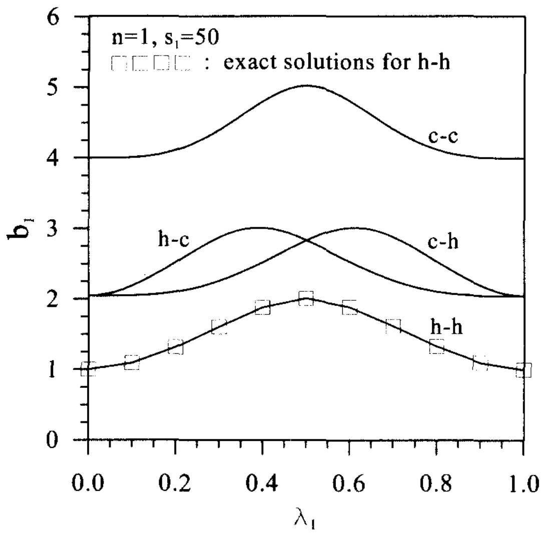

Fig. 9 b_1 versus λ_1 curves

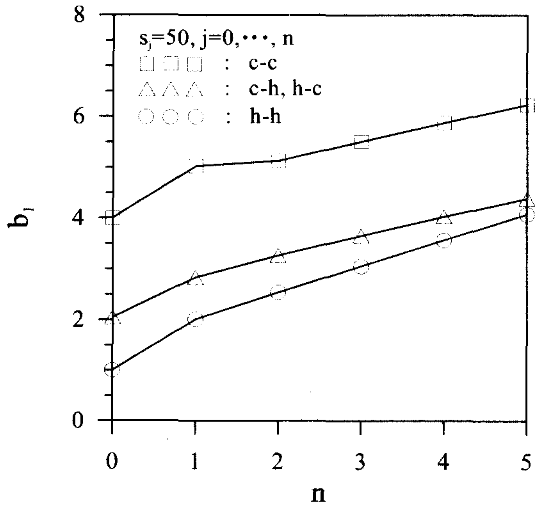

Fig. 10 Variation of b_1 with number of elastic springs equally spaced

Fig. 10 shows the variation of the first buckling load parameters b_1 with the number of elastic springs equally spaced. The b_1 values increase as the number of springs increases, which is expected.

5. Concluding remarks

The numerical methods developed herein for calculating the approximate natural frequencies and buckling loads of the columns with the multiple elastic springs are found to be efficient, accurate and highly versatile. These methods are practically versatile since the natural frequencies can be calculated considering the effects of axial loads and the multiple elastic springs restraining the column laterally, taken separately or in combination. Also the buckling loads can be calculated accurately for such columns.

Acknowledgements

This paper was supported by Wonkwang University in 2000. The first author gives thanks for this financial support. All authors are pleased to do this research co-worked by the Korean and Chinese scholars.

References

(1) Timoshenko, S. P. and Gere, J. M., 1961, *Theory of elastic stability*, Second Edition, McGraw-Hill.

(2) Timoshenko, S. P., Young, D.H. and Weaver, W., 1974, *Vibration problems in engineering*, John Wiley and Sons.

(3) Abbas, B. A. and Thomas, J., 1978, "Dynamic stability of Timoshenko beams on an elastic foundation," *Journal of Sound and Vibration*, Vol. 60, pp. 33~44.

(4) Bokaian, A., 1988, "Natural frequencies of beams under compressive axial loads," Journal of Sound and Vibration, Vol. 126, No. 1, pp. 49~65.

(5) Pavlovic, G. M. and Wylie, G. B., 1983, "Vibration of beams on non-homogeneous elastic foundation," Earthquake Engineering and Structural Dynamics, Vol. 11, pp. 797~808.

(6) Valsangkar, A. J. and Pradhanang, R.B., 1987, "Free vibration of partially supported piles," Journal of Engineering Mechanics, ASCE, Vol. 113, pp. 1244~1247.

(7) Lee, B. K. and Oh, S. J., 1994, "Free vibrations and buckling loads of beam-column on elastic foundations," Proceedings of the International Conference on Vibration Engineering, Beijing, pp. 1155~1160.

(8) Lee, B. K., Jeong, J. S., Li, G. F. and Jin, T.K., 1999, "Free vibrations of tapered piles embedded partially in elastic foundation," Chinese Journal of Geotechnical Engineering, Vol. 21, No. 5, pp. 609~613.

(9) Lee, B. K. and Oh, S. J., 2000, "Elastica and buckling loads of simple tapered column with constant volume," International Journal of Solids and Structures, Vol. 37, No. 18, pp. 2507~2508.

(10) Oh, S. J., Lee, B. K. and Lee, I.W., 1999, "Natural frequencies of non-circular arches with rotatory inertia and shear deformation," Journal of Sound and Vibration, Vol. 219, No. 1, pp. 23~33.

(11) Mathews, J. H., 1987, Numerical method, Prentice Hall.

(12) Oh, S. J., Lee, B. K. and Lee, I.W., 2000, "Free vibrations of non-circular arches with non-uniform cross-section," International Journal of Solids and Structures, Vol. 37, No. 36, pp. 4871~4891.