Journal of the Korean Statistical Society (2000), **29**: *4, pp 489–499*

Parameter Estimation in a Complex Non-Stationary and Nonlinear Diffusion Process †

Beong Soo So[1]

ABSTRACT

We propose a new instrumental variable estimator of the complex parameter of a class of univariate complex-valued diffusion processes defined by the possibly non-stationary and/or nonlinear stochastic differential equations. On the basis of the exact finite sample distribution of the pivotal quantity, we construct the exact confidence intervals and the exact tests for the parameter. Monte-Carlo simulation suggests that the new estimator seems to provide a viable alternative to the maximum likelihood estimator(MLE) for nonlinear and/or non-stationary processes.

Keywords: Ito process; exact procedure; Instrumental variable estimator.

1. INTRODUCTION

Recently , modeling of the evolution of various random system by the continuous-time stochastic processes has become widespread in such diverse areas as optimal control theory in engineering and financial economics. For example Ito processes are used most often in economics as models of asset-price behavior as in the case of the well-known Black-Scholes stock option pricing model.

Accordingly, a considerable amount of research in the statistics literature has been devoted to the statistical estimation problems associated with such continuous-time processes. See chap.9 of Basawa and Rao(1980) for recent survey and extensive bibliography in this area. However, their discussion focuses exclusively on the asymptotic properties of the maximum likelihood estimation(MLE) for the special class of stationary Ito processes and relatively little research was done for the systematic study of small sample results in the non-stationary processes. This is also true of the most of the other studies which consider MLE of diffusion processes such as those by Liptser and Schryayev(1978), Le Breton(1976) , Loges(1984) and Tugnait(1985).

†This research was supported by a grant from BK21 project of Korea Science Foundation.

[1]Dept. of Statistics, Ewha University, Seoul, 120-750, Korea.

In order to circumvent the enormous difficulty of finding the exact finite-sample distribution of MLE for nonlinear and/or non-stationary case, we consider alternative methods of estimation with a reasonable efficiency in the stationary case having a nice finite sample property for possibly nonlinear and/or non-stationary processes. Specifically, we will consider the method of instrumental variables (IV) estimation from a continuously sampled data and identify a special type of instrumental variable estimator whose pivotal statistic has an exact *finite sample* normal distribution regardless of stationarity of the underlying Ito processes. Of course, our goal is more modest than those of the aforementioned papers: we should expect some loss of efficiency of the IV estimator in the strictly stationary case in compensation for the nice exact finite sample normality and the robustness to possible departure from stationarity.

In this paper, we consider parameter estimation problem from a continuously sampled data $\{z(t); t \in [0,T]\}$ of a complex-valued continuous-time stochastic process $\{z(t)\}$ generated by the following type of possibly non-stationary and/ or nonlinear stochastic differential equation(SDE) ;

$$dz(t) = a(t, z(t))dt + \theta b(t, z(t))dt + \sigma dW(t)\ ,\ t \in [0,T],\quad z(0) = z_o \qquad (1.1)$$

where $z(t) = x(t) + iy(t)$, $a(t,\cdot)$,$b(t,\cdot)$ are fixed complex-valued functions, $\theta = \alpha + i\beta$ is an unknown complex parameter of interest with $i^2 = -1$, $\sigma > 0$ is a known constant, $W(t) = U(t) + iV(t)$ and $U(t), V(t)$ are two independent standard Brownian motion processes.

Let us now briefly describe the structure of the paper. In Section 2 we state model assumptions explicitly and introduce a new instrumental variable estimator of θ. Then in Section 3. we prove a key lemma which establishes the exact finite sample distribution of the pivotal statistic and construct the exact confidence regions and the exact tests of the hypotheses on the parameter θ . In Section 4, applications to some diffusion models are considered and some simulation results are given which compares favorably the performance of the new estimator with those of MLE in the small-sample nonlinear and/or non-stationary cases. Section 5 concludes with some discussions on extensions to other models.

2. NEW ESTIMATOR

We assume the following conditions for the existence and uniqueness of the solution of the SDE (1.1).

