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REGULAR CLOSED BOOLEAN ALGEBRA IN SPACE WITH ONE POINT LINDELÖFFICATION TOPOLOGY

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ABSTRACT. Let (X^*, τ^*) be the space with one point Lindelöffication topology of space (X, τ). This paper offers the definition of the space with one point Lindelöffication topology of a topological space and proves that the retraction regular closed function $f : K^*(X^*) \to K(X)$ defined by $f(A^*) = A^*$ if $p \notin A^*$ or $f(A^*) = A^* - \{p\}$ if $p \in A^*$ is a homomorphism. There are two examples in this paper to show that the retraction regular closed function f is neither a surjection nor an injection.

1. One point Lindelöffication topology

Definition 1. Let (X, τ) be an arbitrary topological space. A subset A of X is called *Lindelöf subset* if A as the subspace of (X, τ) is a Lindelöf space (see Dow and Vermeer [1]). In (X, τ), the family of *closed sets* is

$$\Omega = \{B : B \subset X \text{ and } X - B \in \tau\}$$

and the family of *Lindelöf subsets* is

$$L = \{A : A \subset X \text{ and } A \text{is a Lindelöf subset in } (X, \tau)\}.$$

We assume that p is a point not in X and $X^* = X \cup \{p\}$. Suppose that $\tau^* = \tau \cup \tau_1$ where $\tau_1 = \{E : E \subset X^* \text{ and } X^* - E \in \Omega \cap L\}$. (X^*, τ^*) is called the space with *one point Lindelöffication topology* of topological space (X, τ).

It is clear that $X^* \in \tau_1$ since $\emptyset \in \Omega \cap L$. From Definition 1 one can assert that the family of closed sets in (X^*, τ^*) is $\Omega^* = \{A^* : A^* = A \cup \{p\}$ where $A \in \Omega\} \cup \{B^* : B^* \in \Omega \cap L\}$.

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For any $x \in X$, we denote the neighborhood system of x in (X, τ) by Ξ_x.

2. Lemmas and propostions

Lemma A. *For any $x \in X$, $U^* \in \Xi_x$ or $U^* - \{p\} \in \Xi_x$ if and only if U^* is a neighborhood of x.*

Proof. If U^* is a neighborhood of x in (X^*, τ^*), then $U^* \in \Xi_x$ or $\exists A^* \in \tau_1$ such that $x \in A^* \subset U^*$. In the second case, there is $B \in \Omega \cap L$ satisfying $A^* = X^* - B$ so $x \in X - B \subset A^* \subset U^*$. It follows that $U^* - \{p\} \in \Xi_x$. According to the definition of neighborhood, the converse is obvious. □

Lemma B. *For any $U^* \subset X^*$, if $p \in U^* \in \Xi_p^*$, then there is $A \in \tau$ such that $A \subset U^* - \{p\}$.*

Proof. If $U^* \in \Xi_p^*$, then there is $A^* \in \tau_1$ such that $p \in A^* \subset U^*$ so there is $B \in \Omega \cap L$ satisfying $X^* - A^* = B$. It follows that $X - B \in \tau$ and $X - B = A^* - \{p\} \subset U^* - \{p\}$. □

The neighborhood system for any $x \in X$ in (X^*, τ^*) can be also denoted by $\Xi_x^* = \Xi_x^0 \cup \Xi_x^1 \cup \Xi_x^2$, where Ξ_x^0 is the neighborhood system Ξ_x for x in (X, τ),

$$\Xi_x^1 = \{U \cup \{p\} : U \in \Xi_x\}$$

and

$$\Xi_x^2 = \{U^* : \exists A^* \in \tau_1 \text{ such that } x \in A^* \subset U^* \subset X^*\}.$$

The neighborhood system for p is $\Xi_p^* = \{U^* : \exists A^* \in \tau_1 \text{ such that } A^* \subset U^* \subset X^*\}$. That is, for any $U^* \in \Xi_p^*$, there is a set $B \in \Omega \cap L$ such that $X^* - U^* \subset B$.

Throughout this paper, we denote the closure and the interior of set A in (X, τ) by $c(A)$ and $i(A)$ respectively. The closure and the interior of set A in (X^*, τ^*) are represented by $c^*(A)$ and $i^*(A)$ respectively. The family of regular closed sets in (X, τ) and in (X^*, τ^*) are denoted by $K(X)$ and $K^*(X^*)$ respectively, that is, for any $A \in K(X)$, $c(i(A)) = A$ and for any $A^* \in K^*(X^*)$, $c^*(i^*(A^*)) = A^*$.

