J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 7, Number 2 (November 2000), Pages 137–143

THE CONVERGENCE THEOREMS FOR THE McSHANE-STIELTJES INTEGRAL

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ABSTRACT. In this paper, we define the uniformly sequence for the vector valued McSane-Stieltjes integrable functions and prove the dominated convergence theorem for the McShane-Stieltjes integrable functions .

1. Introduction and Preliminaries

It is well known that the Riemann-Stieltjes integral is not adequate for advanced mathematics, since there are many functions that are not Riemann-Stieltjes integrable, and since the integral does not possess sufficiently strong convergence theorems. In the late 1960's, McShane [8] proved that the Lebesque integral is indeed equivalent to a modified version of the Henstock integral (cf. Henstock [5]). Yoon, Eun and Lee [9] defined the McShane-Stieltjes integral for real-valued function which is the generalization of the McShane integral and investigated some properties of this integral. Gordon [3] generalized the definition of the McShane integral for real-valued functions to functions taking values in Banach spaces and investigated some of its properties. Many authors have studied McShane integral (cf. [3], [4]).

In this paper, we define the uniformly sequence for the Banach-valued McShane-Stieltjes integrable functions and prove the dominated convergence theorem for the McShane-Stieltjes integrable functions. Throughout this paper, X is a Banach space and we always assume that α is an increasing function on $[a, b]$ unless otherwise stated. We begin with some definitions.

Definition 1.1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A *free tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [a, b]$.

Received by the editors October 17, 2000.
2000 *Mathematics Subject Classification.* Primary 28B05.
Key words and phrases. McShane-Stieltjes integrable.

The free tagged interval $(x, [c, d])$ is *subordinate to* δ if

$$[c, d] \subseteq (x - \delta(x), x + \delta(x)).$$

The letter $\mathcal{P}$ will be used to denote finite collections of non-overlapping free tagged intervals. Let $\mathcal{P} = \{(x_i, [c_i, d_i] : 1 \leq i \leq n\}$ be such a collection in $[a, b]$. We adopt the following terminology.

(1) The points $\{x_i : 1 \leq i \leq n\}$ are the tags of $\mathcal{P}$ and the intervals $\{[c_i, d_i] : 1 \leq i \leq n\}$ are the intervals of $\mathcal{P}$.
(2) If $(x_i, [c_i, d_i])$ is subordinate to δ for each i, then $\mathcal{P}$ is subordinate to δ.
(3) If $\mathcal{P}$ is subordinate to δ and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to δ.

Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping free tagged intervals in $[a, b]$, let $f : [a, b] \to \mathbb{R}$, and let α be an increasing function on $[a, b]$. We will use the following notations:

$$f(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(d_i - c_i)$$

and

$$f^{\alpha}(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(\alpha(d_i) - \alpha(c_i)).$$

Definition 1.2. The function $f : [a, b] \to X$ is the *McShane integrable* on $[a, b]$ if there exists a vector z in X with the following property:

For each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $\|f(\mathcal{P}) - z\| < \varepsilon$ whenever $\mathcal{P}$ is subordinate to δ on $[a, b]$.

The function f is *McShane integrable* on a measurable set $E \subseteq [a, b]$ if the function $f\chi_E$ is McShane integrable on $[a, b]$.

Definition 1.3. The function $f : [a, b] \to X$ is the McShane-Stieltjes integrable function with respect to α if for each $\varepsilon > 0$ there exists a positive function δ on $[a, b]$ such that $| f^{\alpha}(\mathcal{P}) - z | < \varepsilon$ whenever $\mathcal{P} = \{(t_i, [a_i, b_i]) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ that is subordinate to δ. In this case, we write $z = \int_a^b f d\alpha$. A function $f : [a, b] \to X$ is McShane-Stieltjes integrable with respect to α on a measurable set $E \subseteq [a, b]$ if $f\chi_E$ is McShane-Stieltjes integrable with respect to α on $[a, b]$.

2. The Convergence Theorems for the McShane-Stieltjes Integral

We now mention Henstock's Lemma for real-valued McShane integrable functions. For the proof, see Gordon [4].

Lemma 2.1 (Saks-Henstock Lemma). *Let $f : [a,b] \to \mathbb{R}$ be McShane integrable on $[a,b]$. Let $F(x) = \int_a^x f$ for each $x \in [a,b]$, and let $\varepsilon > 0$. Suppose that δ is a positive function on $[a,b]$ such that $\mid f(\mathcal{P}) - F(\mathcal{P}) \mid < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ. If $\mathcal{P}_0 = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ is subordinate to δ, then*

$$\sum_{i=1}^{n} \Big| f(x_i)(d_i - c_i) - (F(d_i) - F(c_i)) \Big| \le 2\varepsilon.$$

We state a weak version of Saks-Henstock Lemma which holds for the real-valued McShane-Stieltjes integrable functions, whose proof is identical to the real-valued McShane-integrable function case (See [4, Theorem 3.7]).

