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A STUDY ON THE BOOLEAN ALGEBRAS IN THE FRAENKEL-MOSTOWSKI TOPOS

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Abstract. The purpose of this paper is to show that in the Fraenkel-Mostowski topos, the category of the Boolean algebras has enough injectives.

1. Introduction

Banaschewski [1] showed that the category of Boolean algebras in the Blass topos has no non-trivial injectives. But Ebrahimi [4] showed that the category of Boolean algebras in the topos *M-Set* has enough injectives, also he showed that the relation between injectivity and internal completeness in the topos *M-Set*. In this paper we show that in the Fraenkel-Mostowski topos the category of Boolean algebras has enough injectives and an injective Boolean algebras object is internally complete. Also we show underlying set functor on $E(\mathcal{F})$ preserves internal complete posets.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results. In original definition of an elementary topos by Lawvere [8] and Tierney [11] it was required by axiom that colimits also exist. But Mikkelsen ([5, p. 84], [10]) discovered that the existence of colimits is implied by the rest of axioms. We use the definition of elementary topos, modified by Mikkelsen.

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Definition 2.1. An *elementary topos* is a category E that satisfies the following conditions;

(T1) E is finitely complete,
(T2) E has exponentiation, and
(T3) E has a subobject classifier.

Definition 2.2. An object Q in a category C is *injective* if the contravariant *hom* functor $hom_C(-, Q) : C^{op} \to Set$ preserves epimorphisms. A category C is said to have *enough injectives* if every object of C admits a monomorphism to an injective object.

Blass [3] suggested a topos, Fraenkel-Mostowski topos, on $E(\mathcal{F})$ in his unpublished manuscript.

Let $(G, \cdot, e)$ be a group and $\mathcal{F}$ be a normal filter of subgroups of G. That is, $\mathcal{F}$ is a nonempty collection of subgroups such that

(1) $G_M, G_N \in \mathcal{F} \Longrightarrow G_M \cap G_N \in \mathcal{F}$
(2) $G_M \subseteq G_N, G_M \in \mathcal{F} \Longrightarrow G_N \in \mathcal{F}$ (so, in particular, $G \in \mathcal{F}$)
(3) $G_M \in \mathcal{F},\ g \in G \Longrightarrow gG_Mg^{-1} \in \mathcal{F}$.

By an *$\mathcal{F}$-set*, we mean a triple (X, G_X, ψ_X) where X is a set, $G_X \in \mathcal{F}$ and $\psi_X : G_X \times X \to X$ is an (left) action of G_X on X. That is, $\psi_X : G_X \times X \to X$ is a function such that $\psi_X(e, x) = x$ for all $x \in X$, $\psi_X(m \cdot n, x) = \psi_X(m, \psi_X(n, x))$ for all $m, n \in G_X$ and for all $x \in X$. When no confusion is likely, we just write X for $\mathcal{F}$-set (X, G_X, ψ_X) and we write (m, x) for $\psi_X(m, x)$.

For each $x \in X$ let $fix(x)$ be the Isotropy group $\{m \in G_X \mid \psi_X(m, x) = x\}$ of x.

We call an $\mathcal{F}$-set (X, G_X, ψ_X) *almost trivial* if $fix(x) \in \mathcal{F}$ for all $x \in X$ and a map $f : X \to Y$ from $\mathcal{F}$-set (X, G_X, ψ_X) to $\mathcal{F}$-set (Y, G_Y, ψ_Y) is *almost equivariant* if, some $G_N \in \mathcal{F}, G_N \subseteq G_X \cap G_Y$,

$$f(\psi_X(n, x)) = \psi_Y(n, f(x)) \text{ for all } n \in G_N \text{ and for all } x \in X.$$

The almost trivial $\mathcal{F}$-sets and almost equivalent maps between them constitute a category, denoted by $E(\mathcal{F})$.

