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A VERSION OF A CONVERSE MEASURABILITY FOR WIENER SPACE IN THE ABSTRACT WIENER SPACE

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ABSTRACT. Johnson and Skoug [Pacific J. Math. **83** (1979), 157–176] introduced the concept of scale-invariant measurability in Wiener space. And they applied their results in the theory of the Feynman integral. A converse measurability theorem for Wiener space due to Köehler and Yeh-Wiener space due to Skoug [Proc. Amer. Math. Soc. **57** (1976), 304–310] is one of the key concept to their discussion.

In this paper, we will extend the results on converse measurability in Wiener space which Chang and Ryu [Proc. Amer. Math. Soc. **104** (1998), 835–839] obtained to abstract Wiener space.

1. Introduction and preliminaries

Let H be an infinite dimensional real separable Hilbert space with norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. And let $\mathcal{P} = \mathcal{P}(H)$ be the class of orthogonal projections on H with finite dimensional range. Then for $P \in \mathcal{P}$,

$$\mathcal{C}_P := \{P^{-1}B : B \text{ is a Borel set in the range of P}\}$$

is a σ-field. And the sets in $\mathcal{C}_P$ are called cylinder sets with base P.

Let $\mathcal{C} = \bigcup \mathcal{C}_P$. Then $\mathcal{C}$ is a field but is not a σ-field. Let μ be the cylinder set measure on H defined by

$$\mu(E) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_F \exp\left(-\frac{|x|^2}{2}\right) dx$$

where $E = P^{-1}(F)$, F is a Borel set in the image of an n-dimensional projection P in H and dx is Lebesgue measure in PH.

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Definition 1.1. *A norm $||\cdot||$ on H is said to be measurable with respect to μ if for every $\epsilon > 0$, there exists $P_\epsilon \in \mathcal{P}$ such that*

$$\mu(\{x \in H : ||Px|| > \epsilon\}) < \epsilon$$

for all P is a orthogonal to P_ϵ, $P \in \mathcal{P}$.

Let P, $P_\epsilon \in \mathcal{P}$ with $\dim P_\epsilon H = n$ and $\dim PH = k$ and $P > P_\epsilon$(means that $P(H) \supset P_\epsilon(H)$), then $P - P_\epsilon$ is orthogonal to P_ϵ and $P - P_\epsilon \in \mathcal{P}$. Further $\dim(P - P_\epsilon)H = k - n$, and

$$\begin{aligned}
&\mu(\{x \in H : |(P - P_\epsilon)(x)| > \epsilon\}) \\
&\quad = \mu(\{x \in H : |(P - P_\epsilon)(x)|^2 > \epsilon^2\}) \\
&\quad = 1 - \mu(\{x \in H : |(P - P_\epsilon)(x)|^2 < \epsilon^2\}) \to 1 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Thus the Definition 1.1 shows that the Hilbertian norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$ is not a measurable norm.

A measurable norm is necessarily weaker than the given Hilbertian norm $|\cdot|$. Indeed, if $||\cdot||$ is a measurable norm, then there exists a constant c such that $||h|| \le c|h|$ for all $h \in H$ and H is not complete with respect to $||\cdot||$ (cf. [8]).

Let B denote the Banach space which is the $||\cdot||$-completion of H. Let $\gamma : H \to B$ denote the natural injection (so that $\gamma(h) = h$). Then γ is continuous and $\gamma(H)$ is dense in B. The adjoint operator γ^* is one-to-one and maps B^* continuously onto a dense subset of H^*. Since H is a Hilbert space, H^* can be identified with H. Thus we have a triple, $B^* \subset H^* = H \subset B$ and $\langle x, y \rangle = (x, y)$ for all x in H and y in B^*, where (x, y) denote the action of an element y in B^* on an element x in B.

Let $\mathcal{B}_0$ denote the set of the form

$$\{x \in B : ((x, y_1), (x, y_2), \cdots, (x, y_k)) \in E\}$$

for $k \ge 1$, $y_i \in B^*$, $E \in \mathcal{B}(\mathbb{R}^k)$, the Borel σ-field of $\mathbb{R}^k$.

Using this we can see that $\gamma^{-1}(\mathcal{B}_0) \subseteq \mathcal{C}$. By a well-known result of Gross [5], $\mu \circ \gamma^{-1}$ has a unique countably additive extension ν to the Borel σ-field $\mathcal{B}(B)$ of B. The triple (H, B, ν) is called an abstract Wiener space.

Since $\gamma^*(B^*)$ is dense in $H^* = H$, we can choose a complete orthonormal system $\{e_j : j \ge 1\}$ of H such that $\{e_j : j \ge 1\} \subseteq \gamma^*(B^*)$.