C : There exists some constant $K > 0$ such that the complex-valued functions $a(t,x)$ and $b(t,x)$, satisfy the following conditions for all $x, y \in S \subset R^2$ and $s,t \in [0,T]$:

$$a) \quad |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq K|x-y|,$$
$$b) \quad |a(s,x) - a(t,x)| + |b(s,x) - b(t,x)| \leq K|s-t|,$$
$$c) \quad |a|^2(t,x) + |b|^2(t,x) \leq K^2(1+|x|^2).$$

where $|u+iv| = (u^2+v^2)^{1/2}$ for $u, v \in R$.

For the special model of the type (1.1), both least squares estimator(LSE) and MLE of the unknown complex parameter θ are the same and is given by

$$\hat{\theta}_o = \int_0^T \bar{b}(t,z(t))[dz(t) - a(t,z(t))dt] / \int_0^T |b(t,z(t))|^2 dt \tag{2.1}$$

from the Girsanov formula which in particular gives an explicit expression of the density (with respect to Wiener measure) of the measure in $C[0,T]$ induced by the solution of the SDE (1.1). See chap.9 of Basawa and Rao(1980) for more details. Here $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$. In order to motivate our new IV estimator, we note that the martingale property of the Brownian motion $W(t)$ implies

$$E[dW(t)|F_t] = 0 \ , \ t \geq 0$$

where F_t is the σ-field generated by $\{z(s); s \in [0,t]\}$ for $t \geq 0$. This in turn implies that

$$E[dW(t) sign(\bar{b}(t,z(t)))] = 0 \ , \ t \geq 0 \tag{2.2}$$

for any complex-valued function $b(,)$, $sign(z) = z/|z|$ for $z \neq 0$ in R^2 and $sign(0) = 0$.

Now the corresponding estimating equation follows directly from the sample analog of (2.2) and is given by;

$$T^{-1} \int_0^T sign[\bar{b}(t,z(t))][dz(t) - a(t,z(t))dt - \theta b(t,z(t))dt] = 0.$$

This defines the new estimator

$$\hat{\theta}_c = \int_0^T sign[\bar{b}(t,z(t))][dz(t) - a(t,z(t))dt] / \int_0^T |b(t,z(t))| dt. \tag{2.3}$$

The estimator (2.3) is an instrumental variable estimator based on the special instrument $sign[\bar{b}(t,z(t))]$ instead of usual instrument $\bar{b}(t,z(t))$ which defines the

LSE $\hat{\theta}_o$. In view of the formal similarity with the sign-based instrumental variables estimator for linear regression model first proposed by Cauchy(1836), we may call the new estimator as Cauchy estimator. See So and Shin(1999) for more details in the discretely sampled models. The corresponding pivotal quantity τ_c based on $\hat{\theta}_c$ is defined by

$$\tau_c = [\int_0^T |b(t,z(t))|dt](\hat{\theta}_c - \theta)/\sigma T^{1/2} = \int_0^T sign[\bar{b}(t,z(t))]dW(t)/T^{1/2}. \qquad (2.4)$$

In the next Section, we will establish the exact finite sample distribution of the pivotal quantity τ_c.

3. MAIN RESULTS

In the following, $A \sim B$ denotes that A and B have the same distribution, $N_2(\mu, \Sigma)$ denotes the bivariate normal distribution with a mean μ and a covariance matrix Σ and I_2 is the identity matrix of order 2.

Lemma 1. Let$\{v(t)\}$ be a non-anticipating process adapted to the σ-fields F_t satisfying the condition

A : $v(t)$ has no atom at 0, i.e., $sign(v(t)) \neq 0$ $a.s.$ If we let for $t \geq 0$

$$M_t = \int_0^t sign(v(s))dW(s)$$
$$M_t^* = \int_0^t dW(s),$$

then

$$a) \quad \{M_t, F_t\}_{t=0}^T \text{ is a martingale,}$$
$$b) \quad \{M_t\}_{t=0}^T \sim \{M_t^*\}_{t=0}^T,$$
$$c) \quad M_T \sim N_2(0, TI_2).$$

Furthermore, if both $\{v(t)\}$ and $\{W(t)\}$ are real-valued processes , then

$$d) \quad M_T \sim N(0,T).$$

holds instead of c).