Proposition 2.3. *$E(\mathcal{F})$ is a topos.*

Proof. $(\{*\}, G, \psi_{\{*\}})$, where $\psi_{\{*\}}$ is the trivial action, is terminal object. The pullback of (X, G_X, ψ_X) and (Y, G_Y, ψ_Y) is $(X \times Y, G_X \cap G_Y, \psi_{X\times Y})$ where

$$\psi_{X\times Y}(n, (x, y)) = (\psi_X(n, x), \psi_Y(n, y)).$$

Let $2 = \{0, 1\}$ be adoubleton in Set. Then $\Omega = (2, G, \psi_2)$, where ψ_2 is the trivial action of G fixing both elements, is a subobject classfier. For (X, G_X, ψ_X) and (Y, G_Y, ψ_Y), we define an almost trivial object $(Y^X, G_X \cap G_Y, \psi_{Y^X})$ where $\psi_{Y^X}(t, f)(x) = \psi_Y(t, f(\psi_X(t^{-1}, x)))$. □

Definition 2.4. An internal poset P is internally complete if there is an order preserving map $\bigcup : \Omega^P \to P$ which is internally left adjoint to the morphism $\downarrow seg$.

3. Main Parts

Theorem 3.1. *The category of Boolean algebras in $E(\mathcal{F})$ has enough injectives.*

Proof. For any Boolean algebra object K is $E(\mathcal{F})$, Dedekind-MacNeille completion of K is also a Boolean algebra object in $E(\mathcal{F})$ by [1] and there exists an embedding of K into the Dedekind-MacNeille completion of K in $E(\mathcal{F})$. So we only show that complete Boolean algebra object is injective in $E(\mathcal{F})$.

Given Boolean algebra homomorphism $f : A \to C$ where A is a sub Boolean algebra object of a Boolean algebra object B and C is a complete Boolean algebra object in $E(\mathcal{F})$. By definition of the complete Boolean algebra we can define $h : B \to C$ where $h(b) = \bigvee\{f(a) \mid a \in A, a \leq b\}$. Then, for all $k \in G_A \cap G_C$,

$$\begin{aligned} kh(b) &= k\bigvee\{f(a) \mid a \in A, a \leq b\} \\ &= \bigvee\{kf(a) \mid a \in A, a \leq b\} \\ &= \bigvee\{f(ka) \mid a \in A, a \leq b\} \\ &= \bigvee\{f(t) \mid t \in A, t \leq kb\} = h(kb). \end{aligned}$$

Thus h is almost equivariant. Also

$$\begin{aligned} h(b \bigwedge c) &= \bigvee\{f(a) \mid a \in A, a \leq (b \bigwedge c)\} \\ &= [\bigvee\{f(a) \mid a \in A, a \leq b\}] \bigwedge [\bigvee\{f(a) \mid a \in A, a \leq c\}] \\ &= h(b) \bigwedge h(c). \end{aligned}$$

Thus h preserves meets, similarly h preserves joins.

Let p and q be identities on $\bigvee$ and $\bigwedge$ in B respectively. Then, for all $x \in C$,

$$h(p) \bigvee x = [\bigvee\{f(a) \mid a \in A, a \leq p\}] \bigvee x = f(p) \bigvee x = x.$$

Thus $h(p)$ is an identity on $\bigvee$ in C, similarly $h(q)$ is an identity on $\bigwedge$ in C. Also h preserves the complement and extends f since $h(a) = \bigvee\{f(d) \mid d \in A, d \leq a\} = f(a)$ for all $a \in A$. $\square$

Proposition 3.2. *If A is injective Boolean algebra object in $E(\mathcal{F})$ then it is internally complete.*

Proof. Let $\phi : X \to A$ and $\mu : X \to \Omega^A$ be any two generalized elements. Then, by exponential transpose, for any $\mu : X \to \Omega^A$ there exists a monomorphism $y : Y \to X \times A$ such that

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "y"'] & 1 \arrow[d, "T"] \\
X \times A \arrow[r, "\chi_y"'] \arrow[d, "\mu \times i_A"'] & \Omega \arrow[d, "i_\Omega"] \\
\Omega^A \times A \arrow[r, "ev"] & \Omega
\end{tikzcd}$$

commutes.

For the morphism $\downarrow seg : A \to \Omega^A$ where $\Omega = \{2, G, \psi_2\}$ with 2 is a doubleton set and $\psi_2 : G \times 2 \to 2$ is trivial action, since A is injective we obtain $\bigcup : \Omega^A \to A$ such that $\bigcup \downarrow seg = i_A$.