Let $\{y_j : j \geq 1\} \subseteq B^*$ be such that $e_j = \gamma^*(y_j)$. For each h in H and x in B, let

$$L(h)(x) = \begin{cases} \sum_{j=1}^{\infty} \langle h, e_j \rangle (x, e_j) & \text{if the series converges,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

By the choice of $\{y_j : j \geq 1\}$, $\{y_j\}$ is a sequence of independent identically distributed random variables on $(B, \mathcal{B}(B), \nu)$ with mean zero and unit variance. Thus the series in (1.1) converges a.e. x.

Furthermore $L(h)$ is a Borel measurable on B and if both h and x are in H, Parseval's identity gives $L(h)(x) = \langle h, x \rangle$ (cf. [9]). We have the following facts from Kallianpur and Bromley [7].

Lemma 1.2. *Let (H, B, ν) be an abstract Wiener space. Then*

(a) *for each $h(\neq 0) \in H$, $L(h)$ is Gaussian with mean zero and variance $|h|^2$; and*

(b) *if $\{h_1, h_2, \cdots, h_n\}$ is an orthonormal set in H, then the random variables $L(h_j)$ are independent.*

2. Converse measurability for (H, B, ν)

A probability measure P on a σ-field $\mathcal{S}$ containing the Borel sets in a topological space S is called *tight* if for every $\epsilon > 0$ and for any $E \in \mathcal{S}$ there exists a compact set K such that $P(E \backslash K) < \epsilon$. It is well-known that any probability measure on the Borel class of a complete separable metric space is tight (cf. [9]).

Let P be a tight measure on $\mathcal{B}(S)$ and m be a measure on $\mathcal{B}(T)$ where S and T are topological spaces, respectively. Let $(S, \overline{\mathcal{B}(S)}, \overline{P})$ and $(T, \overline{\mathcal{B}(T)}, \overline{m})$ be the completion of $(S, \mathcal{B}(S), P)$ and $(T, \mathcal{B}(T), m)$, respectively. Then $\overline{P}$ is also a tight measure on $\overline{\mathcal{B}(S)}$ (cf. [2]). And we have the following lemma which is an extension of the result in Chang and Ryu [2].

Lemma 2.1. *Let $f : S \to T$ be a Borel measurable function and let*

$$\mathcal{U} = \{E \subset T : f^{-1}(E) \text{ is } \overline{P}\text{-measurable}\} \tag{2.1}$$

Then $(T, \mathcal{U}, \mu)$ is a complete tight measure space where $\mu := \overline{P} \circ f^{-1}$ is a set function on $\mathcal{U}$.

Proof. It is easy to see that $\mathcal{U}$ is a σ-field which contains the Borel sets in T and $(T, \mathcal{U}, \mu)$ is a complete measure space.

To show that μ is tight on $\mathcal{U}$, let $E \in \mathcal{U}$, and $\epsilon > 0$ be given. Since $E \in \mathcal{U}$, there exists a Borel subset $\tilde{E}$ of $f^{-1}(E)$ such that $\overline{P}(\tilde{E}) = \overline{P}(f^{-1}(E))$. And since f is a Borel measurable, it follows from a generalization of Lusin's theorem [3], there exists a compact subset K_ϵ of $\tilde{E}$ such that $\overline{P}(\tilde{E}) < \overline{P}(K_\epsilon) + \epsilon$ and f is continuous on K_ϵ, where $\tilde{E}$ is a Borel subset of $f^{-1}(E)$. Thus $f(K_\epsilon)$ is compact, $f(K_\epsilon) \subset E$ and

$$\begin{aligned}\mu(E\backslash f(K_\epsilon)) = \overline{P}(f^{-1}(E\backslash f(K_\epsilon))) &\le \overline{P}(f^{-1}(E)\backslash K_\epsilon) \\ &= \overline{P}(f^{-1}(E)) - \overline{P}(K_\epsilon) = \overline{P}(\tilde{E}) - \overline{P}(K_\epsilon) < \epsilon.\end{aligned}$$

Therefore μ is a tight measure on $\mathcal{U}$. □

By the similar arguments as in Chang and Ryu [2], we have the following lemma.

Lemma 2.2. *Let $\mathcal{U}$ be defined as in (2.1), then $\mathcal{U} = \overline{\mathcal{B}(T)}$ under the following assumption: N is $\bar{m}$-null set if and only if N is μ-null set.*

We can now prove a version of converse measurability theorem for Wiener space (cf. [2]) in the setting of abstract Wiener space.

