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Stochastic Comparisons of Markovian Retrial Queues†

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ABSTRACT

We consider a Markovian retrial queue with waiting space in which the service rates and retrial rates depend on the number of customers in the service facility and in the orbit, respectively. Each arriving customer from outside or orbit decide either to enter the facility or to join the orbit in Bernoulli manner whose entering probability depend on the number of customers in the service facility. In this paper, a stochastic order relation between two bivariate processes $(C(t), N(t))$ representing the number of customers $C(t)$ in the service facility and one $N(t)$ in the orbit is deduced in terms of corresponding parameters by constructing the equivalent processes on a common probability space. Some applications of the results to the stochastic bounds of the multi-server retrial model are presented.

Keywords: multi-server retrial queue; waiting room; stochastic comparison

1. INTRODUCTION

We consider a Markovian retrial queue with service facility. Customers arrive from outside according to a Poisson process with rate λ. An arriving customer who finds k customers in the service facility enters the facility with probability p_k or joins the orbit with probability $1-p_k$ and retries to get service after exponential time. If returning customer from orbit finds k customers in the service facility, then the customer enters the facility with probability u_k or joins the orbit again with probability $v_k = 1 - u_k$ and retries to get service after exponential time. Let θ_k be the total retrial rate when k customers are in the orbit. When there are n customers at the service facility, we assume that the time until the next service completion is exponential with rate μ_n.

It is easily seen that choosing the parameters appropriately, the model described above contains various retrial queueing models as special cases such as

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not only standard $M/M/c/K$ retrial queue but also $M/M/c/K$ retrial queue with limited retrial rates $\theta_j = \min(j, M)\theta$ and multi-server retrial queue with unlimited waiting place considered in Artalego (1995). However, even for the standard $M/M/c/K$ retrial queue, analytical results are available only for some particular cases (Hanschke (1987) and Pearce (1989)). So, several approximations and bounding methods of the steady state distributions for multi-server retrial queues have been proposed (e.g. Artalego (1995), Greenberg and Wolff (1987), for the comprehensive references see Falin and Templeton (1997) and Artalego (1999)). Neuts and Rao (1990) suggested a numerical algorithm without analytical proof of the convergence for the stationary distribution in $M/M/c/c$ retrial queue by restricting the retrial rates in the orbit.

We are interested in the bivariate process $X(t) = (C(t), N(t))$, called $CN-process$ (Kulkarni and Liang (1997)), where $C(t)$ and $N(t)$ represent the numbers of customers in the service facility and the orbit, respectively, at time t. Obviously, the process $X = \{X(t),\ t \geq 0\}$ corresponding to our model is a Markov chain with the lattice set $E = Z^+ \times Z^+$, where $Z^+ = \{0, 1, 2, \cdots\}$, as the state space. In this paper, instead of studying performance measures in a quantitative fashion, we investigate a stochastic order between two $CN-processes$ in terms of parameters by constructing equivalent processes on a common probability space. We also give an analytical proof of the convergence of the algorithm in Neuts and Rao (1990).

The constructive approach has been used early in various applications by many authors (e.g. Bhaskaran (1986), Miller (1979), Sonderman (1979a, 1979b), and see Stoyan (1983) and Shaked and Shanthikumar (1994) for applications and detailed references). In particular, applications of stochastic order relation to retrial model can be found in Falin and Templeton (1997) for $M/G/1$ retrial queue and multi-server Markovian retrial queue with no waiting room and Liang and Kulkarni (1993) for $G/G/1/K$ retrial queue with phase type retrial time.

This paper is organized as follows. In section 2, we construct two CN-processes on a probability space by keeping the order relation between sample paths of each process under suitable conditions of parameters and apply the results and method to compare the specific multi-server retrial queues. In section 3 we prove the convergence of the bounding method. Section 4 is devoted to present the proofs of the results given in section 2.

2. CONSTRUCTION AND COMPARISONS

Define a relation $\prec$ on $E = Z^+ \times Z^+$ by $(i,j) \prec (k,l)$ if and only if $j \leq l$ and $i+j \leq k+l$, then it is immediate that $\prec$ is a partial order on E. Indeed,

$$(i) \qquad (i,j) \prec (i,j), \text{ for all } (i,j) \in E;$$
$$(ii) \qquad (i,j) \prec (k,l) \text{ and } (k,l) \prec (i,j) \text{ implies } (i,j) = (k,l);$$
$$(iii) \qquad (i,j) \prec (k,l) \text{ and } (k,l) \prec (m,n) \text{ implies } (i,j) \prec (m,n).$$

There can be considered two types of inequality for processes $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ with state space $(E, \prec)$:

(i) X is strictly stochastically smaller than Y with respect to $\prec$, written $X \prec_{str} Y$ if and only if for every positive integer n and $0 \leq t_0 < t_1 < \cdots < t_n$ and increasing real valued function $f : E^{n+1} \longrightarrow R$ it follows that $P\{f(X(t_0), \cdots, X(t_n)) > 0\} \leq P\{f(Y(t_0), \cdots, Y(t_n)) > 0\}$.

(ii) If X and Y are defined on the same probability space then X is almost surely smaller than Y with respect to $\prec$, written $X \prec_{as} Y$ if and only if $P(X(t) \prec Y(t) \text{ for all } t \geq 0) = 1$.

Note that the order relation $\prec_{str}$ and $\prec_{as}$ are equivalent in the sense that if $X \prec_{str} Y$, then there exists a probability space upon which it is possible to construct versions $\hat{X}$ and $\hat{Y}$ of X and Y such that $\hat{X} \prec_{as} \hat{Y}$ and clearly $X \prec_{as} Y$ implies that $X \prec_{str} Y$ (cf. Kamae et al. (1977)). Thus if $X \prec_{as} Y$ (or equivalently $X \prec_{str} Y$) and $X(t)$ and $Y(t)$ have weak limits $X(\infty)$ and $Y(\infty)$ as $t \to \infty$, then $X(\infty) \prec Y(\infty)$ in distribution. Very general results for stochastic orders can be found in Kamae et al. (1977) and a comprehensive account of stochastic orders and its applications to queue are given in Stoyan (1983).

