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CHARACTERIZATION OF *CR* SUBMANIFOLD IN A COMPLEX PROJECTIVE SPACE IN TERMS OF RICCI TENSORS

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ABSTRACT. Let M be an n-dimensional *CR* submanifold of *CR* dimension $n-1$ of a complex projective space M. We charaterize M of $\overline{M}$ in terms of an estimations of the length of the derivative of Ricci tensor or of the length of Ricci tensor.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a complex manifold $\overline{M}$ with complex structure J. If the maximal J-invariant subspace $JT_x(M) \cap T_x(M)$ of $T_x(M)$ has constant dimension for any $x \in M$, then M is calld a *CR submanifold* and the constant is called the *CR dimension* of M [2, 10]. Now let M be a *CR* submanifold of *CR* dimension $n-1$ of $\overline{M}$. Then M admits an induced almost contact structure (cf. [11, 13]). A typical example of *CR* submanifold of *CR* dimension $n-1$ is a real hypersurface. Hereby we may expect to generalize some results which are valid in real hypersurface to *CR* submanifold of *CR* dimension $n-1$. When the *ambient* manifold $\overline{M}$ is a complex projective space, real hypersurfaces are investigated by many authors (cf. [1, 4, 5, 6, 7, 8, 9, 12]).

On the other hand, Kimura and Maeda provided some characterizations of geodesic hyperspheres in complex projective space in terms of Ricci tensor S. They obtained an estimate of $\|\nabla S\|$ which characterized geodesic hyperspheres in complex projective space. We here recall their work.

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Theorem A [4]. *Let M be a real hypersurface with constant mean curvature in $P^{\frac{n+1}{2}}(C), n \geq 5$. Then*

$$|\nabla S|^2 \geq \frac{4(n+1)}{n-1}(\operatorname{tr} A_1 - u^1(A_1U_1)) \times \left\{\frac{n+1}{2}(\operatorname{tr} A_1 - u^1(A_1U_1)) - \operatorname{tr}(FA_1\nabla_{U_1}A_1)\right\}. \tag{1}$$

Moreover, the equality of (1) *holds if and only if M is locally congruent to a geodesic hypersphere of $P^{\frac{n+1}{2}}(C)$ provided that $u^1(AU_1)$ is constant.*

Here we review the work of Cecil and Ryan [1], and Kon [8]. They defined *pseudo-Einstein* real hypersurface M in $P^{\frac{n+1}{2}}(C)$, that is,

$$SX = aX + bg(X, J\xi_1)J\xi_1 \tag{2}$$

for some smooth functions a and b on M. The theorem is as follows:

Theorem B [1, 4]. *Let M be a connected real hypersurface $P^{\frac{n+1}{2}}(C), n \geq 5$, which Ricci tensor S satisfies the above equation* (2). *Then M is locally congruent to one of the following:*

(i) *a geodesic hypersphere,*
(ii) *a tube of radius r over a totally geodesic $P^k(C)$, $0 < k < \frac{n-1}{2}$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = k/((n-1)/2 - k)$,*
(iii) *a tube of radius r over a complex quadric $Q^{(n-1)/2}$, where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = \frac{n-3}{2}$.*

The purpose of the present paper is to study some characterizations of *CR* submanifold in $P^{\frac{n+p}{2}}(C)$ in terms of an estimate of $\|\nabla S\|$, that is, the length of the derivative of the Ricci tensor (cf. Theorem 1) and in terms of an estimate of $\|S\|$, the length of the Ricci tensor (cf. Theorem 2).

2. Preliminaries

Let $(\overline{M}, J, \overline{g})$ be an $(n+p)$-dimensional almost Hermitian manifold and let M be a connected n-dimensional submanifold of $\overline{M}$ with induced metric g. For $x \in M$ we denote by $T_x(M)$ and $T_x^{\perp}(M)$ the tangent space and normal space of M at x, respectively. Next, we assume that

$$\dim(JT_x(M) \cap T_x(M)) = n-1,$$

that is, M is *CR* submanifold of *CR* dimension $n-1$. This implies real dimension of M is odd [2, 11].