Theorem 2.3. *Let (H, B, ν) be an abstract Wiener space, and let $\{h_1, h_2, \cdots, h_n\}$ be a linearly independent subset of H. Let E be any subset of $\mathbb{R}^n$ and let L be defined as in (1.1). Then E is Lebesgue measurable in $\mathbb{R}^n$ if and only if $f^{-1}(E)$ is abstract Wiener measurable, where*

$$f(x) = (L(h_1)(x), L(h_2)(x), \cdots, L(h_n)(x)). \tag{2.2}$$

Proof. Let

$$\mathcal{U} = \{E \subset \mathbb{R}^n : f^{-1}(E) \text{ is abstract Wiener measurable}\}.$$

To show that $\mathcal{U} \subset \mathcal{L}(\mathbb{R}^n)$, the class of Lebesgue measurable sets, by Lemma 2.2, it suffice to show that if N is a μ-null set then N is an l-null set, where l is the Lebesgue measure on $\mathbb{R}^n$ and μ is the measure defined by $\mu = \nu \circ f^{-1}$.

Assume that N is μ-null set and is not l-null set. If N is l-measurable, then there exists a Borel set $G \subset N$ such that $l(G) = l(N) > 0$. By above Lemma 1.2, f is an n-dimensional Gaussian random vector on B with mean zero and covariance (v_{ij}), where $v_{ij} = \langle h_i, h_j \rangle$ for $i, j = 1, 2, \cdots, n$. Thus we have

$$\mu(G) = \nu \circ f^{-1}(G) = \nu(f^{-1}(G)) > 0.$$

Since $G \subset N$, we have $\mu(N) \geqslant \mu(G) > 0$, which is a contradiction.

And if N is not l-measurable, that is $N \notin \mathcal{L}(\mathbb{R}^n)$ then $l(G) > 0$ for every Borel set $G \supset N$. Since by Lemma 2.1, $\mu = \nu \circ f^{-1}$ is tight, there exists a compact set $K_n \subset N^c$ such that $\mu(N^c \backslash K_n) < \frac{1}{n}$ for each n.

Let $K = \cup_{n=1}^{\infty} K_n$. Then K^c is a Borel set and $N \subset K^c$,

$$\mu(K^c) = 1 - \mu(K) = \mu(N^c) - \mu(K) \leq \mu(N^c \backslash K) \leq \mu(N^c \backslash K_n) < \frac{1}{n}$$

for each n. Hence $\mu(K^c) = 0$. Since $\mu(G) > 0$ for any Borel set $G \supset N$, which is a contradiction. Thus every μ-null set is also an l-null set.

Conversely, suppose that E is a Lebesgue measurable set in $\mathbb{R}^n$. Then there exist a Borel set G and a subset N_1 of a Borel null set N in $\mathbb{R}^n$ such that $E = G \cup N_1$. Since f is Borel measurable, it follows that $f^{-1}(G)$ and $f^{-1}(N)$ are in $\mathcal{B}(B)$. Since $\mu(N) = \nu \circ f^{-1}(N) = 0$, $F^{-1}(N_1)$ is in $\overline{\mathcal{B}(B)}^{\nu}$ and hence $f^{-1}(E) = f^{-1}(G) \cup f^{-1}(N_1)$ is in $\overline{\mathcal{B}(B)}^{\nu}$. □

Remark. Let

$$H = \{x : [a,b] \to \mathbb{R}^N : \ x(t) = (x^1(t), \cdots, x^N(t)); \int_a^b (Dx^j(s))^2 ds < \infty, \ 1 \leq j \leq N\}$$

where $x^j(t) = \int_a^t Dx^j(s)ds, 1 \leq j \leq N$, and $D = \frac{d}{ds}$. Then H is a Hilbert space with inner product

$$\langle x, \hat{x} \rangle := \sum_{j=1}^{N} \int_a^b Dx^j(s) D\hat{x}^j(s) ds$$

and the norm $|||x||| = \sqrt{\langle x, x \rangle}$.

Let $||x||_1 := \sup_{a \leq t \leq b} (\sum_{j=1}^N (x^j(t))^2)^{\frac{1}{2}}$. Then $\overline{H}^{||\cdot||_1} = C_0([a,b], \mathbb{R}^N) := B$, the separable Banach space of continuous functions from $[a,b]$ into $\mathbb{R}^N$ which is vanish at a (cf. [8]). And it is called the N-dimensional Wiener space.