For any $A, B \in K(X)$ we define $A \odot B = c(i(A \cap B))$, $A' = c(i(X - A))$ and $A \circ B = (A \odot B') \cup (B \odot A')$. In order to distinguish an operation in $K^*(X^*)$ from in $K(X)$, we write $A \odot B$ for $A, B \in K(X)$ and the operation $\odot$ is operated in $K(X)$, $A \odot^* B$ for $A, B \in K^*(X^*)$ and the operation in $K^*(X^*)$, $A' = c(i(X - A))$,

$A'_* = c^*(i^*(X^* - A))$, $A \circ B = (A \odot B') \cup (B \odot A')$ and $A \circ^* B = (A \odot^* B'_*) \cup (B \odot^* A'_*)$, etc.

Proposition 1. *$c^*(A) = c(A) \cup \{p\}$ if $A \subset X$, and, for any $B \in \Omega \cap L$, $A \cap (X - B) \neq \emptyset$.*

Proof. According to Dugundji [2, p. 77], $c(A) \subset c^*(A)$ since (X, τ) is a subspace of (X^*, τ^*). It is clear that $p \notin A$ if $A \subset X$. For any $U^* \in \Xi^*_p$, $U^* \cap A \neq \emptyset$ since $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$. It follows that $p \in c^*(A)$.On the other hand, for any $x \in X$, if $x \notin c(A)$, then there is $U \in \Xi^0_x$ such that $U \cap A = \emptyset$ so $x \notin c^*(A)$ because $\Xi^0_x \subset \Xi^*_x$. From above one can assert that $c^*(A) = c(A) \cup \{p\}$. □

Corollary 1. *For any $A \subset X$, $p \notin A$ and the following two conclusions hold:*

(1) *$c^*(A) = c(A)$ if $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$.*
(2) *$c^*(A) = c(A) \cup \{p\}$ if $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$.*

An apparent fact is $c^*(X) = X^*$ if (X, τ) is not a Lindelöf space. In this case (X^*, τ^*) is right the one point Lindelöffication of (X, τ) from Definition 1. So in the rest of this paper we assume that (X, τ) is always a non-Lindelöf space unless stated otherwise.

Proposition 2. *If $p \in A$, then $c^*(A) = c(A - \{p\}) \cup \{p\}$.*

Proof. It is clear that $c^*(\{p\}) = \{p\}$ since $X^* - \{p\} = X \in \tau^*$. So $c^*(A) = c^*(A - \{p\}) \cup c^*(\{p\}) = c(A - \{p\}) \cup \{p\}$ from Corollary 1. □

Corollary 2. *For any $A \subset X^*$, the folowings are hold.*

(1) *If $p \notin A$ and $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$, then $c^*(A) = c(A - \{p\})$;*
(2) *If $p \in A$ or $p \notin A$ and $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$, then $c^*(A) = c(A - \{p\}) \cup \{p\}$.*

Proposition 3. *For any $A \subset X$, $i^*(A) = i(A)$.*

Proof. Since $p \in X^* - A = (X - A) \cup \{p\}$, $i^*(A) = X^* - c^*((X - A) \cup \{p\}) = X^* - (c(X - A) \cup \{p\}) = i(A)$ from Proposition 2. □

Proposition 4. *If $p \in A \subset X^*$, then the folowings are hold.*

(1) *$i^*(A) = i(A - \{p\}) \cup \{p\}$ provided that $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$.*
(2) *$i^*(A) = i(A - \{p\})$ provided that $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$.*

Proof. It is clear that $p \notin X^* - A$ since $p \in A$ so $X^* - A = X - (A - \{p\})$.

(1) Since there is $B \in \Omega \cap L$ satisfying $A \cap (X - B) = \emptyset$, $i^*(A) = X^* - c^*(X - [A - \{p\}]) = X^* - c(X - [A - \{p\}]) = i(A - \{p\}) \cup \{p\}$ from (1) of Corollary 1.