We note that the definition of *CR* submanifold of *CR* dimension $n-1$ meets the definition of *CR* submanifold in the sense of Bejancu [14].

Furthermore, our hypothesis implies that there exists a unit vector field ξ_1 normal to M such that $JT(M) \subset T(M) \oplus \operatorname{span}\{\xi_1\}$. Hence, for any tangent vector field X and for a local orthonormal basis $\{\xi_\beta;\ \beta = 1, \cdots, p\}$ of normal vectors to M, we have the following decomposition in tangential and normal components:

$$JX = FX + u^1(X)\xi_1 \text{ and } J\xi_\beta = -U_\beta + P\xi_\beta,\ \beta = 1, \cdots, p. \tag{3}$$

Then it is easily seen that F and P are skew-symmetric endomorphisms acting on $T_x(M)$ and $T_x^\perp(M)$, respectively. Moreover, the Hermitian property of J implies

$$g(FU_\beta, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\beta), \tag{4}$$

$$g(U_\beta, U_\gamma) = \delta_{\beta\gamma} - \overline{g}(P\xi_\beta, P\xi_\gamma). \tag{5}$$

From $\overline{g}(JX, \xi_\beta) = -\overline{g}(X, J\xi_\beta)$, we get

$$g(X, U_\beta) = u^1(X)\delta_{1\beta},$$

and hence

$$g(U_1, X) = u^1(X) \text{ and } U_\beta = 0; \quad \beta = 2, \cdots, p.$$

Next, applying J to (3) and using (4), the first equation of (3) yields

$$F^2X = -X + u^1(X)U_1,\ u^1(X)P\xi_1 = -u^1(FX)\xi_1. \tag{6}$$

Since P is skew-symmetric, the second equation of (6) gives

$$u^1(FX) = 0,\ P\xi_1 = 0,\ FU_1 = 0. \tag{7}$$

So, the second equation of (3) may be written in the form

$$J\xi_1 = -U_1 \text{ and } J\xi_\beta = P\xi_\beta; \quad \beta = 2, \cdots, p \tag{8}$$

and further, we may put

$$P\xi_\beta = \sum_{\gamma=2}^{p} P_{\beta\gamma}\xi_\gamma,\ \ \beta = 2, \cdots, p$$

where $(P_{\beta\gamma})$ is a skew-symmetric matrix which satisfies

$$\sum_{\gamma} P_{\beta\gamma}P_{\gamma\mu} = -\delta_{\beta\mu}.$$

These results imply that (F, U_1, u^1, g) defines an almost contact metric structure on (M, g) [13].

Now, let $\overline{\nabla}$ and ∇ denote the Levi Civita connection on $\overline{M}$ and M, respectively and denote by D the normal connection induced from $\overline{\nabla}$ in the normal bundle $T^{\perp}(M)$ of M. The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \text{ and } \overline{\nabla}_X \xi_\beta = -A_\beta X + D_X \xi_\beta, \quad \beta = 1, \cdots, p$$

for any tangent vectors X, Y to M. Here h denotes the second fundamental form and A_β is the shape operator corresponding to ξ_β. They are related by

$$h(X,Y) = \sum_{\beta=1}^{p} g(A_\beta X, Y)\xi_\beta.$$

Furthermore, putting

$$D_X \xi_\beta = \sum_{\gamma=1}^{p} s_{\beta\gamma}(X)\xi_\gamma,$$

it is easy to show that $(s_{\beta\gamma})$ is the skew-symmetric matrix of connection forms of D.

