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Intrinsic Priors for Testing Two Exponential Means with the Fractional Bayes Factor

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ABSTRACT

This article addresses the Bayesian hypothesis testing for the comparison of two exponential means. Conventional Bayes factors with improper noninformative priors are not well defined. The fractional Bayes factor (FBF) of O'Hagan (1995) is used to overcome such a difficulty. We derive proper intrinsic priors, whose Bayes factors are asymptotically equivalent to the corresponding FBFs. We demonstrate our results with three examples.

Keywords: Default Bayes factor, Fractional Bayes factor, Intrinsic prior, Noninformative prior.

1. INTRODUCTION

Bayes factors under proper priors or informative priors have been successful in testing or model selection problems. However, limited information and time constraints often require the use of noninformative priors such as Jeffreys's priors (Jeffreys, 1961) or reference priors (Berger and Bernardo, 1992).

Suppose the data $\mathbf{x}$ has a parametric distribution with density $f(\mathbf{x}|\theta_i)$, where θ_i is a vector of unknown parameters, $i = 1, 2$. Let Θ_i be the parameter space for θ_i, $i = 1, 2$. Let $\pi_i^N(\theta_i)$ be the improper prior density. The Bayes factor B_{21}^N of model H_2 to model H_1 is

$$B_{21}^N = \frac{m_2^N(\mathbf{x})}{m_1^N(\mathbf{x})} = \frac{\int_{\Theta_2} f(\mathbf{x}|\theta_2)\pi_2^N(\theta_2)d\theta_2}{\int_{\Theta_1} f(\mathbf{x}|\theta_1)\pi_1^N(\theta_1)d\theta_1}, \tag{1.1}$$

where $m_1^N(\mathbf{x})$ and $m_2^N(\mathbf{x})$ are the marginal densities under H_1 and H_2 respectively. Since $\pi_i^N(\theta_i)$ (with i being 1 or 2) is improper, it is defined only up to

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an arbitrary constant c_i. Thus, B_{21}^N is defined only up to (c_2/c_1), which is also arbitrary so that the resulting Bayes factor is not well defined. This issue has been initially addressed by several authors including Geisser and Eddy (1979), Spiegelhalter and Smith (1982), and San Martini and Spezzaferri (1984).

Recently two methods have been proposed and often served as default Bayes factors. These methods are the fractional Bayes factor (FBF) of O'Hagan (1995) and the intrinsic Bayes factor (IBF) of Berger and Pericchi (1996). These methodologies provide fully authentic Bayes factors in the absence of subjective prior information (possibly proper prior distributions).

The exponential distribution is probably the most commonly used parametric distribution in life testing, reliability, and other related fields of application. A fairly common example is to see the difference of two treatment groups where the response measurements follow exponential distributions, and the main interest is to compare the means of each group. There are several articles dealing with the comparison of two exponential means using the IBF (cf. Kim (2000); Kim and Sun (2000); Kim *et al.* (2000)). In this article, we conduct a Bayesian test using the FBF criterion for both one sided and two sided hypotheses testing problems.

The format of the paper is organized as follows. In Section 2, we review the concept of the fractional Bayes factor and the intrinsic prior. In Section 3, we derive a class of intrinsic priors for comparing two exponential means. A real dataset and a simulated dataset are analized in Section 4.

2. PRELIMINARIES

It has been seen that the Bayes factor B_{21}^N in (1.1) involves arbitrary constants. One possibility for removing this arbitrariness is to use a portion of the likelihood with a so-called the fraction δ. O'Hagan (1995) proposed the fractional Bayes factor (FBF) as a default Bayes factor. The FBF of model H_2 to model H_1 is

$$B_{21}^F = B_{21}^N \cdot CFA(\delta), \tag{2.1}$$

where the correction $CFA(\delta)$ is defined as

$$CFA(\delta) = \frac{\int_{\Theta_1} L_1^\delta(\theta_1)\pi_1^N(\theta_1)d\theta_1}{\int_{\Theta_2} L_2^\delta(\theta_2)\pi_2^N(\theta_2)d\theta_2}.$$

Here, $L_i(\theta_i)$ is the likelihood function under model H_i, $i = 1, 2$ and δ is a fraction of the likelihood. A commonly suggested choice is $\delta = m/n$, where m is the size of the minimal training sample advocated by Berger and Pericchi (1996) and n is the size of the sample. We will use this choice in our problems. However, the choice of δ may vary to specify for obtaining a stable Bayes factor (cf. O'Hagan (1995); Berger and Pericchi (1998)).

It is of quite interest to find reasonable priors, often called intrinsic priors, so that the regular Bayes factors under these priors (possibly proper) are asymptotically equivalent to the default Bayes factors when the sample size n is large enough. This issue was initiated by Berger and Pericchi (1996), and several intrinsic priors were derived in various settings. See Lingham and Sivaganesan (1997), Berger and Mortera (1999), and Kim (2000) for related work.