(2) Since $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$,

$$i^*(A) = X^* - c^*(X - [A - \{p\}]) = X^* - \{c(X - [A - \{p\}]) \cup \{p\}\} = i(A - \{p\})$$

from (2) of Corollary 1. □

Corollary 3. *For any $A \subset X^*$, the folowings are hold.*

(1) *If $p \in A$ and $\exists B \in \Omega \cap L$ such that $A \cap (X - B) = \emptyset$, then $i^*(A) = i(A - \{p\}) \cup \{p\}$;*

(2) *If $p \notin A$ or $p \in A$ and $A \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$, then $i^*(A) = i(A - \{p\})$.*

Proposition 5. *For any $A \subset X^*$, the folowings are hold.*

(1) $c^*(i^*(A)) = c(i(A)) \cup \{p\}$ *if $p \notin A$ and for any $B \in \Omega \cap L$, $i(A) \cap (X - B) \neq \emptyset$.*

(2) $c^*(i^*(A)) = c(i(A))$ *if $p \notin A$ and there is $B \in \Omega \cap L$, $i(A) \cap (X - B) = \emptyset$.*

(3) $c^*(i^*(A)) = c(i(A - \{p\})) \cup \{p\}$ *if $p \in A$ and there is $B \in \Omega \cap L$, $i(A - \{p\}) \cap (X - B) = \emptyset$.*

(4) $c^*(i^*(A)) = c(i(A - \{p\}))$ *if $p \in A$ and for any $B \in \Omega \cap L$, $i(A - \{p\}) \cap (X - B) \neq \emptyset$.*

Proof. (1) Since $p \notin A$, $p \notin i(A)$ and $i^*(A) = i(A)$ from Proposition 3. So $c^*(i^*(A)) = c^*(i(A)) = c(i(A)) \cup \{p\}$ according to (2) of Corollary 2.

(3) Since $p \in A$, $i^*(A) = i(A - \{p\}) \cup \{p\}$ from (1) of Proposition 4. It is obvious that $p \notin i(A - \{p\})$. So $c^*(i^*(A)) = c^*(i(A - \{p\})) \cup \{p\} = c(i(A - \{p\})) \cup \{p\}$ according to (1) of Corollary 2.

The proofs for (2) and (4) are similar with that for (1) and (3) respectively. □

Proposition 6. *If $p \notin A$ and $A \in K^*(X^*)$, then $A \in K(X)$.*

Proof. According to (1) and (2) of Proposition 5, $A = c(i(A))$ since $p \notin A$ and $A = c^*(i^*(A))$ from $A \in K^*(X^*)$. □

Proposition 7. *If $p \in A^* \in K^*(X^*)$, then $A^* - \{p\} \in K(X)$.*

Proof. From (3) and (4) of Proposition 5, $A^* - \{p\} = c(i(A^* - \{p\}))$ so $A^* - \{p\} \in K(X)$. □

Proposition 8. *If $A \in K(X)$, then $A \in K^*(X^*)$ and $A \cup \{p\} \in K^*(X^*)$ if and only if there is $B \in \Omega \cap L$, such that $i(A) \cap (X - B) = \emptyset$.*

Proof. It is easy to see that $c^*(i^*(A)) = A$ and $c^*(i^*(A \cup \{p\})) = A \cup \{p\}$ from (2) and (3) of Proposition 5 respectively. □

Proposition 9. *$\{p\} \in K^*(X^*)$ if and only if (X, τ) is a Lindelöf space.*

Proof. If $\{p\} \in K^*(X^*)$, then $c^*(i^*(\{p\})) = \{p\}$. It follows that $i^*(\{p\}) = \{p\}$ so $\{p\}$ is an open set in (X^*, τ^*). Moreover, $X = X^* - \{p\} \in \Omega \cap L$ so X is a Lindelöf space.

Conversely, if X is a Lindelöf space, then $\{p\}$ is an open and closed set in (X^*, τ^*) so $c^*(i^*(\{p\})) = \{p\}$. That is, $\{p\} \in K^*(X^*)$. □

Proposition 10. *$X \in K^*(X^*)$ if and only if (X, τ) is a Lindelöf space.*

Proof. If $X \in K^*(X^*)$, then there is $B \in \Omega \cap L$ satisfying $i(X) \cap (X - B) = \emptyset$ from Proposition 5 so $X \subset B$. On the other hand, $B \subset X$. Finally, $X = B$ is a Lindelöf set in (X, τ). Conversely, if (X, τ) is a Lindelöf space, then $X \in \Omega \cap L$ so X is a closed set in (X^*, τ^*). Since $X \in \tau \subset \tau^*$, X is open in (X^*, τ^*). Hence $X \in K^*(X^*)$. □