Finally, if the ambient space $\overline{M}$ is a Kaehler manifold of constant holomorphic sectional curvature 4, the Gauss, Codazzi, Ricci equations, Ricci tensor and the scalar curvature are respectively given by

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX - g(FX,Z)FY \\ &\quad - 2g(FX,Y)FZ + \sum g(A_\beta Y, Z)A_\beta X - \sum g(A_\beta X, Z)A_\beta Y, \end{aligned}$$

$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1,$$

$$\overline{g}(R^{\perp}(X,Y)\xi_\beta, \xi_1) = g([A_1, A_\beta]X, Y) \text{ for } \beta = 2, \cdots, p, \tag{9}$$

$$\begin{aligned} &S(X,Y) \\ &= (n+2)g(X,Y) - 3u^1(X)u^1(Y) + \sum(\operatorname{tr} A_\beta)g(A_\beta Y, X) - \sum g(A_\beta{}^2 Y, X), \end{aligned}$$

and

$$\rho = (n+3)(n-1) + \sum (\operatorname{tr} A_\beta)^2 - \sum \operatorname{tr} A_\beta{}^2, \tag{10}$$

for any tangent vector fields X, Y, Z to M [2, 3, 11]. Here R denotes the Rimannian curvature tensor of M and $R^\perp$ is the curvature tensor of the normal connection D.

3. Submanifolds of $P^{\frac{n+p}{2}}(C)$ in terms of ∇S

In this section we consider the case of a complex projective space $\overline{M} = P^{\frac{n+p}{2}}(C)$ of constant holomorphic sectional curvature 4. Then by differentiating (3) and (4) covariantly, using $\overline{\nabla} J = 0$, and by comparing the tangential and normal parts, we obtain

$$(\nabla_Y F)X = u^1(X)A_1Y - g(A_1X, Y)U_1, \tag{11}$$

$$(\nabla_Y u^1)(X) = g(FA_1Y, X), \tag{12}$$

$$\nabla_X U_1 = FA_1X \tag{13}$$

and

$$g(A_\beta U_1, X) = -\sum_{\gamma=2}^{p} s_{1\gamma}(X)P_{\gamma\beta}; \quad \beta = 2, \cdots, p \tag{14}$$

for any tangent vectors X, Y to M.

On the other hand, the almost contact metric structure (F, U_1, u^1, g) is said to be *normal* if the tensor field N defined by

$$N(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] + 2du^1(X,Y)U_1 \tag{15}$$

vanishes identically [11, 14]. By using (7), (8), (11), (12) and (15), we can easily prove the following lemma.

Lemma A [3, 11]. *Let M be an n-dimensional CR submanifold of CR dimension $n-1$ in a complex space form. If the normal vector field ξ_1 is parallel with respect to the normal connection, then (F, U_1, u^1, g) is normal if and only if A_1 and F commute.*

From the proof of Lemma A it follows that $A_1U_1 \in \ker F$ and hence we have

Lemma B [11]. *Under the hypothesis of Lemma A, U_1 is an eigenvector of A_1 for any $x \in M$. Therefore, we put*

$$A_1 U_1 = \alpha U_1.$$

In what follows we suppose that M is an n-dimensional submanifolds of $P^{\frac{n+p}{2}}(C)$ with parallel normal vector field ξ_1 with respect to the normal connections, that is, $D_X \xi_1 = 0$. Consequently, we get

$$s_{1\gamma} = 0, \quad \gamma = 2, \cdots, p$$

and hence, from (14), we have

$$A_\beta U_1 = 0, \quad \beta = 2, \cdots, p. \tag{16}$$

Theorem C [4]. *Let M be a real hypersurface of $P^{\frac{n+1}{2}}(C)$. Then M is locally congruent to a geodesic hypersphere in $p^{\frac{n+1}{2}}(C)$ if and only if the Ricci tensor S of M satisfies*

$$(\nabla_X S)Y = c(g(FX, Y)U_1 + u^1(Y)FX) \quad \text{for any } X, Y \in T(M),$$

where c is a non-zero constant.