And it can be shown that $||\cdot||_1$ is a measurable norm (cf. [8]). Thus if $\gamma : H \to B$ denote the natural injection, then (H, B, ν) is an abstract Wiener space, where ν is the corresponding abstract Wiener measure. And it is well-known that the dual B^* of B is given by

$$B^* = \{\theta = (\theta^1, \cdots, \theta^N) : \theta^j \text{ is a finite signed measure on } [a,b], 1 \leq j \leq N\}$$

and the action (x,θ) is given by

$$(x,\theta)=\sum_{j=1}^{N}\int_a^b x^j(u)d\theta^j(u), \quad x=(x^1,\cdots,x^N).$$

Thus we have $\nu=m_w$, where m_w is the standard Wiener measure on $(B,\mathcal{B}(B))$.

And if we let $L(h)(x)=\sum_{j=1}^{\infty}<h,\theta_j>(x,\theta_j)$. Then we have

$$L(h)(x)=\int_a^b Dh\cdot dx \quad \text{a.e. } x \tag{2.3}$$

Let $a=t_0<t_1<\cdots<t_n=b$ be a subdivision of $[a,b]$. Let E be any subset of $\mathbb{R}^n$ and define $J: C_0([a,b],\mathbb{R})\to\mathbb{R}^n$ by

$$J(x)=(x(t_1),x(t_2),\cdots,x(t_n)). \tag{2.4}$$

Then J is continuous on $C_0([a,b],\mathbb{R})$ with respect to the uniform topology.

Chang and Ryu [2] established Theorem 2.4 below, which is called an *converse measurability theorem* for Wiener space. Now, we prove Theorem 2.4 as a corollary of Theorem 2.3.

Theorem 2.4 (Köehler). *Let $\sigma: a=t_0<t_1<\cdots<t_n=b$ be a subdivision of $[a,b]$. Let E be any subset of $\mathbb{R}^n$ and J be defined as in* (2.4). *Then E is Lebesgue measurable if and only if $J^{-1}(E)$ is Wiener measurable.*

Proof. For $N=1$ in the above remark, $(H,C_0([a,b],\mathbb{R}),m_w)$ is an abstract Wiener space. Let $h_j^\sigma(s)=\int_a^s\chi_{[a,t_j]}(u)du$ for $j=1,2,\cdots,n$. Then $\{h_j^\sigma\}$ is clearly a linearly independent subset in H. From (2.2), (2.3) and (2.4) we have,

$$\begin{aligned}J(x)&=(x(t_1),x(t_2),\cdots,x(t_n))\\&=(L(h_1^\sigma)(x),L(h_2^\sigma)(x),\cdots,L(h_n^\sigma)(x))=f(x).\end{aligned}$$

Hence by Theorem 2.3, E is Lebesgue measurable if and only if $J^{-1}(E)$ is Wiener measurable. □

REFERENCES

1. K. S. Chang, *Converse measurability theorems for Yeh-Wiener space*, Pacific J. Math. **97** (1981), 59–63. MR **83g:**60067.
2. K. S. Chang and K. S. Ryu, *A generalized converse measurability theorem*, Proc. Amer. Math. Soc. **104** (1988), 835–839. MR **89e:**28021.
3. D. L. Cohn, *Measure Theory*, Birkhäuser, Boston, Mass., 1980. MR **81k:**28001.
4. G. B. Folland, *Real analysis*, Modern techniques and their applications. Pure and Applied Mathematics, John Wiley and Sons, Inc., New York, 1984. MR **86k:**28001.
5. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1, Univ. California Press, Berkeley, Calif., 1967, pp. 31–42. MR **35**#3027.
6. G. W. Johnson and D. L. Skoug, *Scale-invariant measurability in Wiener space*, Pacific J. Math. **83** (1979), 157–176. MR **81b:**28016.
7. G. Kallianpur and C. Bromley, *Generalized Feynman integrals using analytic continuation in several complex variables*, Stochastic analysis and applications (M. Pinsky, ed.), Marcel-Dekker, New York, 1984, pp. 217–267. MR **86m:**60007.
8. H. H. Kuo, *Gaussian measures in Banach space*, Lecture Notes in Mathematics, No.463, Springer, Berlin, 1975. MR **57**#1628.
9. K. R. Parthasarathy, *Probability measures on metric space*, Academic Press, New York, 1967. MR **37**#2271.
10. D. L. Skoug, *Converses to measurability theorems for Yeh-Wiener space*, Proc. Amer. Math. Soc. **57** (1976), 304–310. MR **54**#10549.
11. I. Yoo, *The analytic Feynman integral over paths in abstract Wiener space*, Commun. Korean Math. Soc. **10** (1995), 93–107. MR **98g:**28015.

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