In the following of this section we just describe the results and their proofs are given in section 4.

Theorem 2.1. *Let $X^{(i)} = \{X^{(i)}(t),\ t \geq 0\}$, $i = 1, 2$ be the CN-processes of the queueing model described in the previous section with arrival rates $\lambda^{(i)}$, service rates $\mu_n^{(i)}$ with $\mu_0^{(i)} = 0$, retrial rates $\theta_n^{(i)}$ with $\theta_0^{(i)} = 0$ and the probabilities $p_n^{(i)}$ and $u_n^{(i)}$ of entering the service facility of customers from outside and from the* *orbit, respectively. Suppose that $X^{(i)}$, $i = 1, 2$ are regular and*

$$\begin{array}{ll} (i) & X^{(1)}(0) = (i,j) \prec X^{(2)}(0) = (i',j'), \\ (ii) & \lambda^{(1)} \le \lambda^{(2)}, \\ (iii) & \mu_n^{(1)} \ge \mu_n^{(2)} \text{ and } \mu_n^{(i)} \le \mu_{n+1}^{(i)},\ i = 1,2,\ n = 0,1,2,\cdots, \\ (iv) & \theta_n^{(1)} \ge \theta_n^{(2)},\ n \ge 0, \\ (v) & p_n^{(1)} \ge p_n^{(2)} \text{ and } p_n^{(i)} \ge p_{n+1}^{(i)},\ i = 1,2,\ n = 0,1,2,\cdots, \\ (vi) & u_n^{(1)} \ge u_n^{(2)} \text{ and } u_n^{(i)} \ge u_{n+1}^{(i)},\ i = 1,2,\ n = 0,1,2,\cdots. \end{array}$$

Then there are versions $\hat{X}^{(i)}$ of $X^{(i)}$, $i = 1, 2$ on a common probability space such that

$$\hat{X}^{(1)} \prec_{as} \hat{X}^{(2)}.$$

Let $\Sigma(c, K, M)$ be the $M/M/c/K$ retrial queue in which at most M customers among them in orbit can retry. Then the standard $M/M/c/K$ retrial queue is denoted by $\Sigma(c, K, \infty)$ and classical $M/M/c$ queue is equivalent to $\Sigma(c, \infty, \infty)$. The next corollary is immediate from Theorem 2.1 by letting the parameters $\mu_n = \min\{c, n\}\mu$, $\theta_n = \min\{M, n\}\theta$ and $p_n = u_n = 1$ for $n \le K - 1$ and $p_n = u_n = 0$ for $n \ge K$.

Corollary 2.2. *Let $\lambda^{(k)}$, $\mu_n^{(k)}$ and $\theta_n^{(k)}$ be the parameters corresponding to the retrial queues $\Sigma(c_k, K_k, M_k)$, $k = 1, 2$ satisfying $\lambda^{(1)} \le \lambda^{(2)}$, $\mu^{(1)} \ge \mu^{(2)}$ and $\theta^{(1)} \ge \theta^{(2)}$. Then for $c_1 \ge c_2$, $K_1 \ge K_2$ and $M_1 \ge M_2$ and the initial values (i_k, j_k) of the systems $\Sigma(c_k, K_k, M_k)$, $k = 1, 2$, satisfying $(i_1, j_1) \prec (i_2, j_2)$, we have*

$$\Sigma(c_1, K_1, M_1) \prec_{as} \Sigma(c_2, K_2, M_2)$$

where $\prec_{as}$ means that the corresponding CN-processes are related by the partial order $\prec_{as}$.

We denote $\tilde{\Sigma}(c, K, M)$ (with $K \le M$) by the retrial queue with c parallel servers and waiting space K including service space in which the retrial rates becomes infinity as soon as the number of customers in orbit reaches the level M. Falin and Templeton (1997) considered $\tilde{\Sigma}(c, c, M)$ to approximate the system $\Sigma(c, c, \infty)$, but the convergence of the approximation and the stochastic order relation between two systems was not dealt. Here we give a comparison result between $\tilde{\Sigma}(c, K, M)$ and $\Sigma(c, K, \infty)$ and the convergence is proved in section 3.

Corollary 2.3. *Let $\tilde{\Sigma}(c_k, K_k, M_k)$, $k = 1, 2$ be the retrial queues described above with arrival rate $\lambda^{(k)}$, service rate $\mu^{(k)}$ of each server and retrial rate $\theta^{(k)}$, $k = 1, 2$ satisfying $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$ and $\theta^{(1)} \geq \theta^{(2)}$. Let (i_k, j_k) be the initial states of the system $\tilde{\Sigma}(c_k, K_k, M_k)$, $k = 1, 2$. If $c_1 \geq c_2$, $K_1 \geq K_2$, $M_1 \leq M_2$, and $(i_1, j_1) \prec (i_2, j_2)$, then we have*

$$\tilde{\Sigma}(c_1, K_1, M_1) \prec_{as} \tilde{\Sigma}(c_2, K_2, M_2).$$

Corollary 2.4. *Let $\tilde{\Sigma}(c_1, K_1, M_1)$ and $\Sigma(c_2, K_2, \infty)$ be the retrial queue described above with arrival rate $\lambda^{(i)}$, service rate $\mu^{(i)}$ of each serve and retrial rate $\theta^{(i)}$, $i = 1, 2$ satisfying $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$ and $\theta^{(1)} \geq \theta^{(2)}$. Let (i, j) and (i', j') be the initial state of the system $\tilde{\Sigma}(c_1, K_1, M_1)$ and $\Sigma(c_2, K_2, \infty)$, respectively. If $c_1 \geq c_2$, $K_1 \geq K_2$ and $(i, j) \prec (i', j')$, then we have*

$$\tilde{\Sigma}(c_1, K_1, M_1) \prec_{as} \Sigma(c_2, K_2, \infty).$$

We consider a retrial queue with c exponential servers and waiting room of size K in service facility and at most M customers in the orbit can attempt to get service. If an arriving customer from outside finds available capacity in service facility, then the customer enters to the facility and if the customer finds the service facility is full, then the customer either joins the orbit with probability α or leaves the system with probability $1 - \alpha$. Customers in orbit return to the service facility after an exponential time. The returning customers from the orbit behave similar fashion to the arriving customer from outside with corresponding probability β. We denote this system by $\Sigma_I(c, K, M)$. Note that letting $\alpha = \beta = 1$, $\Sigma_I(c, K, M)$ becomes $\Sigma(c, K, M)$.