Lemma C [6, 7, 9]. *If ξ_1 is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Thus we have the main theorem:

Theorem 1. *Let M be a CR submanifold of $P^{\frac{n+p}{2}}(C)$, $n \geq 5$ with constant $h_\beta =$ tr $A_\beta; \beta = 1, \cdots, p$. If U_1 is principal of A_1 and ξ_1 is parallel normal vector field with respect to the normal connection. Then the following inequality holds:*

$$\begin{aligned} \|\nabla S\|^2 \geq\ & 30 \operatorname{tr}(A_1 F)^2 + \sum h_\beta{}^2 \operatorname{tr}(\nabla_i A_\beta)^2 - 4 \sum h_\beta \operatorname{tr}(A_\beta (\nabla_i A_\beta)^2) \\ & + 2 \sum \operatorname{tr}(A_\beta{}^2 (\nabla_i A_\beta)^2) + 2 \sum \operatorname{tr}(A_\beta (\nabla_i A_\beta))^2 \\ & - 12[(h_1 - \alpha)\{\operatorname{tr}(A_1 F^2) - \operatorname{tr}(A_1 F \nabla_{U_1} A_1)\} + \sum_{\beta=2}^{p} h_\beta \operatorname{tr}(F^2 A_1{}^2 A_\beta) \\ & - \sum_{\beta=2}^{p} \operatorname{tr}(F^2 A_1{}^2 A_\beta{}^2) - \operatorname{tr}(A_1 F A_1 (\nabla_{U_1} A_1))]. \end{aligned} \tag{17}$$

In the case $p = 1$, the equality in (17) *holds if and only if M is locally congruent to a geodesic hypershere of $P^{\frac{n+1}{2}}(C)$.*

Proof. Firstly let us suppose that h_β is constant for any $\beta = 1, \cdots, p$. Throughout this paper, we regard that any X and Y belong to $T(M)$. From (10) we have

$$SX = (n+2)X - 3u^1(X)U_1 + \sum(\operatorname{tr} A_\beta)A_\beta X - \sum {A_\beta}^2 X. \tag{18}$$

Differentiating (18) covariantly, we obtain

$$\begin{aligned}(\nabla_X S)Y = -3g(Y, \nabla_X U_1)U_1 - 3u^1(Y)\nabla_X U_1 + \sum h_\beta(\nabla_X A_\beta)Y \\ - \sum(\nabla_X A_\beta)A_\beta Y - \sum A_\beta(\nabla_X A_\beta)Y.\end{aligned} \tag{19}$$

Using (13), we get

$$\begin{aligned}(\nabla_X S)Y = -3g(Y, FA_1X)U_1 - 3u^1(Y)FA_1X + \sum h_\beta(\nabla_X A_\beta)Y \\ - \sum(\nabla_X A_\beta)A_\beta Y - \sum A_\beta(\nabla_X A_\beta)Y.\end{aligned} \tag{20}$$

Putting $X = e_i$ and $Y = U_1$ in (20), we have

$$\begin{aligned}(\nabla_i S)U_1 = -3g(U_1, FA_1e_i)U_1 - 3u^1(U_1)FA_1e_i + \sum h_\beta(\nabla_i A_\beta)U_1 \\ - \sum(\nabla_i A_\beta)A_\beta U_1 - \sum A_\beta(\nabla_i A_\beta)U_1.\end{aligned}$$

From which, using (8) we have

$$\begin{aligned}(\nabla_i S)U_1 = -3FA_1e_i + \sum h_\beta(\nabla_i A_\beta)U_1 \\ - \sum(\nabla_i A_\beta)A_\beta U_1 - \sum A_\beta(\nabla_i A_\beta)U_1.\end{aligned}$$

Let $e_1, \cdots, e_n$ be local fields of orthonormal vectors on M. Making use of (20), we define the following tensor T on M by

$$\begin{aligned}T(X,Y) = (\nabla_X S)Y + 3g(Y, FA_1X)U_1 + 3u^1(Y)FA_1X - \sum h_\beta(\nabla_X A_\beta)Y \\ + \sum(\nabla_X A_\beta)A_\beta Y + \sum A_\beta(\nabla_X A_\beta)Y.\end{aligned}$$