Lemma 2.2 (Weak Saks-Henstock Lemma). *Let $f : [a,b] \to \mathbb{R}$ be McShane-Stieltjes integrable on $[a,b]$ with respect to α. Let $F^\alpha(x) = \int_a^x f d\alpha$ for each $x \in [a,b]$, and let $\varepsilon > 0$. Suppose that f is a positive function on $[a,b]$ such that $\mid f^\alpha(\mathcal{P}) - F^\alpha(\mathcal{P}) \mid < \varepsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ.*

If $\mathcal{P}_0 = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ is any collection of non-overlapping free tagged intervals that is subordinate to δ, then

$$\sum_{i=1}^{n} \Big| f(x_i)(\alpha(d_i) - \alpha(c_i)) - (F^\alpha(b) - F^\alpha(a)) \Big| \le 2\varepsilon.$$

We define the uniform McShane-Stieltjes integrability for a sequence of McShane-Stieltjes integrable functions.

Definition 2.3. Let α be an increasing function on $[a,b]$ and let $\{f_n\}$ be a sequence of vector-valued McShane-Stieltjes integrable functions on $[a,b]$ with respect to α. The sequence $\{f_n\}$ is uniformly McShane-Stieltjes integrable functions on $[a,b]$ with respect to α if for each $\varepsilon > 0$, there exists a positive function δ defined on $[a,b]$ such that $\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n d\alpha\| < \varepsilon$ for all n whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ.

Theorm 2.4. *Let $\{f_n\}$ be a sequence of vector-valued McShane-Stieltjes integrable functions defined on $[a,b]$ and suppose that $\{f_n\}$ converses pointwise to f on $[a,b]$.*

If $\{f_n\}$ is uniformly McShane-Stieltjes integrable on $[a,b]$ with respect to α, then f is McShane-Stieltjes integrable on $[a,b]$ with respect to α and

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha.$$

Proof. Let α be an increasing function on $[a,b]$. Since $\{f_n\}$ is uniformly McShane-Stieltjes integrable on $[a,b]$ with respect to α, there exists a free tagged partition $\mathcal{P}_0$ of $[a,b]$ such that $\mid f_n^\alpha(\mathcal{P}_0) - \int_a^b f_n d\alpha \mid < \varepsilon$ for all n. Since $\{f_n\}$ converges pointwise to f on $[a,b]$, for a free tagged partition $\mathcal{P}_0 = \{(x_i,(c_i,d_i)) : i = 1,\cdots,k\}$

$$\begin{aligned}\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| &= \Big\| \sum_{i=1}^k f_n(x_i)(\alpha(d_i) - \alpha(c_i)) - \sum_{i=1}^k f_m(x_i)(\alpha(d_i) - \alpha(c_i)) \Big\| \\ &\le \sum_{i=1}^k \Big\| f_n(x_i) - f_m(x_i) \Big\| (\alpha(b) - \alpha(a)) \\ &= (\alpha(b) - \alpha(a)) \sum_{i=0}^k \Big\| f_n(x_i) - f_m(x_i) \Big\|.\end{aligned}$$

For each x_i, there exists s positive integer $K_i(x_i)$ such that

$$\|f_n^\alpha(x_i) - f_m^\alpha(x_i)\| \le \frac{\varepsilon}{k} \quad \text{for all } n,m \ge K_i.$$

Set $N = \max\{K_i : 1 \le i \le k\}$. Then

$$\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| < \varepsilon \text{ for all } m,n \ge N.$$

There exists a positive integer N such that $\|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| < \varepsilon$ for all $m,n \ge N$. Then

$$\begin{aligned}&\Big\| \int_a^b f_n d\alpha - \int_a^b f_m d\alpha \Big\| \\ &= \Big\| \int_a^b f_n d\alpha - f_n^\alpha(\mathcal{P}_0) \Big\| + \|f_n^\alpha(\mathcal{P}_0) - f_m^\alpha(\mathcal{P}_0)\| + \Big\| f_m^\alpha(\mathcal{P}_0) - \int_a^b f_m\, d\alpha \Big\| \\ &< 3\varepsilon\end{aligned}$$

for all $m,n \ge N$. It follows that $\left\{\int_a^b f_n d\alpha\right\}$ is a Cauchy sequence in Banach space X. Let $L = \lim_{n\to\infty} \int_a^b f d\alpha$. We need to show that $\int_a^b f d\alpha = L$. Hence, it is sufficient to show that $\int_a^b f d\alpha = L$. Let $\varepsilon > 0$. By hypothesis, there exists a positive function δ on $[a,b]$ such that $\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n d\alpha\| < \varepsilon$ for all n whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ. Since $\{f_n\}$ converges pointwise to f, there exists $k \geq N$ such that $\|f^\alpha(\mathcal{P}) - f_k^\alpha(\mathcal{P})\| < \varepsilon$. Hence

$$\|f^\alpha(\mathcal{P}) - L\| \leq \|f^\alpha(\mathcal{P}) - f_k^\alpha(\mathcal{P})\| + \left\|f_k^\alpha(\mathcal{P}) - \int_a^b f_k\, d\alpha\right\| + \left\|\int_a^b f_k\, d\alpha - L\right\| < 3\varepsilon.$$

It follows that f is McShane-Stieltjes integrable on $[a,b]$ with respect to α and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$. □

Now, we will prove the *Dominated Convergence Theorem* for McShane-Stieltjes integrable functions.