Corollary 2.5. *Let $\Sigma_I^{(i)}(c, K, M_i)$ be retrial queues with parameters $\lambda^{(i)}$,$\mu^{(i)}$, $\theta^{(i)}$, $\alpha^{(i)}$ and $\beta^{(i)}$, $i = 1, 2$. Suppose that $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \geq \mu^{(2)}$, $\theta^{(1)} \geq \theta^{(2)}$, $\alpha^{(1)} \leq \alpha^{(2)}$ and $\beta^{(1)} \leq \beta^{(2)}$. Then for $M_1 \geq M_2$, we have*

$$\Sigma_I^{(1)}(c, K, M_1) \prec_{as} \Sigma_I^{(2)}(c, K, M_2).$$

Remark 2.1. Let $\hat{\Sigma}(c, c, M)$ denote the $M/M/c/c$ retrial queue with finite buffer M of orbit, in which if the number of customers in orbit equals M then the blocked customers are lost and have no influence on the functioning of the system. Falin and Templeton (1997) used the system $\hat{\Sigma}(c, c, M)$ to approximate the system $\Sigma(c, c, \infty)$ and showed that

$$\hat{\Sigma}(c, c, M) \leq_{as} \hat{\Sigma}(c, c, M + 1) \leq_{as} \Sigma(c, c, \infty),$$

where the partial order $(i,j) \leq (i',j')$ means $i \leq i'$ and $j \leq j'$ which implies the order $\prec$. However, it can be seen from the proof of Theorem 2.1 that the results in this section do not hold for the relation $\leq_{as}$.

3. CONVERGENCE OF BOUNDING METHOD

Recalling Corollary 2.2 – Corollary 2.4 we obtain the relation

$$\tilde{\Sigma}(c,K,M) \prec_{as} \Sigma(c,K,\infty) \prec_{as} \Sigma(c,K,M'),\ \ M,M' \geq 0.$$

It can be seen that a sufficient condition for the systems $\tilde{\Sigma}(c,K,M)$, $\Sigma(c,K,\infty)$ and $\Sigma(c,K,M')$ with arrival rate λ and service rate μ of each server to be ergodic is $\rho = \frac{\lambda}{c\mu} < 1$, by taking the Lyapunov function $f(i,j) = ai + j$, $\rho < a < 1$, $(i,j) \in E$. We assume that $\rho < 1$ and let $\tilde{p}_{ij}^{(M)}$, p_{ij} and $p_{ij}^{(M)}$ be the stationary distributions of the CN-processes in $\tilde{\Sigma}(c,K,M)$, $\Sigma(c,K,\infty)$ and $\Sigma(c,K,M')$, respectively. Next theorem guarantees the convergence of the algorithm that uses the stationary distribution of $\tilde{\Sigma}(c,K,M)$ or $\Sigma(c,K,M')$ to approximate that of $\Sigma(c,K,\infty)$ (e.g. Neuts and Rao (1990) and Falin and Templeton (1997)).

Theorem 3.1.

$$\lim_{M\to\infty} \tilde{p}_{ij}^{(M)} = p_{ij} = \lim_{M\to\infty} p_{ij}^{(M)},\ \ 0 \leq i \leq K,\ j \geq 0$$

Proof: Let $I_{(i,j)} = \{(i',j') : (i,j) \prec (i',j'),\ 0 \leq i', 0 \leq j'\}$ be the increasing set at the point (i,j) and $\bar{p}_{ij}^{(M)} = \sum_{(i',j')\in I_{(i,j)}} p_{i'j'}^{(M)}$ and $\bar{p}_{ij}$ the corresponding one to $\{p_{ij}\}$. We have from Corollary 2.2 that

$$\bar{p}_{ij}^{(M)} \geq \bar{p}_{ij}^{(M+1)} \geq \bar{p}_{ij},$$

and hence that for each $(i,j) \in E$, the sequence $\{\bar{p}_{ij}^{(M)}, M = 1,2,\cdots\}$ is monotone and bounded. Thus the limit $a_{ij} = \lim_{M\to\infty} \bar{p}_{ij}^{(M)}$ exists and the inequality $a_{ij} \geq \bar{p}_{ij}$ hods. Since

$$p_{ij}^{(M)} = \bar{p}_{ij}^{(M)} - \bar{p}_{i-1,j+1}^{(M)} - \bar{p}_{i+1,j}^{(M)} + \bar{p}_{i,j+1}^{(M)},$$

there exists the limits $\lim_{M\to\infty} p_{ij}^{(M)} = b_{ij}$ and we have the relations

$$b_{ij} = a_{ij} + a_{i,j+1} - a_{i-1,j+1} - a_{i+1,j}$$

and $\bar{b}_{ij} = a_{ij}$. Note that

$$\bar{b}_{00} = a_{00} = \lim_{M\to\infty} \bar{p}_{00}^{(M)} = 1.$$

It is easily seen that $p_{ij}^{(M)}$ satisfies the following balance equations : for $0 \le i \le K-1$ and $j \ge 0$,

$$(\lambda + \mu_i + \theta_j^{(M)})p_{ij}^{(M)} = \lambda p_{i-1,j}^{(M)} + \theta_{j+1}^{(M)} p_{i-1,j+1}^{(M)} + \mu_{i+1} p_{i+1,j}^{(M)},$$

and for $i = K, j \ge 0$

$$(\lambda + \mu_K)p_{Kj}^{(M)} = \lambda p_{K-1,j}^{(M)} + \theta_{j+1}^{(M)} p_{K-1,j+1}^{(M)},$$

where $\mu_i = \min(i,c)\mu$ and $\theta_j^{(M)} = \min(j,M)\theta$ and $p_{-1,j}^{(M)} = 0$. Taking the limits $M \to \infty$ of both sides in the above balance equations, we get the quantities $\{b_{ij}\}$ satisfy the balance equations for $\Sigma(c,K,\infty)$:

$$\begin{aligned}(\lambda + \mu_i + \theta_j)p_{ij} &= \lambda p_{i-1,j} + \theta_{j+1} p_{i-1,j+1} + \mu_{i+1} p_{i+1,j}, \quad 0 \le i \le K-1, \ j \ge 0, \\ (\lambda + \mu_K)p_{Kj} &= \lambda p_{K-1,j} + \theta_{j+1} p_{K-1,j+1}, \quad i = K, \ j \ge 0,\end{aligned}$$

where $\theta_j = j\theta$ and $p_{-1,j} = 0$. Since $\bar{b}_{00} = 1$, we conclude that $b_{ij} = p_{ij}$. By the similar way, we can show that $\lim_{M\to\infty} \tilde{p}_{ij}^{(M)} = p_{ij}$. □

Remark 3.1. The stationary distribution of $\Sigma_I(c,K,M)$ converges to that of $\Sigma_I(c,K,\infty)$ as $M \to \infty$ can be proved by the same procedure in the proof of Theorem 3.1.

4. PROOFS

In this section we present proofs of theorem and corollaries in section 2. The corollaries can be proved by following the same procedure of Theorem 2.1. Thus we prove Theorem 2.1 in detail and just sketch the proof of Corollary 2.3 and we omit the proofs of Corollary 2.4 and Corollary 2.5.

Proof of Theorem 2.1 : We prove the theorem by the way of introducing auxiliary Poisson processes and Bernoulli processes and constructing two stochastic processes $\hat{X}^{(k)}$, $k = 1, 2$ keeping the order $\hat{X}^{(1)}(t) \prec \hat{X}^{(k)}(t)$ and finally showing that $\hat{X}^{(k)}$ has the same distribution as $X^{(k)}$, $k = 1, 2$.

(1) *Construction of auxiliary processes* :

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which the following independent Poisson processes are defined:

1. $A^{(2)} = \{A^{(2)}(t), t \ge 0\}$ with rate $\lambda^{(2)}$.
2. $S_{ij} = \{S_{ij}(t), t \ge 0\}$ with rate $\mu_{ij} = \max\{\mu_i^{(1)}, \mu_j^{(2)}\}$, $(i,j) \in E$.

3. $R_{ij} = \{R_{ij}(t), t \geq 0\}$ with rate $\theta_{ij} = \max\{\theta_i^{(1)}, \theta_j^{(2)}\}$, $(i,j) \in E$.

We construct a set of Poisson processes:

4. $A^{(1)} = \{A^{(1)}(t), t \geq 0\}$ with rate $\lambda^{(1)}$ by thinning $A^{(2)}$.

5. $S_{ij}^{(1)}$ with rate $\mu_i^{(1)}$ and $S_{ij}^{(2)}$ with rate $\mu_j^{(2)}$ by thinning S_{ij}, $(i,j) \in E$.

6. $R_{ij}^{(1)}$ with rate $\theta_i^{(1)}$ and $R_{ij}^{(2)}$ with rate $\theta_j^{(2)}$ by thinning R_{ij}, $(i,j) \in E$.

Then it is easily seen that for each i, $S_{ij}^{(1)}$ (resp. $R_{ij}^{(1)}$), $j = 0, 1, 2, \cdots$, are independent Poisson processes and $S_{ij}^{(1)}$ has the same distribution as the Poisson process, say $S_i^{(1)}$ (resp. $R_j^{(1)}$) with rate $\mu_i^{(1)}$ (resp. $\theta_i^{(1)}$). Similarly, we denote $S_j^{(2)}$ and $R_j^{(2)}$ by the representatives of the Poisson processes $S_{ij}^{(2)}$ and $R_{ij}^{(2)}$ with rates $\mu_j^{(2)}$ and $\theta_j^{(2)}$, respectively. The processes $A^{(k)}$, $S_{ij}^{(k)}$ and $R_{ij}^{(k)}$ correspond to the arrival process, service process and retrial process, respectively. Let $a^{(k)}(t)$, $d_{ij}^{(k)}(t)$ and $r_{ij}^{(k)}(t)$ be the forward recurrence times of the processes $A^{(k)}$, $S_{ij}^{(k)}$ and $R_{ij}^{(k)}$, respectively, at time t. We see from the above observations that the distributions of $d_{ij}^{(1)}(t)$ and $r_{ij}^{(1)}(t)$ (resp. $d_{ij}^{(2)}(t)$ and $r_{ij}^{(2)}(t)$) do not depend on the index j (resp. i). Let $\mathcal{H}(t)$ be the completion of the σ-field generated by $\{A^{(k)}(s), S_{ij}(s), S_{ij}^{(k)}(s), R_{ij}(s), R_{ij}^{(k)}(s),\ 0 \leq s \leq t,\ k = 1, 2,\ (i,j) \in E\}$ and $\mathcal{H}(\infty) = \bigvee_{t \geq 0} \mathcal{H}(t)$ the σ-field generated by $\mathcal{H}(t)$, $t \geq 0$.

For the entering processes to the service facility, on the probability space $(\Omega, \mathcal{F}, P)$, we introduce a set of independent Bernoulli processes which are also independent of $\mathcal{H}(\infty)$:

7. $\{B_{ij}(n), n = 1, 2, \cdots\}$ with the success probabilities $p_{ij} = \max\{p_i^{(1)}, p_j^{(2)}\}$, $(i,j) \in E$

8. $\{U_{ij}(n), n = 1, 2, \cdots\}$ with the success probabilities $u_{ij} = \max\{u_i^{(1)}, u_j^{(2)}\}$, $(i,j) \in E$.

The Bernoulli processes $\{B_{ij}(n)\}$ and $\{U_{ij}(n)\}$ will be used in the later for entering processes from outside and orbit, respectively. By thinning $\{B_{ij}(n)\}$, we define two Bernoulli processes $\{B_{ij}^{(k)}(n), n = 1, 2, \cdots\}$, $k = 1, 2$ as follows.