Corollary 4. *$\{p\} \in K^*(X^*)$ if and only if $X \in K^*(X^*)$.*

According to Kuratowski and Mostowski [3, p. 39], $K(X)$ is a Boolean algebra with a unit with respect to the operations $\circ$ and $\odot$ as well as that $K^*(X^*)$ a Boolean algebra with a unit with respect to the operations $\circ^*$ and $\odot^*$. It is easy to see that $K(X)$ is a sub-algebra of $K^*(X^*)$ if and only if X is a Lindelöf space from Propositions 6, 7 and 10.

Proposition 11. *If $A \subset X$ and $A^* = A \cup \{p\}$ then*

(1) *$(A^*)'_* = A' \cup \{p\}$ provided that $i(X - A) \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$;*
(2) *$(A^*)'_* = A'$ provided that there is $B \in \Omega \cap L$ satisfying $i(X - A) \cap (X - B) = \emptyset$;*
(3) *$A'_* = A' \cup \{p\}$ provided that there is $B \in \Omega \cap L$ satisfying $i(X - A) \cap (X - B) = \emptyset$;*
(4) *$A'_* = A'$ provided that $i(X - A) \cap (X - B) \neq \emptyset$ for any $B \in \Omega \cap L$.*

Proof. (1) From (1) of Proposition 5, $(A^*)'_* = c^*(i^*(X^* - A^*)) = c(i(X - A)) \cup \{p\} = A' \cup \{p\}$.

(2) From (2) of Proposition 5, $(A^*)'_* = c^*(i^*(X^* - A^*)) = c(i(X - A)) = A'$.

(3) From (3) of Proposition 5, $A'_* = c^*(i^*(X^* - A^*)) = c(i(X - A)) \cup \{p\} = A' \cup \{p\}$.

(4) From (4) of Proposition 5, $A'_* = c^*(i^*(X^* - A)) = c(i(X - A)) = A'$. □

3. The retraction regular closed function

Definition 2. The *retraction regular closed function* $f : K^*(X^*) \to K(X)$ is defined for any $A^* \in K^*(X^*)$ by $f(A^*) = A^*$ if $p \notin A^*$ or $f(A^*) = A^* - \{p\}$ if $p \in A^*$.

Proposition 12. *If $p \in A^* \in K^*(X^*)$, then $f((A^*)'_*) = (f(A^*))'$.*

Proof. Let $A \subset X$ and $A^* = A \cup \{p\}$. If for any $B \in \Omega \cap L$, $i(X - A) \cap (X - B) \neq \emptyset$, then $f((A^*)'_*) = f(A' \cup \{p\}) = A' = (f(A^*))'$ from (1) of Proposition 11 and Definition 2. On the other hand, if there is $B \in \Omega \cap L$ satisfying $i(X - A) \cap (X - B) = \emptyset$, then $f((A^*)'_* = f(A') = A' = (f(A^*))'$ from (2) of Proposition 11 and Definition 2. □

Proposition 13. *If $p \notin A \in K^*(X^*)$, then $f(A'_*) = (f(A))'$.*

Proof. From (3) of Proposition 11 and Definition 2, if there is $B \in \Omega \cap L$ satisfying $i(X - A) \cap (X - B) = \emptyset$, then $f(A'_*) = f(A' \cup \{p\}) = A' = (f(A))'$. On the other hand, if for any $B \in \Omega \cap L$, $i(X - A) \cap (X - B) \neq \emptyset$, then $f(A'_*) = f(A') = A' = (f(A))'$ from (4) of Proposition 11 and Definition 2. □

Corollary 5. *For any $A^* \in K^*(X^*)$, $f((A^*)'_*) = (f(A^*))'$.*

Proposition 14. *For any $A^*, B^* \in K^*(X^*)$,*

(1) *If $p \in A^* - B^*$ and for any $C \in \Omega \cap L$, $i(A^* \cap B^*) \cap (X - C) \neq \emptyset$, then $A^* \odot^* B^* = ((A^* - \{p\}) \odot B^*) \cup \{p\}$;*