Now we have then, by a straightforward computation

$$\begin{aligned}\|T\|^2 &= \sum_{i,j} g(T(e_i, e_j),\, T(e_i, e_j)) \\ &= \sum g((\nabla_i S)e_j,\, (\nabla_i S)e_j) + 9\sum g^2(FA_1e_i,\, e_j) \\ &\quad + 9\sum (u^1(e_j))^2 g(FA_1e_i,\, FA_1e_i) + \sum {h_\beta}^2 g((\nabla_i A_\beta)e_j,\, (\nabla_i A_\beta)e_j) \\ &\quad + \sum g((\nabla_i A_\beta)A_\beta e_j,\, (\nabla_i A_\beta)A_\beta e_j) + \sum g(A_\beta(\nabla_i A_\beta)e_j,\, A_\beta(\nabla_i A_\beta)e_j) \\ &\quad + 6\sum g((\nabla_i S)e_j,\, U_1)g(FA_1e_i,\, e_j) + 6\sum u^1(e_j)g((\nabla_i S)e_j,\, FA_1e_i)\end{aligned}$$

$$
\begin{aligned}
&-2\sum h_\beta g((\nabla_i S)e_j,\ (\nabla_i A_\beta)e_j)+2\sum g((\nabla_i S)e_j,\ (\nabla_i A_\beta)A_\beta e_j)\\
&+2\sum g((\nabla_i S)e_j, A_\beta(\nabla_i A_\beta)e_j)+18\sum u^1(e_j)g(FA_1e_i,e_j)g(U_1,FA_1e_i)\\
&-6\sum h_\beta g(FA_1e_i,e_j)g((\nabla_i A_\beta)e_j,U_1)+6\sum g(FA_1e_i,e_j)g(U_1,(\nabla_i A_\beta)A_\beta e_j)\\
&+6\sum g(FA_1e_i,e_j)g(U_1,A_\beta(\nabla_i A_\beta)e_j)+6\sum u^1(e_j)g(FA_1e_i,(\nabla_i A_\beta)A_\beta e_j)\\
&-6\sum h_\beta u^1(e_j)g(FA_1e_i,(\nabla_i A_\beta)e_j)+6\sum u^1(e_j)g(FA_1e_i,A_\beta(\nabla_i A_\beta)e_j)\\
&-2\sum h_\beta g((\nabla_i A_\beta)e_j,(\nabla_i A_\beta)A_\beta e_j)+2\sum g((\nabla_i A_\beta)A_\beta e_j,A_\beta(\nabla_i A_\beta)e_j)\\
&-2\sum h_\beta g((\nabla_i A_\beta)e_j,A_\beta(\nabla_i A_\beta)e_j). \qquad (21)
\end{aligned}
$$

From (8) and the Codazzi equation (9), we get, for each i,

$$(\nabla_i A_1)U_1=(\nabla_{U_1}A_1)e_i-Fe_i. \qquad (22)$$

Also, we have from (13) and (16)

$$(\nabla_i A_\beta)U_1=-A_\beta FA_1e_i,\quad \beta=2,\cdots,p. \qquad (23)$$

Then we have, by using (21), (22) and (23),

$$
\begin{aligned}
\|T\|^2=\ &\|\nabla S\|^2-30\mathrm{t}r(A_1F)^2+12(h_1-\alpha)\,\mathrm{tr}\,(A_1F^2)\\
&-\sum {h_\beta}^2\,\mathrm{tr}\,(\nabla_i A_\beta)^2)+4\sum h_\beta\,\mathrm{tr}\,(A_\beta(\nabla_i A_\beta)^2)\\
&-2\sum \mathrm{tr}\,({A_\beta}^2(\nabla_i A_\beta)^2)-2\sum \mathrm{tr}\,(A_\beta(\nabla_i A_\beta))^2\\
&+12\sum_{\beta=2}^{p} h_\beta\,\mathrm{tr}\,({A_1}^2FA_\beta F)-12\sum_{\beta=2}^{p}\mathrm{tr}\,({A_1}^2F{A_\beta}^2F)\\
&-12\,\mathrm{tr}\,(A_1FA_1(\nabla_{U_1}A_1))-12(h_1-\alpha)\,\mathrm{tr}\,(A_1F\nabla_{U_1}A_1).
\end{aligned}
$$