9. If $B_{ij}(n) = 1$, let

$$B_{ij}^{(1)}(n) = \begin{cases} 1, & \text{if } p_{ij} = p_i^{(1)} \\ 1, & \text{with probability } \frac{p_i^{(1)}}{p_{ij}} \text{ if } p_{ij} > p_i^{(1)} \\ 0, & \text{with probability } 1 - \frac{p_i^{(1)}}{p_{ij}} \text{ if } p_{ij} > p_i^{(1)} \end{cases}$$

and

$$B_{ij}^{(2)}(n) = \begin{cases} 1, & \text{if } p_{ij} = p_j^{(2)} \\ 1, & \text{with probability } \frac{p_j^{(2)}}{p_{ij}} \text{ if } p_{ij} > p_j^{(2)} \\ 0, & \text{with probability } 1 - \frac{p_j^{(2)}}{p_{ij}} \text{ if } p_{ij} > p_j^{(2)}. \end{cases}$$

If $B_{ij}(n) = 0$, then let $B_{ij}^{(k)}(n) = 0$, $k = 1, 2$.

Then for each $(i, j) \in E$, we see that $P(B_{ij}^{(1)}(n) = 1) = p_i^{(1)}$ and $P(B_{ij}^{(2)}(n) = 1) = p_j^{(2)}$ and we have from the assumption $p_i^{(k)} \geq p_{i+1}^{(k)}$ that $B_{ij}^{(1)}(n) \geq B_{ij}^{(2)}(n)$ for all $j \geq i \geq 0$. Note that for each i (resp. j), the sequence $\{B_{ij}^{(1)}\}$ (resp. $\{B_{ij}^{(2)}\}$) of Bernoulli processes are infinite independent copies of the Bernoulli process, say $\{B_i^{(1)}(n), n = 1, 2, \cdots\}$ (resp. $\{B_j^{(2)}(n), n = 1, 2, \cdots\}$) with success probability $p_i^{(1)}$ (resp. $p_j^{(2)}$).

10. Similarly, we also construct Bernoulli processes $\{U_{ij}^{(k)}(n), n = 1, 2, \cdots\}$, $k = 1, 2$ with the success probabilities $u_i^{(1)}$ and $u_j^{(2)}$, respectively and let $U_i^{(1)}$ and $U_j^{(2)}$ be the representative processes of $U_{ij}^{(1)}$ and $U_{ij}^{(2)}$, respectively.

(2) *Construction of two (C, N) processes* :

Now we construct two bivariate processes $\hat{X}^{(k)} = \{(\hat{C}^{(k)}(t), \hat{N}^{(k)}(t)),\ t \geq 0\}$, $k = 1, 2$. Let $\hat{X}^{(k)}(0) = (C^{(k)}(0), N^{(k)}(0))$, $k = 1, 2$ and we define random variables τ_n, $n \geq 1$ and T_n, $n \geq 0$ with $T_0 \equiv 0$ recursively as follows: On $\{\hat{X}^{(1)}(T_{n-1}) = (i, j),\ \hat{X}^{(2)}(T_{n-1}) = (i', j')\}$, define

$$\tau_n = \inf\{a^{(k)}(T_{n-1}),\ d_{ii'}^{(k)}(T_{n-1}),\ r_{jj'}^{(k)}(T_{n-1}),\ k = 1, 2\},\ n \geq 1$$

and the nth transition time of $\hat{X}^{(1)}$ or $\hat{X}^{(2)}$

$$T_n = T_{n-1} + \tau_n,\ n \geq 1.$$

For the notational simplicity we let

$$\begin{aligned} a_n^{(k)} &= a^{(k)}(T_{n-1}), \\ d_n^{(k)} &= d_{(\hat{C}^{(1)}(T_{n-1}), \hat{C}^{(2)}(T_{n-1}))}^{(k)}(T_{n-1}), \\ r_n^{(k)} &= r_{(\hat{N}^{(1)}(T_{n-1}), \hat{N}^{(2)}(T_{n-1}))}^{(k)}(T_{n-1}). \end{aligned}$$

We define

$$\hat{X}^{(k)}(t) = \hat{X}^{(k)}(T_{n-1}),\ \text{for } T_{n-1} \leq t < T_n,\ k = 1, 2$$

and on $\{\hat{X}^{(1)}(T_{n-1}) = (i,j),\ \hat{X}^{(2)}(T_{n-1}) = (i',j')\}$,

$$\hat{X}^{(1)}(T_n) = \begin{cases} (i + B_{ii'}^{(1)}(n), j+1-B_{ii'}^{(1)}(n)), & \text{if } \tau_n = a_n^{(1)} \\ (i-1,j), & \text{if } \tau_n = d_n^{(1)} \\ (i+U_{ii'}^{(1)}(n), j-U_{ii'}^{(1)}(n)), & \text{if } \tau_n = r_n^{(1)} \\ (i,j), & \text{otherwise,} \end{cases}$$

$$\hat{X}^{(2)}(T_n) = \begin{cases} (i' + B_{ii'}^{(2)}(n), j'+1-B_{ii'}^{(2)}(n)), & \text{if } \tau_n = a_n^{(2)} \\ (i'-1,j'), & \text{if } \tau_n = d_n^{(2)} \\ (i'+U_{ii'}^{(2)}(n), j'-U_{ii'}^{(2)}(n)), & \text{if } \tau_n = r_n^{(2)} \\ (i',j'), & \text{otherwise.} \end{cases}$$

We now prove that if $\hat{X}^{(1)}(T_{n-1}) = (i,j) \prec \hat{X}^{(2)}(T_{n-1}) = (i',j')$, then $\hat{X}^{(1)}(T_n) \prec \hat{X}^{(2)}(T_n)$. Note that two pairs (i,j) and (i',j') in E with $(i,j) \prec (i',j')$ satisfy one of the two cases (i) $i \le i'$ and $j \le j'$ or (ii) $i > i'$ and $j+1 \le j+(i-i') \le j'$. It is noted from the construction of $a_n^{(k)}$, $k=1,2$ that $a_n^{(2)} \le a_n^{(1)}$. Thus at the epoch T_n, we have one of the following five possibilities, (i) $\tau_n = a_n^{(2)} \le a_n^{(1)}$, (ii) $\tau_n = d_n^{(1)} \le d_n^{(2)}$, (iii) $\tau_n = d_n^{(2)} < d_n^{(1)}$, (iv) $\tau_n = r_n^{(1)} \le r_n^{(2)}$, (v) $\tau_n = r_n^{(2)} < r_n^{(1)}$. Comparing the states of $\hat{X}^{(k)}(T_n)$ in each case, we have the following.