(2) *If $p \in B^* - A^*$ and there is $C \in \Omega \cap L$ satisfying $i(A^* \cap B^*) \cap (X - C) = \emptyset$, then $A^* \odot^* B^* = A^* \odot (B^* - \{p\})$;*

(3) *If $p \in A^* \cap B^*$ and for any $C \in \Omega \cap L$, $i((A^* \cap B^*) - \{p\}) \cap (X - C) \neq \emptyset$, then $A^* \odot^* B^* = ((A^* - \{p\}) \odot (B^* - \{p\})$;*

(4) *If $p \in A^* \cap B^*$ and there is $C \in \Omega \cap L$ satisfying $i((A^* \cap B^*) - \{p\}) \cap (X - C) = \emptyset$, then $A^* \odot^* B^* = ((A^* - \{p\}) \odot (B^* - \{p\})) \cup \{p\}$;*

(5) *If $p \notin A^* \cup B^*$ and for any $C \in \Omega \cap L$, $i((A^* \cap B^*)) \cap (X - C) \neq \emptyset$, then $A^* \odot^* B^* = (A^* \odot B^*) \cup \{p\}$;*

(6) *If* $p \notin A^* \cup B^*$ *and there is* $C \in \Omega \cap L$ *satisfying* $i(A^* \cap B^*) \cap (X - C) = \emptyset$, *then* $A^* \odot^* B^* = A^* \odot B^*$.

Proof. (1) From (1) of Proposition 5, $A^* \odot^* B^* = c(i((A^* - \{p\}) \cap B^*)) \cup \{p\}$.

(2) From (2) of Proposition 5, $A^* \odot^* B^* = c(i(A^* \cap (B^* - \{p\})))$.

(3) From (4) of Proposition 5, $A^* \odot^* B^* = c(i((A^* \cap B^*) - \{p\})) = c(i((A^* - \{p\}) \cap (B^* - \{p\})))$.

(4) From (3) of Proposition 5, $A^* \odot^* B^* = c(i((A^* \cap B^*) - \{p\})) \cup \{p\} = c(i((A^* - \{p\}) \cap (B^* - \{p\})))$.

(5) From (1) of Proposition 5, since $p \notin A^* \cap B^*$, $A^* \odot^* B^* = c(i(A^* \cap B^*)) \cup \{p\}$.

(6) From (2) of Proposition 5, $A^* \odot^* B^* = c(i(A^* \cap B^*))$. □

Proposition 15. *For any* $A^*, B^* \in K^*(X^*)$, $f(A^* \odot^* B^*) = f(A^*) \odot f(B^*)$.

Proof. In order to reduce the length of the proof, we call

$$i(A^* \cap B^*) \cap (X - C) = \emptyset \tag{*}$$

the equality (*) brifly. The situation of the proof can be divided into the following eight cases:

1. $p \in A^* - B^*$, for any $C \in \Omega \cap L$ the equality (*) does not hold.
2. $p \in A^* - B^*$, there is $C \in \Omega \cap L$ satisfying the equality (*).
3. $p \in B^* - A^*$, for any $C \in \Omega \cap L$ the equality (*) does not hold.
4. $p \in B^* - A^*$, there is $C \in \Omega \cap L$ satisfying the equality (*).
5. $p \in A^* \cap B^*$, for any $C \in \Omega \cap L$ the equality (*) does not hold.
6. $p \in A^* \cap B^*$, there is $C \in \Omega \cap L$ satisfying the equality (*).
7. $p \notin A^* \cup B^*$, for any $C \in \Omega \cap L$ the equality (*) does not hold.
8. $p \notin A^* \cup B^*$, there is $C \in \Omega \cap L$ satisfying the equality (*).

The proof for each case is simple from Proposition 14 so it is omitted. □

Proposition 16. *For any* $A^*, B^* \in K^*(X^*)$, $f(A^* \cup B^*) = f(A^*) \cup f(B^*)$.

Proof. It is easy to check the equality from Definition 2. □

Proposition 17. *For any* $A^*, B^* \in K^*(X^*)$, $f(A^* \circ^* B^*) = f(A^*) \circ f(B^*)$.

Proof. It is the result of calculating from Propositions 16 and 15 and Corollary 5. □

From above we obtain the following theorem.