Since $\|T\|^2\geq 0$, we have

$$
\begin{aligned}
\|\nabla S\|^2\geq\ &30\,\mathrm{tr}\,(A_1F)^2-12(h_1-\alpha)\,\mathrm{tr}\,(A_1F^2)+\sum {h_\beta}^2\,\mathrm{tr}\,(\nabla_i A_\beta)^2\\
&-4\sum h_\beta\,\mathrm{tr}\,(A_\beta(\nabla_i A_\beta)^2)+2\sum \mathrm{tr}\,({A_\beta}^2(\nabla_i A_\beta)^2)\\
&+2\sum \mathrm{tr}\,(A_\beta(\nabla_i A_\beta))^2-12\sum_{\beta=2}^{p} h_\beta({A_1}^2FA_\beta F)\\
&+12\sum_{\beta=2}^{p}\mathrm{tr}\,({A_1}^2F{A_\beta}^2F)+12\,\mathrm{tr}\,(A_1FA_1(\nabla_{U_1}A_1))\\
&+12(h_1-\alpha)\,\mathrm{tr}\,(A_1F\nabla_{U_1}A_1). \qquad (24)
\end{aligned}
$$

Furthermore, by Lemma C, (24) can be rewritten as

$$\begin{aligned}\|\nabla S\|^2 \geq\ & 30\,\mathrm{tr}\,(A_1F)^2 - 12(h_1-\alpha)\,\mathrm{tr}\,A_1F^2 + \sum {h_\beta}^2\,\mathrm{tr}\,(\nabla_i A_\beta)^2 \\ & - 4\sum h_\beta\,\mathrm{tr}\,(A_\beta(\nabla_i A_\beta)^2) + 2\sum \mathrm{tr}\,({A_\beta}^2(\nabla_i A_\beta)^2) \\ & + 2\sum \mathrm{tr}\,(A_\beta(\nabla_i A_\beta))^2 - 12\sum_{\beta=2}^{p} h_\beta\,\mathrm{tr}\,(F^2{A_1}^2A_\beta) \\ & + 12\sum_{\beta=2}^{p}\mathrm{tr}\,(F^2{A_1}^2{A_\beta}^2) + 12\,\mathrm{tr}\,(A_1FA_1(\nabla_{U_1}A_1)) \\ & + 12(h_1-\alpha)\,\mathrm{tr}\,(A_1F\nabla_{U_1}A_1) \\ =\ & 30\,\mathrm{tr}\,(A_1F)^2 + \sum {h_\beta}^2\,\mathrm{tr}\,(\nabla_i A_\beta)^2 - 4\sum h_\beta\,\mathrm{tr}\,(A_\beta(\nabla_i A_\beta)^2) \\ & + 2\sum \mathrm{tr}\,({A_\beta}^2(\nabla_i A_\beta)^2) + 2\sum \mathrm{tr}\,(A_\beta(\nabla_i A_\beta))^2 \\ & - 12\{\sum_{\beta=2}^{p} h_\beta\,\mathrm{tr}\,(F^2{A_1}^2A_\beta) - \sum_{\beta=2}^{p}\mathrm{tr}\,(F^2{A_1}^2{A_\beta}^2) \\ & - \mathrm{tr}\,(A_1FA_1(\nabla_{U_1}A_1)) + (h_1-\alpha)\,\mathrm{tr}\,(A_1F^2) \\ & - (h_1-\alpha)\,\mathrm{tr}\,(A_1F\nabla_{U_1}A_1)\}. \end{aligned} \tag{25}$$

Therefore, the required inequality (17) follows from (25). The equality of (17) is given by (19). Hence, in the special case $p=1$, Theorem *C* shows that the equality of (17) holds if and only if M is locally congruent to a geodesic hypersphere. □

From Theorem 1 we have:

Corollary 1. *Let M be a submanifold satisfying the assumption of Theorem* 1. *If M has the normal almost contact metric structure (F, U_1, u^1, g). Then the following inequality holds:*

$$\begin{aligned}\|\nabla S\|^2 \geq & \sum {h_\beta}^2\,\mathrm{tr}\,(\nabla_i A_\beta)^2 - 4\sum h_\beta\,\mathrm{tr}\,(A_\beta(\nabla_i A_\beta)^2) \\ & + 2\sum \mathrm{tr}\,({A_\beta}^2(\nabla_i A_\beta)^2) + 2\sum \mathrm{tr}\,(A_\beta(\nabla_i A_\beta))^2, \end{aligned} \tag{26}$$

where $h_\beta = \mathrm{tr}\,A_\beta$ for $\beta = 1,\cdots,p$.