Proof. a),b) c) follow directly from the spherical symmetry of the isotropic 2-dimensional Brownian motion process $W(t)$ and d) follows from the symmetry of the Brownian motion. See Viterbi(1966) for details.

Note that Lemma 1-a), b) imply that the process $\{M_t\}_{t=0}^{\infty}$ is an isotropic Brownian motion on the complex plane R^2.

Lemma 1-c), d) enables us to establish the exact distribution of the pivotal quantity $\tau_c(\theta)$ in (2.4) for any fixed real or complex parameter θ and for any possibly non-stationary and/or nonlinear diffusion models of the type (1.1) as is to be shown in Theorem 1.

Theorem 1. Consider model (1.1). Let $\{b(t, z_t)\}$ be a process satisfying A. Then, for any complex θ , we have

$$\tau_c(\theta) \sim N_2(0, I_2)$$

and

$$t_c(\theta) \sim N(0,1),$$

where $t_c(\theta) = [\int_0^T |b(t, z(t))|dt](a\hat{\alpha}_c + b\hat{\beta}_c - a\alpha - b\beta)/\sigma T^{1/2}(a^2 + b^2)^{1/2}$ for fixed real numbers a, b.

Furthermore, if $\{a(t,\cdot), b(t,\cdot), z(t), W(t)\}$ and θ are real-valued, then

$$\tau_c(\theta) \sim N(0,1).$$

Proof. This is immediate from Lemma 1.

Using the result of Theorem 1, we can construct an exact simultaneous confidence region for θ and an exact confidence interval for an arbitrary linear function $a\alpha + b\beta$ of θ for some fixed real constants α, β .

Exact Confidence Region for complex θ:

$$R_c : \{\theta \in R^2 | [\int_0^T |b(t, z(t))|dt](\hat{\theta}_c - \theta) \leq \sigma T^{1/2} \chi_\alpha(2)\}$$

where $\chi^2_\alpha(2)$ is an upper α-th quantile of the χ^2-distribution with degrees of freedom 2.

Exact Confidence Interval for $a\alpha + b\beta$:

$$I_c : (a\hat{\alpha}_c + b\hat{\beta}_c) \pm [\int_0^T |b(t, z(t))|dt]^{-1} \sigma T^{1/2} (a^2 + b^2)^{1/2} z_{\alpha/2}$$

where z_α is an upper α-th quantile of the standard normal distribution. In contrast, the corresponding LSE-based approximate confidence region and intervals are given by

$$R_o : \{\theta \in R^2 | [\int_0^T |b(z(t))|^2 dt]^{1/2} (\hat{\theta}_o - \theta) \leq \sigma(\chi_\alpha(2) + o(1))\}$$

and

$$I_o : (a\hat{\alpha}_o + b\hat{\beta}_o) \pm [\int_0^T |b(z(t))|^2 dt]^{-1/2} \sigma (a^2 + b^2)^{1/2} (z_{\alpha/2} + o(1)).$$

respectively for the stationary process . This follows directly from the asymptotic distribution of the pivotal statistics τ_o ;

$$[\int_0^T |b(z(t))|^2 dt]^{1/2} (\hat{\theta}_o - \theta)/\sigma = \int_0^T \bar{b}(z(t)) dW(t) / [\int_0^T |b(z(t))|^2 dt]^{1/2}$$
$$\Rightarrow N_2(0, I_2)$$

which follows from the continuous-time martingale CLT (Hall and Heyde(1980)) as $T \to \infty$.

One immediate consequence of the Theorem 1 is that Cauchy estimator of $a\alpha + b\beta$ is exactly median-unbiased but this is not true for LSE in most cases as is evident from the simulation results for $P[\hat{\alpha}_o < \alpha]$ in Table 1 below.