To prove that $\bigcup$ is internally left adjoint to $\downarrow seg$, let $\mu \leq \downarrow seg\, \phi$. Since $\bigcup$ is order preserving, this implies $\bigcup \mu \leq \phi$. Conversely, let $\bigcup \mu \leq \phi$. Then for any $\downarrow seg\, \phi : X \to \Omega^A$, there is a monomorphism $z : Z \to X \times A$ such that

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "z"'] & 1 \arrow[d, "T"] \\
X \times A \arrow[r, "\chi_z"'] \arrow[d, "\downarrow seg\, \phi \times i_A"'] & \Omega \arrow[d, "i_\Omega"] \\
\Omega^A \times A \arrow[r, "ev"] & \Omega
\end{tikzcd}$$

commutes.

And, for a morphism $(m, \phi m) : M \to X \times A$ where $m : M \to X$ is a monomorphism, we get two pullback squares, that is, the squares

$$\begin{tikzcd} W \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "y"] \\ M \arrow[r, "(m,\phi m)"'] & X \times A \end{tikzcd}$$

and

$$\begin{tikzcd} S \arrow[r, "q_1"] \arrow[d, "q_2"'] & Z \arrow[d, "z"] \\ M \arrow[r, "(m,\phi m)"'] & X \times A \end{tikzcd}$$

commute.

Since upper square is pullback and $\bigcup \downarrow seg = i_A$, $\bigcup \mu \leq \phi$ implies $W \leq M$. Since lower square is pullback, this implies $W \leq S$. By above two pullback squares, this implies $Y \leq Z$. By exponential transpose, this implies $\mu \leq \downarrow seg\ \phi$. Therefore we have the desired internal adjunction. □

Proposition 3.3. *Underlying set functor $W : E(\mathcal{F}) \to Set$ preserves internally complete posets.*

Proof. Let A be a internally complete poset in $E(\mathcal{F})$ and $\rho : B \to \Omega^{WA}$ be a generalized element in Set. Then, by exponential transpose, there is a unique morphism $(a, b) : X \to B \times WA$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "(a,b)"'] & 1 \arrow[d, "T"] \\ B \times WA \arrow[r, "\chi_{(a,b)}"'] \arrow[d, "\rho \times i_{WA}"'] & \Omega \arrow[d, "i_\Omega"] \\ \Omega^{WA} \times WA \arrow[r, "ev"] & \Omega \end{tikzcd}$$

commutes.

Since W has a left adjoint V, there is a morphism $(Va, c) : VX \to VB \times A$ where c is the transpose of b. And the morphism $(Va, c) : VX \to VB \times A$ has an epi-mono factorization by the properties of functor and factorization. That is, the diagram

$$\begin{array}{ccc} VX & \xrightarrow{(Va,c)} & VB \times A \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle m} \\ I & \xrightarrow[i_I]{} & I \end{array}$$

commutes.

And, for above monomorphism $m : I \to VB \times A$ there is a unique morphism $\eta : VB \to \Omega^A$ such that the diagram

$$\begin{array}{ccc} I & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle T} \\ VB \times A & \xrightarrow[\chi_m]{} & \Omega \\ {\scriptstyle \eta\times i_A}\downarrow & & \downarrow{\scriptstyle i_\Omega} \\ \Omega^A \times A & \xrightarrow{ev} & \Omega \end{array}$$

commutes.

Hence, by hyphothesis, we get $\bigcup'\eta : VB \to A$ where $\bigcup' : \Omega^A \to A$ is the internally left adjoint to $\downarrow seg' : A \to \Omega^A$ in $E(\mathcal{F})$. Also we obtain a morphism $\bigcup : \Omega^{WA} \to WA$ because $\bigcup'\eta$ is the transpose of $B \to WA$. To show that $\bigcup$ is internally left adjoint to $\downarrow seg$, let $\psi : B \to WA$ be a generalized element in Set. Then for $\downarrow seg\psi : B \to \Omega^{WA}$, where $\downarrow seg : WA \to \Omega^{WA}$, there is a unique morphism $n : Y \to B \times WA$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle T} \\ B \times WA & \xrightarrow{\chi_n} & \Omega \\ {\scriptstyle \downarrow seg\psi\times i_{WA}}\downarrow & & \downarrow{\scriptstyle i_\Omega} \\ \Omega^{WA} \times WA & \xrightarrow{ev} & \Omega \end{array}$$

commutes.