Theorm 2.5 (Dominated Convergence Theorem). *Let α be an increasing function on $[a,b]$. Let $\{f_n\}$ be a sequence of vector-valued McShane-Stieltjes integrable functions on $[a,b]$ with respect to α and suppose that $\{f_n\}$ converges pointwise to f on $[a,b]$. Let $F_n^\alpha(x) = \int_a^x f_n\, d\alpha$. If there exists a real-valued McShane-Stieltjes integrable function g on $[a,b]$ such that $\|f_n\| \leq g$ for all n and if $\{F_n^\alpha\}$ is a Cauchy sequence in X, then f is McShane-Stieltjes integrable on $[a,b]$ and $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$.*

Proof. Let $\varepsilon > 0$, and $G^\alpha(x) = \int_a^x g\, d\alpha$. Then G is absolutely continuous on $[a,b]$, and there exists $\eta > 0$ such that

$$\left\|\sum_{i=1}^{k}(G^\alpha(d_i) - G^\alpha(c_i))\right\| < \varepsilon$$

whenever $\{[c_i, d_i] : 1 \leq i \leq k\}$ is a finite collection of non-overlapping intervals in $[a,b]$ that satisfy $\sum_{i=1}^k (d_i - c_i) < \eta$. By Egoroff's Theorem, there exists an open set O with the Lebesgue measure $\mu(O) < \eta$ such that $\{f_n\}$ convergence uniformly to f on $[a,b] - O$. Choose a positive integer N such that

$$\left\|\int_a^b f_n\, d\alpha - \int_a^b f_m\, d\alpha\right\| < \varepsilon \quad \text{and} \quad \|f_n(x) - f_m(x)\| < \varepsilon$$

for all $m, n \geq N$ and for all $x \in [a,b] - O$. Let δ_g be a positive function on $[a,b]$ such that

$$|g^\alpha(\mathcal{P}) - \int_a^b g\, d\alpha| < \varepsilon \quad \text{and} \quad \left\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n\, d\alpha\right\| < \varepsilon$$

for $1 \leq n \leq N$ whenever $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ_g. Define a positive function δ on $[a,b]$ by

$$\delta(x) = \begin{cases} \delta_g(x), & \text{if } x \in [a,b] - O \\ \min\{\delta_g(x), \rho(x, O^c)\}, & \text{if } x \in O \end{cases}$$

where $\rho(x, O^c) = \inf\{|x-y| : y \in O^c\}$. Suppose that $\mathcal{P}$ is a free tagged partition of $[a,b]$ that is subordinate to δ and fix $n > N$. Let $\mathcal{P}_1$ be the subset of $\mathcal{P}$ that had tags in $[a,b]-O$ and let $\mathcal{P}_2 = \mathcal{P} - \mathcal{P}_1$. Using the weak Saks-Henstock lemma (Lemma 2.2) and $\mu(\mathcal{P}_2) < \delta$

$$\begin{aligned}\|f_n^\alpha(\mathcal{P}) - f_N^\alpha(\mathcal{P})\| &\le \|f_n^\alpha(\mathcal{P}_1) - f_N^\alpha(\mathcal{P}_1)\| + \|f_n^\alpha(\mathcal{P}_2) - f_N^\alpha(\mathcal{P}_2)\| \\ &\le \varepsilon(\alpha(b)-\alpha(a)) + g^\alpha(\mathcal{P}_2) \\ &\le \varepsilon(\alpha(b)-\alpha(a)) + |g^\alpha(\mathcal{P}_2) - G^\alpha(\mathcal{P}_2)| + |G^\alpha(\mathcal{P}_2)| \\ &\le \varepsilon(\alpha(b)-\alpha(a)) + 2\varepsilon + \varepsilon \\ &= \varepsilon(\alpha(b)-\alpha(a)+3).\end{aligned}$$

Hence,

$$\begin{aligned}&\left\|f_n^\alpha(\mathcal{P}) - \int_a^b f_n d\alpha\right\| \\ &\le \left\|f_n^\alpha(\mathcal{P}) - \int_a^b f_N d\alpha\right\| + \left\|f_N^\alpha(\mathcal{P}) - \int_a^b f_N d\alpha\right\| + \left\|\int_a^b f_N d\alpha - \int_a^b f_n d\alpha\right\| \\ &< \varepsilon(\alpha(b)-\alpha(a)+2) + \varepsilon + \varepsilon.\end{aligned}$$

Hence $\{f_n\}$ is uniformly McShane-Stieltjes integrable on $[a,b]$ with respect to α. By Theorem 2.4, f is McShane-Stieltjes integrable on $[a,b]$ with respect to α and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f d\alpha$. □

ACKNOWLEDGEMENT

The authors thank the referees for valuable suggestions which improved and broadened the results in the final manuscript.

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