Next we address the problem of asymptotic relative efficiency(ARE) of $\hat{\theta}_c$ with respect to $\hat{\theta}_o$ as $T \to \infty$ for stationary processes. For time-homogeneous stationary process $\{z(t)\}$ with $a(z_t), b(z_t)$ independent of t, we can compute asymptotic efficiency of the Cauchy estimator with respect to LSE as follows ;

$$ARE(\hat{\theta}_c; \hat{\theta}_o) = [E|b(z_t)|]^2 / E[|b(z_t)|^2]$$

by the ergodic theorem for the stationary process $\{b(z_t)\}$. For example, stationary Ornstein-Uhlenbeck process is defined by

$$dz_t = \theta z_t dt + \sigma dW_t$$

with $Re(\theta) = \alpha < 0$ and $z_0 \sim N_2(0, \sigma^2/2|\alpha| I_2)$. Thus we have

$$ARE(\hat{\theta}_c; \hat{\theta}_o) = [E|z_t|]^2 / E[|z_t)|^2] = \pi/4 = .785$$

which in turn implies that the asymptotic ratio of the lengths of the confidence intervals based on $\hat{\theta}_o$ and $\hat{\theta}_c$ are $\pi^{1/2}/2 = 0.886$. However, for non-stationary processes, ARE of the Cauchy estimator is not relevant for finite-sample and can be greater than 1 for nearly non-stationary processes. See examples 1 and 2 for details. Furthermore, confidence intervals R_c, I_c based on $\hat{\theta}_c$ have the exact coverage probability $1-\alpha$ for any T but those based on LSE $\hat{\theta}_o$ may suffer from the undesirable distortion in coverage probability as is clear from the simulation results of $p[\alpha \in I_o]$ in Table 1 below.

Now it is straightforward to construct the critical regions for the exact tests of the corresponding hypotheses on θ by inverting the suitable pivotal statistics of the confidence regions .

We reject the null hypothesis $H_o : \theta = \theta_0$ in favor of the alternative hypothesis $H_1 : \theta \neq \theta_0$ if

$$|\tau_c(\theta_0)| \geq \chi_\alpha(2).$$

Similarly critical region for the exact level-α test of the null hypothesis H_o : $a\alpha + b\beta = c$ against alternative $H_1 : a\alpha + b\beta \neq c$ is given by

$$|t_\alpha| = [\int_0^T |b(t, z(t))|dt]|a\hat{\alpha}_c + b\hat{\beta}_c - c|/\sigma T^{1/2}(a^2+b^2)^{1/2} \geq z_{\alpha/2}.$$

4. EXAMPLES

Example 1. (Complex Ornstein-Uhlenbeck Process) Consider the Gaussian Process $\{z(t)\}$ defined by the linear stochastic differential equation

$$dz(t) = \theta z(t)dt + \sigma dW(t) \tag{4.1}$$

which has a unique solution

$$z(t) = e^{\theta t} z_0 + \sigma \int_0^t e^{\theta(t-s)} dW(t) \ , \ t \geq 0.$$

Arato et al.(1962) and Basawa and Rao(1980) considered application of the complex Ornstein-Uhlenbeck process to the important geophysical problem where $z(t) = x(t) + iy(t)$ represents deviation of the instantaneous axis of the earth's rotation from the earth's minor axis and investigated the problem of parameter estimation of the complex parameter $\theta = \alpha + i\beta$ on the basis of the continuously observed data $\{z(t)\}, t \in [0,T]$. Under the stationary condition $Re(\theta) = \alpha < 0$, they derived asymptotic distribution of the LSE $\hat{\theta}_o$ as $T \to \infty$. In general case , we may have $Re(\theta) = \alpha \geq 0$ which implies the non stationarity of the process. In particular if $Re(\theta) = \alpha = 0$, then

$$\{z(t)\} \sim \{e^{i\beta t}[z_o + W(t)]\}$$

from Lemma 1. and $z(t)$ is itself a rotated Brownian motion in R^2. In order to compare finite sample performances of the Cauchy estimator and LSE, we made a Monte-Carlo simulation of the process (4.1) using the simple approximation generated by the following discretized stochastic difference equation:

$$\Delta z_t = \theta z_t \Delta t + \sigma(\Delta t)^{1/2} e_t, \; t = 0 \ldots N, \tag{4.2}$$

where $z_i = z(i\Delta t)$, $N = T/\Delta t$, e_t are i.i.d. standard normal random variables. This procedure seems sensible because the sample paths of the discretization (4.2) and the corresponding estimator converge to those of the continuous-time Ito process $z(t)$ as $\Delta t \to 0$. See Le Breton(1975) for more details. We used the following configuration for the Monte-Carlo simulation: $z_o = 0$, $T = 4$, $\Delta t = 0.01$, $\beta = 0$, number of replications = 10,000. Table 1 below shows the bias , standard deviation, mean absolute deviation(mad) of $\hat{\alpha}$, empirical coverage probability(c.v.) $p[\alpha \in I]$ of the 90-% confidence intervals , $Pr[\hat{\alpha} < \alpha]$, based on LSE $\hat{\alpha}_o$ and Cauchy estimator $\hat{\alpha}_c$ of the damping parameter $\alpha = Re(\theta)$ respectively. We also obtained simulation results for other cases $T = 1, 9$ but omitted them since they were similar to the case $T = 4$.

We can summarize the simulation results as follows. In general, Cauchy estimator has a smaller bias but larger variance than LSE and has a smaller mad for the near non-stationary cases. In contrast, it suffers moderate loss in efficiency $(1.17/1.32 = .886 \sim (\pi/4)^{1/2}$ for $\alpha = -5)$ for the strictly stationary case as expected. However, LSE-based confidence intervals suffer from the undesirable distortion of the coverage probability for α close to 0 due to non-negligible bias which deteriorates the performances of the LSE-based confidence intervals and the corresponding tests.

Table 1. $T = 4,\ z(0) = 0,\ \beta = 0;\ dz(t) = \theta z(t)dt + dW(t)$

	LSE					Cauchy				
α	$E(\hat{\alpha}_o)$	s.d.	mad	c.v.	Pr.	$E(\hat{\alpha}_c)$	s.d.	mad	c.v.	Pr.
0.1	-.09	.40	.28	.880	.631	.02	.40	.27	.905	.500
0.0	-.21	.44	.31	.878	.635	-.13	.45	.30	.898	.503
-0.1	-.32	.45	.33	.890	.633	-.23	.47	.32	.899	.497
-1.0	-1.25	.65	.50	.895	.590	-1.15	.71	.53	.895	.503
-5.0	-5.24	1.17	.93	.904	.548	-5.14	1.32	1.03	.900	.505

In summary, Cauchy estimator $\hat{\theta}_c$ seems to have a comparable efficiency for nearly non-stationary cases as well as reasonable efficiency for stationary case and a nice finite sample property regardless of the stationarity and/or linearity of the process $z(t)$ which greatly simplifies the statistical inference for the possibly non-stationary and/or nonlinear process.

Example 2. (Real-valued Diffusion Process) When $\{a(t,\cdot), b(t,\cdot)\}$ are real-valued functions and $\{z(t), W(t)\}$ are real-valued processes with real-valued parameter θ of interest, It is straightforward to establish the similar finite sample properties of the corresponding real-valued Cauchy estimator to those of the complex-valued estimator. we made a small sample ($T = 4$) Monte-Carlo simulation experiment for the real-valued process $x(t)$ generated by the following nonlinear SDE:

$$dx(t) = \theta x(t)dt/[1 + sign(x(t))/2] + dW(t), \quad t \in [0, T], \quad x(0) = 0.$$

Table 2 below summarizes simulation results and the general pattern of the results are similar to that of the linear model (4.1). For non-stationary or nearly non-stationary models with θ near 0, Cauchy estimator seems to provide a viable alternative to LSE in terms of efficiency, simplicity and robustness to possible departure from stationarity.