Theorem 1. *The retraction regular closed function* $f : K^*(X^*) \to K(X)$ *defined for any* $A^* \in K^*(X^*)$ *by* $f(A^*) = A^*$ *if* $p \notin A^*$ *or* $f(A^*) = A^* - \{p\}$ *if* $p \in A^*$ *from the regular closed algebra* $K^*(X^*)$ *of the space with one point Lindelöffication topology into that algebra* $K(X)$ *of the space* (X, τ) *is a homomorphism.*

Proposition 18. *Retraction regular closed function* f *is a surjective homomorphism if and only if for any* $A \in K(X)$, *there exists* $B \in \Omega \cap L$ *satisfying* $i(A) \cap (X - B) = \emptyset$.

Proof. It is obvious from the theorem, Proposition 8 and Definition 2. □

Definition 3. Let $P = \{(x, y) : x, y \in R, y > 0\}$ be the open upper half plane with the Euclidean topology σ and $R_1 = \{(x, 0) : x \in R\}$ the real axis. The topology τ on $X = P \cup R_1$ is generated by adding to σ all sets of the form $\{(x, 0)\} \cup (P \cap U)$ where U is a Euclidean neighborhood of $(x, 0)$ in the plane. The topological space (X, τ) is called the space with half-disc topology (see Steen and Seebach [3, p. 96-97]).

Definition 4. Let (X, τ) be the space with half-disc topology, q a point not in X and $X^* = X \cup \{q\}$. Suppose that $\tau^* = \tau \cup \tau_1$ where $\tau_1 = \{E : E \subset X^*$ and $X^* - E \in \Omega \cap L\}$. The topological space (X^*, τ^*) is called the space with one point Lindelöffication topology of the space with half-disc topology.

Example 1. The retraction regular closed function f may be not surjective homomorphism. Let (X^*, τ^*) be the space with one point Lindelöffication topology of space (X, τ) with half-disc topology. If

$$A = \{(x, y) : x, y \in R,\ 0 \leq y \leq 1,\ x \in \bigcup_{n=1}^{\infty} [n - \frac{1}{n}, n + \frac{1}{n}]\}$$

then $A \in K(X)$ from Definition 3 and for any $B \in \Omega \cap L$ the equality $i(A) \cap (X - B) = \emptyset$ does not hold. So there is no $A^* \in K^*(X^*)$ satisfying $f(A^*) = A$ from Proposition 8, where A is a closed and non-Lindelöf subset in (X, τ) and not a closed set in (X^*, τ^*). Hence the retraction regular closed function f is not a surjective homomorphism.

Proposition 19. *If retraction regular closed function* f *is a surjective homomorphism, then* (X, τ) *is a Lindelöf space.*

Proof. Since $X \in K(X)$, if f is surjective, then there is $B \in \Omega \cap L$ satisfying $X \cap (X - B) = \emptyset$ from Proposition 18. So $B = X$. That is, X is a Lindelöf space. □

Example 2. The retraction regular closed function f may be not an injective homomorphism. Let (X^*, τ^*) be the space with one point Lindelöffication topology of space (X, τ) with half-disc topology. If

$$A = \{(x, y) : x, y \in R, 1 \leq x \leq 2 \text{ and } 2 \leq y \leq 3\}$$

and

$$B = \{(x, y) : x, y \in R, 0 \leq x \leq 3 \text{ and } 1 \leq y \leq 4\}$$

then $A, B \in K(X)$ from the definition of the space with half-disc topology. Since $B \in \Omega \cap L$ and $i(A) \cap (X - B) = \emptyset$, $A \in K^*(X^*)$ and $A \cup \{q\} \in K^*(X^*)$ from Proposition 8. Finally, $f(A) = f(A \cup \{q\}) = A$, that is, f is not injective.

Proposition 20. *For any $A \subset X$, $A \in K^*(X^*)$ if and only if $A \cup \{p\} \in K^*(X^*)$.*

Proof. If $A \in K^*(X^*)$, then $c^*(i^*(A)) = A$ so there is $B \in \Omega \cap L$ satisfying $i(A) \cap (X - B) = \emptyset$ from Proposition 5. So that $c^*(i^*(A \cup \{p\})) = A \cup \{p\} \in K^*(X^*)$ from Proposition 5 again. Since the process of the proof can be inverted, we omit the proof for the converse. □

From Proposition 20, we can assert that any retraction regular closed function f from the regular closed Boolean algebra $K^*(X^*)$ of the space with one point Lindelöffication topology into that algebra $K(X)$ of topological space (X, τ) must not be injective.

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