(i) $\tau_n = a_n^{(2)} \le a_n^{(1)}$: Since $B_{ii'}^{(1)}(n) \ge B_{ii'}^{(2)}(n)$ for $i \le i'$, it is immediate that $\hat{X}^{(1)}(T_n) \prec \hat{X}^{(2)}(T_n)$ for $i \le i'$. If $i > i'$ and $j+1 \le j+(i-i') \le j'$, then

$$\hat{X}^{(1)}(T_n) \prec (i, j+1) \prec (i'+1, j') \prec \hat{X}^{(2)}(T_n).$$

(ii) $\tau_n = d_n^{(1)} \le d_n^{(2)}$: In this case it is trivial that

$$\hat{X}^{(1)}(T_n) = (i-1, j) \prec (i'-1, j') \prec \hat{X}^{(2)}(T_n).$$

(iii) $\tau_n = d_n^{(2)} < d_n^{(1)}$: From the assumption $\mu_k^{(1)} \ge \mu_k^{(2)}$, this case can occur only when $i < i'$ and hence we have

$$\hat{X}^{(1)}(T_n) = (i, j) \prec (i'-1, j') = \hat{X}^{(2)}(T_n).$$

(iv) $\tau_n = r_n^{(1)} \le r_n^{(2)}$: Note that for $i \le i'$, $U_{ii'}^{(1)}(n) \ge U_{ii'}^{(2)}(n)$ and hence we have $\hat{X}^{(1)}(T_n) \prec \hat{X}^{(2)}(T_n)$ for $i \le i'$. If $i > i'$, then $j+1 \le j+(i-i') \le j'$ and hence

$$\hat{X}^{(1)}(T_n) \prec (i, j) \prec (i'+1, j'-1) \prec \hat{X}^{(2)}(T_n).$$

(*v*) $\tau_n = r_n^{(2)} < r_n^{(1)}$: This case can happen only when $j < j'$ and this implies

$$\hat{X}^{(1)}(T_n) = (i,j) \prec (i'+1, j'-1) = \hat{X}^{(2)}(T_n).$$

In any case we have that $\hat{X}^{(1)}(T_n) \prec \hat{X}^{(2)}(T_n)$. By the mathematical induction we conclude

$$\hat{X}^{(1)}(t) \prec \hat{X}^{(2)}(t), \text{ for all } 0 \le t \le \zeta = \lim_{n\to\infty} T_n.$$

It is noted that from the regularity assumption of $X^{(k)}$, $P(\zeta = \infty) = 1$.

(3) *Equivalence of* $\hat{X}^{(k)}$ *and* $X^{(k)}$, $k = 1, 2$:

It remains to show that $\{\hat{X}^{(k)}(t),\ 0 \le t < \infty\}$ has the same distribution as $\{X^{(k)}(t),\ 0 \le t < \infty\}$, $k = 1, 2$. We prove it only the case of $\hat{X}^{(1)}$, since the case of $\hat{X}^{(2)}$ can be proved by similar way.

Let $\mathcal{B}_n$ and $\mathcal{U}_n$ be the σ-fields generated by $\{B_{ij}(l), B_{ij}^{(k)}(l),\ k = 1,2,\ (i,j) \in E,\ l \le n\}$ and $\{U_{ij}(l), U_{ij}^{(k)}(l),\ k = 1,2,\ (i,j) \in E,\ l \le n\}$, respectively. We denote $\mathcal{F}(t)$ by the completion of the σ-field generated by $\mathcal{H}(t)$, $\mathcal{B}_{m(t)}$ and $\mathcal{U}_{m(t)}$, where $m(t) = \sup\{n : T_n \le t\}$. Then each T_n is a stopping time of the family $\{\mathcal{F}(t),\ t \ge 0\}$ and the processes $A^{(k)}$, $S_{ij}^{(k)}$ and $R_{ij}^{(k)}$ are strong Markov chains with respect to the family $\{\mathcal{F}(t)\}$.

Let $T_n^{(1)}$ be the nth transition time of $\hat{X}^{(1)}$ with $T_0^{(1)} = 0$ and $\tau_n^{(1)} = T_n^{(1)} - T_{n-1}^{(1)}$, $n = 1, 2, \cdots$. Note that the sequence $\{T_n^{(1)}\}$ is a subsequence of $\{T_n\}$ and each $T_n^{(1)}$ is a stopping time with respect to $\mathcal{F}(t)$. For the notational simplicity, we let

$$\hat{X}_n^{(1)} = (\hat{C}_n^{(1)}, \hat{N}_n^{(1)}) = \hat{X}^{(1)}(T_n^{(1)}), \quad \mathcal{F}_n^{(1)} = \mathcal{F}(T_n^{(1)}),\ n \ge 0.$$

In order to prove that $\hat{X}^{(1)}$ and $X^{(1)}$ have the same distribution, it suffices to show the following:

(*i*) Given $\{\hat{X}_n^{(1)},\ n \ge 0\}$, $\{\tau_n^{(1)}, n \ge 1\}$ are conditionally independent, and on $\{\hat{X}_{n-1}^{(1)} = (i,j)\}$,

$$\begin{aligned} P(\tau_n^{(1)} > t \mid \hat{X}_k^{(1)}, 0 \le k \le n-1) &= P(\tau_n^{(1)} > t \mid \hat{X}_{n-1}^{(1)}) \\ &= \exp(-(\lambda^{(1)} + \mu_i^{(1)} + \theta_j^{(1)})t). \qquad (4.1) \end{aligned}$$