Table 2. $T = 4$, $x(0) = 0$; $dx(t) = \theta x(t)dt/[1 + sign(x(t))/2] + dW(t)$

	LSE				Cauchy			
θ	$E(\hat{\theta}_o)$	s.d.	mad	$P[\theta \in I_o]$	$E(\hat{\theta}_c)$	s.d.	mad	$P[\theta \in I_c]$
2.0	1.99	.15	.02	.898	1.98	.15	.03	.899
1.0	0.90	.40	.13	.869	.93	.37	.13	.898
0.1	-.27	.69	.46	.868	-.12	.71	.45	.899
0.0	-.38	.70	.49	.879	-.22	.73	.47	.903
-0.1	-.51	.75	.52	.883	-.33	.79	.51	.907
-1.0	-1.43	.94	.71	.897	-1.25	1.12	.82	.896
-5.0	-5.38	1.57	1.23	.899	-5.19	1.98	1.56	.898

5. CONCLUSION

For the possibly non-stationary and/or nonlinear diffusion model , we have developed a new estimator based on the sign-type instrumental variable. Our estimator has nice finite-sample properties such as exact median-unbiasedness and exact normality of the corresponding pivotal statistics. This enable us to construct an exact confidence intervals and exact tests of hypotheses which are free from any restriction such as stationarity and/or linearity of the process. Monte-Carlo simulation for possibly non-stationary and/or nonlinear processes shows that Cauchy estimator $\hat{\theta}_c$ has a good efficiency for nearly non-stationary cases as well as a reasonable efficiency for stationary case.

We also mention the possible extension of the finite sample results of our paper to other class of multivariate diffusion models. For example, we may consider 4-dimensional quarternion-valued process $z(t)$, $z(t) = x(t) + iy_1(t) + jy_2(t) + ky_3(t)$ driven by the system of possibly non-stationary and nonlinear SDEs:

$$dz(t) = a(t, z(t))dt + \theta b(t, z(t))dt + \sigma dW(t), \ \ t \geq 0$$

where $\theta = \alpha + i\beta_1 + j\beta_2 + k\beta_3$, $W(t) = U(t) + iV_1(t) + jV_2(t) + kV_3(t)$, $U(t), V_i(t)$, $i = 1, 2, 3$ are independent standard Brownian motion processes and $a(t, \cdot), b(t, \cdot)$ are given quarternion-valued functions of $z(t)$. Important topic of extensions of the finite sample results to general multi-parameter, multivariate Ito processes will be pursued in detail in a subsequent paper.

In view of the nice finite sample property, simplicity, and desirable robustness to possible departure from linearity and/or stationarity, the statistical inference based on Cauchy-type estimator seems to be a useful alternative to the LSE-based methods.

References

Arato, M., Kolmogorov, A.N. and Sinai, Ya. G. (1962). Evaluation of the parameters of a complex stationary Gauss-Markov process. *Soviet. Math. Dokl.* 3, 1368-1371.

Basawa, I.V. and Prakasa Rao, B.L.S. (1980). *Statistical inference for stochastic processes* , Academic Press, London.

Cauchy, A.L. (1836). On a new formula for solving the problem of interpolation in a manner applicable to physical investigations, *Philosophical Magazine*, 8,459-486.

Hall, P. and Heyde, C. C. (1980). *Martingale limit theory and its applications* , Academic Press, New York.

Le Breton, A. (1976). On continuous and discrete sampling for parameter estimation in diffusion type processes. *In Mathematical Programming Studies*, 5, 124-144.

Liptser, R.S. and Shiryayev, A.N.(1978). *Statistics of random processes II; Applications*, New York, Springer-Verlag.

Loges, W. (1984). Girsanov's theorem in Hilbert space and an application to the statistics of Hilbert space valued stochastic differential equations. *Stochastic Processes and Their Applications*, 17, 243-263.

So, B. S. and Shin, D. W. (1999). Cauchy estimators for autoregressive processes with applications to unit root tests and confidence intervals, *Econometric Theory* , 15, 165-176.

Tugnait, J.K. (1985). Continuous-time system identification on compact parameter sets. *IEEE transactions on Information Theory*, 31, 652-659.

Viterbi, A.J. (1966). *Principles of coherent communications.*, McGraw-Hill, New York.