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ON THE SUPERCLASSES OF QUASIHYPONORMAL OPERATORS

HYUNG KOO CHA, KYO IL SHIN, AND JAE HEE KIM

ABSTRACT. In this paper, we introduce the classes $H(p,q;k)$, $K(p;k)$ of operators determined by the Heinz-Kato-Furuta inequality and Hölder-McCarthy inequality. We characterize relationship between p-quasihyponormal, k-quasihyponormal and k-p-quasihyponormal operators. And it is proved that every operator in $K(p;1)$ for some $0 < p \leq 1$ is paranormal.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{B}(\mathcal{H})$ is said to be p-quasihyponormal, $0 < p \leq 1$ if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$; or equivalently $\| \, |T^*|^p Tx\| \leq \| \, |T|^p Tx\|$ for all $x \in \mathcal{H}$.

An operator T in $\mathcal{B}(\mathcal{H})$ is called k-quasihyponormal for a positive integer k if $T^{*k}(T^*T - TT^*)T^k \geq 0$; or equivalently $\|T^*T^k x\| \leq \|T^{k+1}x\|$ for all $x \in \mathcal{H}$.

We recall the following extension of the Heinz-Kato inequality due to Furuta [6].

Heinz-Kato-Furuta inequality: Let T be an operator on a Hilbert space $\mathcal{H}$. If A and B are positive operators such that $T^*T \leq A^2$ and $TT^* \leq B^2$, then the following inequality holds for all $x, y \in \mathcal{H}$:

$$|\langle T|T|^{p+q-1}x, y\rangle| \leq \|A^p x\| \, \|B^q y\|$$

for any $p, q \in [0,1]$ with $p + q \geq 1$, where $|T|$ is a positive square root of T^*T.

We define a family of class $H(p,q;k)$ for a nonnegtive integer k and $0 \leq p, q \leq 1$; an operator T is $k-(p,q)$-quasihyponormal, or simply $T \in H(p,q;k)$, if T satisfies

$$|\langle U|T|^{p+q}T^{*k}x, T^k y\rangle| \leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^q T^k y\|$$

for all $x, y \in \mathcal{H}$. For simplicity, we denote by $H(p,p;k) = H(p;k)$.

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McCarthy [7] proposed the following inequalities as an operator variant of the Hölder inequality.

Hölder-McCarthy inequality: Let A be a positive operator on $\mathcal{H}$. Then the following inequalities hold:

(i) $\langle A^r x, x\rangle \le \|x\|^{2(1-r)}\langle Ax, x\rangle^r$ for $x \in \mathcal{H}$ if $0 < r \le 1$.

(ii) $\langle A^r x, x\rangle \ge \|x\|^{2(1-r)}\langle Ax, x\rangle^r$ for $x \in \mathcal{H}$ if $r \ge 1$.

We introduce another class of operators, $K(p;k)$; an operator T on $\mathcal{H}$ belongs to $K(p;k)$ for $0 < p \le 1$ if

$$\langle T^{*k}(TT^*)^p T^k x, x\rangle \le \|T^k x\|^{2(1-p)}\langle T^{*k}(T^*T)T^k x, x\rangle^p \quad \text{for all } x \in \mathcal{H}.$$

In the sequel, we consider some relations among several classes of operators around the quasihyponormal and paranormal operators. We shall discuss certain properties of $H(p;k), K(p;k)$, which are related to the superclasses of quasihyponomal.

2. $H(p,q;k)$ for a nonnegtive integer k and $0 \le p,q \le 1$.

Fujii, Nakamoto and Watanabe [5] discussed a family of classes $H(p,q)$ for $0 \le p \le 1$, $0 \le q \le 1$ with $p+q \ge 1$ determined by the Heinz-Kato-Furuta inequality.

Definition 1. An operator T is said to be (p,q)-*hyponormal*, or simply $T \in H(p,q)$, if T satisfies

$$|\langle T|T|^{p+q-1}x, y\rangle| \le \|\,|T|^p x\|\,\|\,|T|^q y\|$$

for all $x, y \in \mathcal{H}$.