(*ii*) $\{\hat{X}_n^{(1)}, \mathcal{F}_n^{(1)},\ n \ge 0\}$ is a Markov chain with state space E and transition probabilities are

$$P\{\hat{X}_{n+1}^{(1)} = (\nu, l) | \hat{X}_n^{(1)} = (i,j)\}$$
$$= \frac{1}{\lambda^{(1)} + \mu_i^{(1)} + \theta_j^{(1)}} \times \begin{cases} \lambda^{(1)} p_i^{(1)}, & \text{if } (\nu, l) = (i+1, j) \\ \lambda^{(1)}(1 - p_i^{(1)}), & \text{if } (\nu, l) = (i, j+1) \\ \mu_i^{(1)}, & \text{if } (\nu, l) = (i-1, j) \\ \theta_j^{(1)} u_i^{(1)}, & \text{if } (\nu, l) = (i+1, j-1) \\ \theta_j^{(1)}(1 - u_i^{(1)}), & \text{if } (\nu, l) = (i, j). \end{cases} \tag{4.2}$$

Given $\hat{X}^{(1)}(t) = (i,j)$, the time period $\tau^{(1)}(t)$ from t to the epoch that the next transition of $\hat{X}^{(1)}$ occurs is the minimum of the forward recurrence times of the processes $\tilde{A}^{(1)}(t) \equiv \{A^{(1)}(t+u), u \geq 0\}$, $\tilde{S}_i^{(1)}(t) \equiv \{S^{(1)}_{i,\hat{C}^{(2)}(t+u)}(t+u), u \geq 0\}$ and $\tilde{R}_j^{(1)}(t) \equiv \{R^{(1)}_{j,\hat{N}^{(2)}(t+u)}(t+u), u \geq 0\}$. Since the distributions of $S_{ii'}^{(1)}$ and $R_{jj'}^{(1)}$ do not depend on the second indices i' and j', respectively, the forward recurrence times of the modulated processes $\tilde{S}_i^{(1)}(t)$ and $\tilde{R}_i^{(1)}(t)$ have the same distributions as those of $d_{ii'}^{(1)}(t)$ and $r_{jj'}^{(1)}(t)$, respectively. Thus given $\hat{X}^{(1)}(t) = (i,j)$, the distribution of $\tau^{(1)}(t)$ is the same as that $\min\{a^{(1)}(t),\ d_{ii'}^{(1)}(t),\ r_{jj'}^{(1)}(t)\}$ and hence is exponential with parameter $\lambda^{(1)} + \mu_i^{(1)} + \theta_j^{(1)}$. Since each of independent processes $A^{(1)}$, $S_{ii'}^{(1)}$ and $R_{jj'}^{(1)}$ has independent increments and has strong Markov property, given $\hat{X}_{n-1}^{(1)} = (i,j)$, $\tau_n^{(1)}$ is independent of $\{\hat{X}_k^{(1)}, 0 \leq k \leq n-2\}$ and is exponentially distributed with parameter $\lambda^{(1)} + \mu_i^{(1)} + \theta_j^{(1)}$. It is immediate by the construction that when $\{\hat{X}_n^{(1)}, n \geq 0\}$ is given, $\{\tau_n^{(1)},\ n \geq 1\}$ are independent. The conditional probabilities (4.2) is immediate from the distribution of $\tau_n^{(1)}$ and the construction of $\hat{X}_n^{(1)}$. It remains to show that $\{\hat{X}_n^{(1)}, \mathcal{F}_n^{(1)}\}$ is a Markov chain. Let $m_n^{(1)} = m(T_n^{(1)})$ and

$$\tilde{A}_n^{(1)} = \{A^{(1)}(t + T_n^{(1)}), 0 \leq t \leq \tau_{n+1}^{(1)}\},$$
$$\tilde{S}_n^{(1)} = \{S^{(1)}_{(\hat{C}_n^{(1)}, \hat{C}^{(2)}(t+T_n^{(1)}))}(t + T_n^{(1)}), 0 \leq t \leq \tau_{n+1}^{(1)}\},$$
$$\tilde{R}_n^{(1)} = \{R^{(1)}_{(\hat{N}_n^{(1)}, \hat{N}^{(2)}(t+T_n^{(1)}))}(t + T_n^{(1)}), 0 \leq t \leq \tau_{n+1}^{(1)}\},$$
$$\tilde{B}_n^{(1)} = \{B^{(1)}_{(\hat{C}_n^{(1)}, \hat{C}^{(2)}(t+T_n^{(1)}))}(k), 0 \leq t \leq \tau_{n+1}^{(1)},\ m_n^{(1)} \leq k \leq m_{n+1}^{(1)}\},$$
$$\tilde{U}_n^{(1)} = \{U^{(1)}_{(\hat{C}_n^{(1)}, \hat{C}^{(2)}(t+T_n^{(1)}))}(k), 0 \leq t \leq \tau_{n+1}^{(1)},\ m_n^{(1)} \leq k \leq m_{n+1}^{(1)}\}.$$

Note that $\hat{X}_{n+1}^{(1)}$ is completely determined by $\hat{X}_n^{(1)}$, $\tilde{A}_n^{(1)}$, $\tilde{S}_n^{(1)}$, $\tilde{R}_n^{(1)}$, $\tilde{B}_n^{(1)}$ and $\tilde{U}_n^{(1)}$ and $\hat{X}_n^{(1)}$ is $\mathcal{F}_n^{(1)}$ measurable. Recall that the distribution of the independent processes $S_{ij}^{(1)}$, $R_{ij}^{(1)}$, $B_{ij}^{(1)}$ and $U_{ij}^{(1)}$ do not depend on the second indices j and are the same as those of the representatives $S_i^{(1)}$, $R_i^{(1)}$, $B_i^{(1)}$ and $U_i^{(1)}$, respectively. Since Poisson processes $A^{(1)}$, $S_i^{(1)}$, $R_i^{(1)}$ have strong Markov property, given $\mathcal{F}_n^{(1)}$, the joint conditional distribution of $\tilde{A}_n^{(1)}$, $\tilde{S}_n^{(1)}$, $\tilde{R}_n^{(1)}$, $\tilde{B}_n^{(1)}$ and $\tilde{U}_n^{(1)}$, is the same as the conditional distribution of $\tilde{A}_0^{(1)}$, $\tilde{S}_0^{(1)}$, $\tilde{R}_0^{(1)}$, $\tilde{B}_0^{(1)}$ and $\tilde{U}_0^{(1)}$, given initial condition $\hat{X}_n^{(1)}$. Consequently we have that

$$P(X_{n+1}^{(1)} = (\nu, l)|\mathcal{F}_n^{(1)}) = P(X_{n+1}^{(1)} = (\nu, l)|X_n^{(1)})$$

which proves that $\{X_n^{(1)}, \mathcal{F}_n^{(1)},\ n \geq 0\}$ is a Markov chain. □