Also for a morphism $\downarrow seg'\xi : VB \to \Omega^A$, where $\downarrow seg'$ and ξ are the transpose of $\downarrow seg'$ and ψ, respectively, there exists a unique monomorphism $s : J \to VB \times A$

such that the diagram

$$\begin{tikzcd}
J \arrow[r] \arrow[d, "s"'] & 1 \arrow[d, "T"] \\
VB \times A \arrow[r, "\chi_s"] \arrow[d, "\downarrow seg'\xi \times i_A"'] & \Omega \arrow[d, "i_\Omega"] \\
\Omega^A \times A \arrow[r, "ev"] & \Omega
\end{tikzcd}$$

commutes.

Hence we get $\rho \leq \downarrow seg\,\psi$ if and only if $X \leq Y$. Since a morphism $(Va, c) : VX \to VB \times A$ factors through a monomorphism $s : J \to VB \times A$, we get $X \leq Y$ if and only if $I \leq J$. Also, by previous diagrams, we have $I \leq J$ if and only if $\eta \leq \downarrow seg'\xi$. Since A is internally complete in $E(\mathcal{F})$, $\eta \leq \downarrow seg'\xi$ if and only if $\bigcup' \eta \leq \xi$. By transposition, we have $\bigcup' \eta \leq \xi$ if and only $\bigcup \rho \leq \psi$. Therefore $W(A)$ is internally complete. $\square$

References

1. B. Banaschewski, *Injectivity and Modelling in the Blass Topos*, J. Pure Appl. Algebra **49** (1987), 1–10. MR **89a:**18006.
2. M. Barr and C. Well, *Toposes, Triples and Theories*, Grundlehren der Mathematischen Wissenschaften 278, Springer-Verlag, New York, 1985. MR **86f:**18001.
3. A. Blass, *Fraenkel-Mostowski Topoi*, Unpublished Manuscript, 1978.
4. M. M. Ebrahimi, *Internal Completeness and Injectivity of Boolean Algebras in the Topos of M-Sets*, Bull. Austral. Math. Soc. **41** (1990), 323–332. MR **91b:**03102.
5. R. Goldblatt, *Topoi*, The categorical analysis of logic. Studies in Logic and the Foundations of Mathematics 98, North-Holland, Amsterdam, 1979. MR **81a:**03063.
6. H. Herrlich and G. E. Strecker, *Category Theory: An Introduction*, Allyn and Bacon Series in Advanced Mathematics, Allyn and Bacon Inc., Boston, Mass., 1973. MR **50**#2284.
7. P. T. Johnstone, *Topos Theory*, London Mathematical Society Monographs, Vol. 10, Academic Press, New York, 1977. MR **57**#9791.
8. F. W. Lawvere, *Introduction*, Toposes, algebraic geometry and logic, Partial Report on a Conference on Connections between Category Theory and Algebraic Geometry & Intuitionistic Logic, Dalhousie University, Halifax, Nova Scotia, January 16–19, 1971. Edited by F. W. Lawvere. Lecture Notes in Mathematics, Vol. 274, Springer-Verlag, Berlin-New York, 1972, pp. 1–12. MR **48**#8592.
9. S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York, 1971. MR **50**#7275.
10. R. Paré, *Colimits in topoi*, Bull. Amer. Math. Soc. **80** (1974), 556–561. MR **48**#11245.
11. M. Tierney, *Sheaf theory and the continuum hypothesis*, Toposes, algebraic geometry and logic, Partial Report on a Conference on Connections between Category Theory and Algebraic Geometry & Intuitionistic Logic, Dalhousie University, Halifax, Nova Scotia, January 16–19,

1971. Edited by F. W. Lawvere. Lecture Notes in Mathematics, Vol. 274, Springer-Verlag, Berlin-New York, 1972, pp. 13–42. MR **51**#10088.

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