And they characterized p-hyponormal operators.

Theorem 2.1 (Fujii, Nakamoto and Watanabe [5]). *For $0 < p \le 1$, an operator T belongs to $H(p) = H(p,p)$ if and only if T is p-hyponormal.*

Shin [8] discussed a family of classes $QH(p,q)$ for $0 \le p,q \le 1$ using Heinz-Kato-Furuta inequality.

Definition 2. An operator T is said to be (p,q)-quasihyponormal, or simply $T \in QH(p,q)$, if T satisfies

$$|\langle U|T|^{p+q}T^*x, Ty\rangle| \le \|\,|T|^p T^* x\|\,\|\,|T|^q Ty\|$$

for all $x, y \in \mathcal{H}$. For simplicity, we denote by $QH(p) = QH(p,p)$.

An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *k-p-quasihyponormal* if

$$T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0;$$

or equivalently $\| \, |T|^p U^* T^k x\| \leq \| \, |T|^p T^k x\|$ for all $x \in \mathcal{H}$.

If $k = 1$, then T is p-quasihyponormal and if $p = 1$, then T is k-quasihyponormal and if $k = p = 1$, then T is quasihyponormal.

We define a new family of class $H(p, q; k)$ for a nonnegative integer k and $0 \leq p, q \leq 1$.

Definition 3. An operator T is $k-(p,q)$-quasihyponormal, or simply $T \in H(p,q;k)$, if T satisfies

$$|\langle U|T|^{p+q}T^{*k}x, T^k y\rangle| \leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^q T^k y\|$$

for all $x, y \in \mathcal{H}$. For simplicity, we denote by $H(p,p;k) = H(p;k)$.

We shall discuss the characterization of k-p-quasihyponormal operators.

Theorem 2.2. *An operator T belongs to $H(p;k)$ if and only if T is k-p-quasihyponormal.*

Proof. Suppose that $T \in H(p;k)$, i.e.,

$$|\langle U|T|^{2p}T^{*k}x, T^k y\rangle| \leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^p T^k y\| \quad \text{for all } x, y \in \mathcal{H}.$$

For every $y \in \mathcal{H}$, taking $U^*T^k y = T^{*k}x$ for some $x \in \mathcal{H}$, we have

$$\begin{aligned} \| \, |T|^p U^* T^k y\|^2 &= \langle \, |T|^p U^* T^{*k} y, |T|^p U^* T^k y\rangle \\ &= \langle U|T|^{2p}T^{*k}x, T^k y\rangle \\ &\leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^p T^k y\| \\ &= \| \, |T|^p U^* T^k y\| \, \| \, |T|^p T^k y\|. \end{aligned}$$

Hence it implies that $\| \, |T|^p U^* T^k y\| \leq \| |T|^p T^k y\|$, so that T is k-p-quasihyponormal.

Conversely, if T is k-p-quasihyponormal, then we have

$$\begin{aligned} |\langle U|T|^{2p}T^{*^k}x, T^k y\rangle| &= |\langle \, |T|^p T^{*k}x, |T|^p U^* T^k y\rangle| \\ &\leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^p U^* T^k y\| \\ &\leq \| \, |T|^p T^{*k}x\| \, \| \, |T|^p T^k y\|, \end{aligned}$$

which completes the proof. □

Corollary 2.3.

(1) *An operator* T *belongs to* $H(1;1)$ *if and only if* T *is quasihyponormal.*
(2) *An operator* T *belongs to* $H(p;1)$ *if and only if* T *is p-quasihyponormal.*
(3) *An operator* T *belongs to* $H(1;k)$ *if and only if* T *is k-quasihyponormal.*

3. $K(p;k)$ for a nonnegtive integer k and $0 < p \leq 1$.

In this section, we discuss the paranormality of $K(p;1)$. An operator T in $\mathcal{B}(\mathcal{H})$ is paranormal if T satisfies $\|Tx\|^2 \leq \|T^2x\|\,\|x\|$ for all $x \in \mathcal{H}$.