Proof of Corollary 2.3 : The notations used here have the same meaning as those in the proof of Theorem 2.1 unless there are any other comments about them. The state space of the CN-process of the system $\tilde{\Sigma}(c_i, K_i, M_i)$ is $E^{(i)} = E_1^{(i)} \cup E_2^{(i)} \cup E_3^{(i)}$, where $E_1^{(i)} = \{0, 1, \cdots, K_i - 1\} \times \{0, 1, \cdots, M_i - 1\}$, $E_2^{(i)} = \{K_i\} \times \{0, 1, \cdots, M_i - 1\}$ and $E_3^{(i)} = \{K_i\} \times \{j : j \geq M_i\}$, $i = 1, 2$. The parameters in Theorem 2.1 becomes here as follows: $\mu_n^{(i)} = \min(n, c_i)\mu^{(i)}$, $p_n^{(i)} = u_n^{(i)} = 1$ for $n < K_i$ and $p_n^{(i)} = u_n^{(i)} = 0$ for $n = K_i$, $i = 1, 2$ and $\theta_n^{(i)} = n\theta^{(i)}$ for $n < M_i$ and $\theta_n^{(i)} = \infty$ for $n \geq M_i$, $i = 1, 2$.

Let $R_{ij} = \{R_{ij}(t), t \geq 0\}$ be independent Poisson processes with rates $\theta_{ij} = \max\{\theta_i^{(1)}, \theta_j^{(2)}\}$, $i < M_1$ and $j < M_2$. The $A^{(k)}$, S_{ij}, $S_{ij}^{(k)}$ are the same as those in the proof of Theorem 2.1. It is noted that we need not the Bernoulli processes B_{ij} and U_{ij} here. We define random variables τ_n, $n \geq 1$ and T_n, $n \geq 0$ with $T_0 \equiv 0$ recursively as follows:

$$T_n = T_{n-1} + \tau_n,\ n \geq 1$$

with

$$\tau_n = \inf\{a_n^{(k)},\ d_n^{(k)},\ r_n^{(k)},\ k = 1, 2\},$$

where $a_n^{(k)} = a^{(k)}(T_{n-1})$ and on the event $\{\hat{X}^{(1)}(T_{n-1}) = (i, j),\ \hat{X}^{(2)}(T_{n-1}) = (i', j')\}$, $d_n^{(k)}$ and $r_n^{(k)}$ are given by

$$d_n^{(k)} = \begin{cases} d_{ii'}^{(k)}(T_{n-1}), & \text{if } j < M_1,\ j' < M_2 \\ d_{i,i'+j'}^{(k)}(T_{n-1}), & \text{if } j < M_1,\ j' \geq M_2 \\ d_{i+j,i'}^{(k)}(T_{n-1}), & \text{if } j \geq M_1,\ j' < M_2 \\ d_{i+j,i'+j'}^{(k)}(T_{n-1}), & \text{if } j \geq M_1,\ j' \geq M_2, \end{cases}$$

and $r_n^{(k)} = r_{jj'}^{(k)}(T_{n-1})$ for $j < M_1$, $j' < M_2$, and $r_n^{(1)} = \infty$ for $j \geq M_1$, $r_n^{(2)} = \infty$ for $j' \geq M_2$.

We define

$$\hat{X}^{(k)}(t) = \hat{X}^{(k)}(T_{n-1}), \text{ for } T_{n-1} \le t < T_n, \; k = 1, 2$$

and on $\{\hat{X}^{(1)}(T_{n-1}) = (i,j),\, \hat{X}^{(2)}(T_{n-1}) = (i',j')\}$, if $(i,j) \in E_1^{(1)}$, then

$$\hat{X}^{(1)}(T_n) = \begin{cases} (i+1, j), & \text{if } \tau_n = a_n^{(1)} \\ (i-1, j), & \text{if } \tau_n = d_n^{(1)} \\ (i+1, j-1), & \text{if } \tau_n = r_n^{(1)} \\ (i,j), & \text{otherwise,} \end{cases}$$

and if $(i,j) \in E_2^{(1)}$, then

$$\hat{X}^{(1)}(T_n) = \begin{cases} (K_1, j+1), & \text{if } \tau_n = a_n^{(1)} \\ (K_1 - 1, j), & \text{if } \tau_n = d_n^{(1)} \\ (i,j), & \text{otherwise,} \end{cases}$$

and if $(i,j) \in E_3^{(1)}$, then

$$\hat{X}^{(1)}(T_n) = \begin{cases} (K_1, j+1), & \text{if } \tau_n = a_n^{(1)} \\ (K_1, j-1), & \text{if } \tau_n = d_n^{(1)} \\ (i,j), & \text{otherwise} \end{cases}$$

and $\hat{X}^{(2)}(T_n)$ is defined by symmetric way. We can prove that if $\hat{X}^{(1)}(T_{n-1}) = (i,j) \prec \hat{X}^{(2)}(T_{n-1}) = (i',j')$, then $\hat{X}^{(1)}(T_n) \prec \hat{X}^{(2)}(T_n)$ by observing the nine cases obtained by the combinations of three cases $(i,j) \in E_k^{(1)}$, $k = 1, 2, 3$ for (i,j)-pair and three cases $(i',j') \in E_k^{(2)}$, $k = 1, 2, 3$ for (i',j')-pair. The remaining part of the proof of this corollary is quite similar to that of Theorem 2.1, hence we omit it. □

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