Proof. Suppose M has the normal almost contact metric structure (F, U_1, u^1, g). Using Lemma A and Lemma C, we get $A_1F = 0$. Hence, from Theorem 1 we have the required result (26). □

4. Pseudo-Einstein submanifold in $P^{\frac{n+p}{2}}(C)$

Here we shall prove the following theorem:

Theorem 2. *Let M be a CR submanifold of $P^{\frac{n+p}{2}}(C)$, $n \geq 5$. Then the following holds:*

$$\|S\|^2 \geq (u^1(SU_1))^2 + \frac{1}{n-1}(\rho - u^1(SU_1))^2, \tag{27}$$

where ρ is the scalar curvature of M. The equality of (27) holds if and only if M is of pseudo-Einstein.

Proof. We first remark that the following are equivalent:

(A) $SX = aX + bu^1(X)U_1X$ for any $X \in T(M)$,

(B) $g(SX,Y) = \lambda g(X,Y)$ for any $X, Y \perp U_1$ and U_1 is an eigenvector of S.

We here rewrite the condition

$$g(SX,Y) = \lambda g(X,Y) \text{ for any } X, Y \perp U_1$$

as the following propositions:

(I) $g(SX,Y) = \lambda g(X,Y)$ for any $X, Y \perp U_1$, or $g(SX,Y) = \rho_0 g(X,Y)$ for any $X, Y \perp U_1$, where $\rho_0 = \frac{1}{n-1}(\rho - g(SU_1, U_1))$.

(II) $g(SX - u^1(X)SU_1, Y - u^1(Y)U_1) = \rho_0 g(X - u^1(X)U_1, Y - u^1(Y)u_1)$ for any $X, Y \in T(M)$.

(III) $SX - \rho_0 X - u^1(X)SU_1 - u^1(SX)U_1 + (u^1(SU_1)\rho_0)u^1(X)U_1 = 0$ for any $X, Y \in T(M)$.

Now we define the tensor T for any $X, Y \in T(M)$ as follows:

$$\begin{aligned} T(X,Y) =& g(SX,Y) - \rho_0 g(X,Y) - u^1(X)g(SU_1,Y) \\ &- u^1(SX)g(U_1,Y) + (u^1(SU_1) + \rho_0)u^1(SU_1)g(U_1,Y). \end{aligned}$$

Calculating the length of T, we find

$$\begin{aligned} \|T\|^2 =& \|S\|^2 - 2g(SU_1, SU_1) + 2\rho_0 u^1(SU_1) + (n-1)\rho_0^2 - 2\rho_0\rho + (u^1(SU_1))^2 \\ =& \|S\|^2 - \frac{1}{n-1}(\rho - u^1(SU_1))^2 - 2\|SU_1\|^2 + (u^1(SU_1))^2. \end{aligned}$$

Since $\|T\|^2 \geq 0$, we have

$$\|S\|^2 \geq \frac{1}{n-1}(\rho - u^1(SU_1))^2 + 2\|SU_1\|^2 - (u^1(SU_1))^2. \tag{28}$$

Now we calculate $\|SU_1\|^2$.

$$\begin{aligned}\|SU_1\|^2 &= g(SU_1, SU_1) = g(\sum g(SU_1, e_i)e_i, SU_1)\\ &= \sum g^2(SU_1, e_i)\\ &= \sum_{i=1}^{n-1} g^2(SU_1, e_i) + g^2(SU_1, U_1)\\ &= \sum_{i=1}^{n-1} g^2(SU_1, e_i) + (u^1(SU_1))^2.\end{aligned}$$

Thus we get

$$\|SU_1\|^2 \geq (u^1(SU_1))^2. \tag{29}$$

Hence from (28) and (29) the required inequality (27) follows. Now the equality of (27) holds if and only if M is of pseudo-Einstein. □

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