By Hölder-McCarthy inequality, Fujii, Nakamoto and Watanabe [5] introduced the class $K(p)$ of operators for $0 < p \leq 1$.

Definition 4. An operator T on $\mathcal{H}$ *belongs to* $K(p)$ if

$$\langle (TT^*)^p x, x \rangle \leq \|x\|^{2(1-p)} \langle T^*Tx, x \rangle^p \text{ for all } x \in \mathcal{H}.$$

Theorem 3.1 (Fujii, Nakamoto and Watanabe [5]). *If* T *belongs to* $K(p)$ *for some* $0 < p \leq 1$, *then* T *is paranormal.*

Now we introduce another class of operators $K(p;k)$.

Definition 5. An operator T on $\mathcal{H}$ *belongs to* $K(p;k)$ for a nonnegative integer k and $0 < p \leq 1$ if

$$\langle T^{*k}(TT^*)^p T^k x, x \rangle \leq \|T^k x\|^{2(1-p)} \langle T^{*k}(T^*T)T^k x, x \rangle^p \text{ for all } x \in \mathcal{H}.$$

We obtain the following theorem.

Theorem 3.2.

(1) *For* $0 < p \leq 1$, *the class* $H(p;k)$ *is a subclass of* $K(p;k)$.
(2) *The classes* $K(p;k)$*'s are monotone decreasing on* $0 < p \leq 1$, *i.e.,*

$$K(p;k) \subset K(q;k) \text{ if } 0 < q < p \leq 1.$$

Proof. (1) Supppse that $T \in H(p;k)$. Then $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ and so by the Hölder-McCarthy inequality

$$\begin{aligned} \langle T^{*k}(TT^*)^p T^k y, y \rangle &\leq \langle T^{*k}(T^*T)^p T^k y, y \rangle \\ &\leq \|T^k y\|^{2(1-p)} \langle T^{*k}(T^*T)T^k y, y \rangle^p \end{aligned}$$

for all $y \in H$.

(2) Suppose that $T \in K(p;k)$ and $0 < q < p \leq 1$, and put $r = q/p < 1$. Then we have, for all $x \in \mathcal{H}$,

$$\begin{aligned}
\langle T^{*k}(TT^*)^q T^k x, x \rangle &= \langle ((TT^*)^p)^r T^k x, T^k x \rangle \\
&\leq \|T^k x\|^{2(1-r)} \langle (TT^*)^p T^k x, T^k x \rangle^r \\
&\leq \|T^k x\|^{2(1-r)} (\|T^k x\|^{2(1-p)} \langle (T^*T) T^k x, T^k x \rangle^p)^r \\
&= \|T^k x\|^{2(1-pr)} \langle (T^*T) T^k x, T^k x \rangle^{pr} \\
&= \|T^k x\|^{2(1-q)} \langle (T^*T) T^k x, T^k x \rangle^q
\end{aligned}$$

which means that $T \in K(q;k)$. □

In the following we shall denote the point spectrum of the operator T by $\sigma_p(T) \equiv \{\lambda \in \mathbb{C} : \ker(T-\lambda) \neq (0)\}$. We say that the complex number λ, $\lambda \in \mathbb{C}$, is in the joint point spectrum $\sigma_{jp}(T)$ of T there exists a unit vector $x \in \mathcal{H}$ such that $(T-\lambda)x = 0$ and $(T^* - \bar{\lambda})x = 0$. Let $E_\lambda[T] = \{x \in \mathcal{H} : Tx = \lambda x\}$. Note that $E_\lambda[T] \neq \{0\}$ if and only if $\lambda \in \sigma_p(T)$. It is well known that $E_\lambda[T] = E_{\bar{\lambda}}[T^*]$ for every normal operator. In general, the relation $E_\lambda[T] = E_{\bar{\lambda}}[T^*]$ does not hold for non-normal operators.

Remark 3.3.

(1) If T is invertible and $K(p;k)$, then T is $K(p)$.
(2) If T is $K(p;k)$ and $\overline{T(\mathcal{H})} = \mathcal{H}$, then T is $K(p)$.
(3) If T is $K(p;k)$ but not $K(p)$, then $\overline{T(\mathcal{H})} \neq \mathcal{H}$, i.e., $0 \in \sigma_p(T^*)$.

Theorem 3.4 (Xia [9]). *Let $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition $T = U|T|$ and let $\lambda = re^{i\theta}$ be a complex number, $r > 0, |e^{i\theta}| = 1$. Then $\lambda \in \sigma_{jp}(T)$ if and only if there exists $x \neq 0$ such that $Ux = e^{i\theta}x$ and $|T|x = rx$.*

Theorem 3.5. *If T is a $K(p;1)$ for $0 < p \leq 1$, then $\lambda(\neq 0) \in \sigma_p(T)$ implies $\lambda \in \sigma_{jp}(T)$ and hence $E_\lambda[T] = E_{\bar{\lambda}}[T^*]$.*

Proof. By Hölder McCarthy inequality we have

$$\begin{aligned}
\|Tx\| &= \|U|T|x\| \\
&= \|\,|T|x\| \\
&= \|\,|T|^{1-q}|T|^q x\| \\
&\leq \|\,|T|^{1+q}x\|^{1-q}\|\,|T|^q x\|^q \\
&= \|\,|T|^q U^* U|T|x\|^{1-q}\|\,|T|^q x\|^q,
\end{aligned}$$

for $0 < q \leq \frac{1}{2}$. Since $T \in K(q;1)$ for all $0 < q \leq p$, using the definition of $K(q;1)$, we obtain

$$\begin{aligned} \|\,|T|^q U^*U|T|x\|^2 &= \langle\, |T|^q U^*U|T|x, |T|^q U^*U|T|x \,\rangle \\ &= \langle\, U|T|^{2q}U^*Tx, Tx \,\rangle \\ &\leq \|Tx\|^{2(1-q)} \langle\, |T|^2 Tx, Tx \,\rangle^q \\ &= \|Tx\|^{2(1-q)} \|T^2x\|^{2q}. \end{aligned}$$

Hence

$$\|Tx\| \leq \|Tx\|^{(1-q)^2} \|T^2x\|^{q(1-q)} \|\,|T|^q x\|^q.$$

If $\lambda = re^{i\theta} \in \sigma_p(T)$, this implies

$$r\|x\| \leq r^{1-q^2} \|x\|^{1-q} \|\,|T|^q x\|^q \quad \text{or} \quad r^{q^2}\|x\|^q \leq \|\,|T|^q x\|^q.$$

Also, by Hölder-McCarthy inequality, we have

$$\begin{aligned} \|\,|T|^q x\|^2 &= \langle\, |T|^{2q}x, x \,\rangle \\ &\leq \|x\|^{2(1-q)} \langle\, |T|^2 x, x \,\rangle^q \\ &= \|x\|^{2(1-q)} \|\,|T|x\|^{2q} \\ &= \|x\|^{2(1-q)} \|Tx\|^{2q} \\ &= r^{2q}\|x\|^2. \end{aligned}$$

Thus, $\|\,|T|^q x\| = r^q\|x\|$ for all $0 < q \leq p$.

Choosing $q = \frac{p}{2}$ and $q = p$, then $\langle\, |T|^p x, x \,\rangle = \langle\, |T|^{2q}x, x \,\rangle = \|\,|T|^q x\|^2 = r^{2q}\|x\|^2 = r^p\|x\|^2$ and $\langle\, |T|^{2p}x, x \,\rangle = \|\,|T|^p x\|^2 = r^{2p}\|x\|^2$, respectively. Hence, we have

$$\begin{aligned} 0 &\leq \||T|^p x - r^p x\|^2 \\ &= \langle\, |T|^p x - r^p x, |T|^p x - r^p x \,\rangle \\ &= \langle\, |T|^p x, |T|^p x \,\rangle - \langle\, r^p x, |T|^p x \,\rangle - \langle\, |T|^p x, r^p x \,\rangle + \langle\, r^p x, r^p x \,\rangle \\ &= \||T|^p x\|^2 + r^{2p}\|x\| - r^p \langle\, x, |T|^p x \,\rangle - r^p \langle\, |T|^p x, x \,\rangle \\ &= r^{2p}\|x\|^2 + r^{2p}\|x\|^2 - r^{2p}\|x\|^2 - r^{2p}\|x\|^2 \\ &= 0. \end{aligned}$$

Therefore $|T|^p x = r^p x$.

Since there is no loss of generality in assuming $p = 2^{-n}$ for some integer $n \geq 1$, this implies

$$|T|x = rx.$$

And, since $Tx = U|T|x = re^{i\theta}x$, we have

$$Ux = e^{i\theta}x.$$

Therefore, $\lambda \in \sigma_{jp}(T)$ and so $E_\lambda[T] = E_{\bar{\lambda}}[T^*]$. □

Theorem 3.6. *If T belongs to $K(p;1)$ for some $0 < p \leq 1$, then T is paranormal.*

Proof. Suppose that $T \in K(p;1)$. For any $x \in \mathcal{H}$, we have

$$\begin{aligned} \langle (T^*T)^{p+1}x, x \rangle &= \langle (TT^*)^pTx, Tx \rangle \\ &= \|Tx\|^{2(1-p)}\|T^2x\|^{2p}, \end{aligned}$$

and moreover the Hölder-McCarthy inequality (ii) implies that

$$\begin{aligned} \langle (T^*T)^{p+1}x, x \rangle &\geq \|x\|^{-2p}\langle T^*Tx, Tx \rangle^{p+1} \\ &\leq \|x\|^{-2p}\|Tx\|^{2p+2}, \end{aligned}$$

Hence we have

$$\|x\|^{-2p}\|Tx\|^{2p+2} \leq \|Tx\|^{2(1-p)}\|T^2x\|^{2p},$$

so that

$$\|Tx\|^2 \leq \|T^2x\|\,\|x\|.$$

Therefore T is paranormal. □

We have the following corollary.

Corollary 3.7. *Every $K(p;1)$ is normaloid.*

References

1. S. L. Campbell and B. C. Gupta, On k-quasihyponormal operators, *Math. Japon.* **23** (1978), 185–189. MR **80b**:47030
2. B. P. Duggal, On a super class of p-hyponormal operators, *Hokkaido Math. J.* **25** (1996), 97–106. MR **97b**:47018
3. M. Fujii, C. Himeji and A. Matsumoto, Theorems of Ando and Saito for p-hyponormal operators, *Math. Japon.* **39** (1994), 595–598. MR **95b**:47022
4. M. Fujii, S. Izumino and R. Nakamoto, Classes of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality, *Nihonkai Math. J.* **5** (1994), 61–67. MR **95e**:47029
5. M. Fujii, R. Nakamoto and H. Watanabe, The Heinz-Kato-Furuta inequality and hyponormal operators, *Math. Japon.* **40** (1994), 469–472. MR **95k**:47033

6. T. Furuta, An extension of the Heinz-Kato theorem, *Proc. Amer. Math. Soc.* **120** (1994), 785–787. MR **94e**:47034
7. C. A. McCarthy, c_ρ, *Israel J. Math.* **5** (1967), 249–271. MR **37**#735
8. K. I. Shin, *On the superclass of the quasihyponormal operators on a Hilbert space*, Ph. D. Thesis, Hanyang Univ., Seoul, 1998.
9. D. Xia, *Spectral Theory of Hyponormal Operators*, Operator Theory: Advances and Applications, Vol. 10, Birkhäuser Verlag, Basel, 1983. MR **87j**:47036

(H. K. CHA) DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEONGDONG-GU, SEOUL 133-791, KOREA

(K. I. SHIN) DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEONGDONG-GU, SEOUL 133-791, KOREA
E-mail address: kishin@fermat.hanyang.ac.kr

(J. H. KIM) DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEONGDONG-GU, SEOUL 133-791, KOREA