Under the regularity conditions in Berger and Pericchi (1996), a set of intrinsic priors denoted by (π_1^I, π_2^I) is a solution of the following system of equations:

$$\begin{cases} \dfrac{\pi_2^I(\phi_2(\theta_1))\pi_1^N(\theta_1)}{\pi_2^N(\phi_2(\theta_1))\pi_1^I(\theta_1)} = B_1^*(\theta_1), \\ \dfrac{\pi_2^I(\theta_2)\pi_1^N(\phi_1(\theta_2))}{\pi_2^N(\theta_2)\pi_1^I(\phi_1(\theta_2))} = B_2^*(\theta_2), \end{cases} \tag{2.2}$$

where for $i = 1, 2$,

$$B_i^*(\theta_i) = \lim_{n\to\infty} CFA(\delta) \text{ under } H_i,$$

and for $i \neq j$,

$$\phi_i(\theta_j) = \lim_{n\to\infty} E_{\theta_j}^{H_j}(\hat{\theta}_i) \text{ under } H_j,$$

with $\hat{\theta}_i$ being the MLE under H_i.

Remark 1. The noninformative priors $\pi_1^N(\theta_1)$ and $\pi_2^N(\theta_2)$ are called starting priors. We note that solutions are not necessarily unique nor proper. It is of interest to find proper intrinsic priors for given starting priors. Once we derive proper intrinsic priors, the fractional Bayes factor B_{21}^F can be replaced by the ordinary Bayes factors B_{21}^I computed with intrinsic priors at least asymptotically.

3. TESTING EXPONENTIAL MEANS

Let $Exp(\mu)$ denote the exponential distribution with mean μ. Suppose that we have independent observations $x_{ij} \sim Exp(\mu_i)$, $i = 1, 2$; $j = 1, 2, \ldots, n_i$. We use the following notation throughout this paper. Let $N = n_1 + n_2$, and let $x_{i\cdot} = \sum_{j=1}^{n_i} x_{ij}$. Assume that $n_1/N \to a$ as $N \to \infty$. Consider the following testing problems,

$$H_1 : \mu_1 = \mu_2, \text{ vs } H_2 : \mu_1 \neq \mu_2,$$

and

$$H_1 : \mu_1 = \mu_2, \text{ vs } H_3 : \mu_1 < \mu_2.$$

The multiple test including all three hypotheses could be possible. However, we could not find intrinsic priors in this setting. (This should be the ultimate goal.) Let μ denote the common value of μ_i under H_1. We employ Jeffreys's priors as starting priors for each model. They are

$$\begin{cases} \pi_1^N(\mu) = 1/\mu \mathbf{1}_{(\mu>0)}, \\ \pi_2^N(\mu_1, \mu_2) = 1/(\mu_1\mu_2)\mathbf{1}_{(\mu_1 \neq \mu_2)}, \\ \pi_3^N(\mu_1, \mu_2) = 1/(\mu_1\mu_2)\mathbf{1}_{(\mu_1 < \mu_2)}. \end{cases}$$

3.1 Test for H_1 versus H_2

Note that the minimal training sample is size of 2. So, the correction factor at $\delta = 2/N$ is

$$CFA_{12}(\frac{2}{N}) = \frac{1}{\Gamma(2n_1/N)\Gamma(2n_2/N)} \cdot \frac{x_{1\cdot}^{2n_1/N} x_{2\cdot}^{2n_2/N}}{(x_{1\cdot} + x_{2\cdot})^2}.$$

Thus, the fractional Bayes factor B_{21}^F of H_2 to H_1 is

$$B_{21}^F = B_{21}^N \cdot CFA_{12}(\frac{2}{N}), \tag{3.1}$$

where the Bayes factor for the full sample is

$$B_{21}^N = \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(N)} \cdot \frac{(x_{1\cdot} + x_{2\cdot})^N}{x_{1\cdot}^{n_1} x_{2\cdot}^{n_2}}.$$

We need to compute $B_1^*(\mu)$ and $B_2^*(\mu_1, \mu_2)$.

Proposition 1. *The quantities $B_1^*(\mu)$ and $B_2^*(\mu_1, \mu_2)$ are given by*

$$B_1^*(\mu) = \frac{a^{2a} b^{2b}}{\Gamma(2a)\Gamma(2b)}, \tag{3.2}$$

and

$$B_2^*(\mu_1, \mu_2) = \frac{a^{2a} b^{2b}}{\Gamma(2a)\Gamma(2b)} \cdot \frac{\mu_1^{2a} \mu_2^{2b}}{(a\mu_1 + b\mu_2)^2}, \tag{3.3}$$

where $b = 1 - a$.

Proof: The result immediately follows from the strong law of large numbers. □

Lemma 1. $B_2^*(\mu_1, \mu_2) \to B_1^*(\mu)$ *as* $(\mu_1, \mu_2) \to (\mu, \mu)$.

Proof: Since $\mu_1^{2a}\mu_2^{2b}/(a\mu_1 + b\mu_2)^2 \to 1$ as $(\mu_1, \mu_2) \to (\mu, \mu)$ in (3.3), we have the result. □

After taking the limit, the system of equations (2.2) becomes

$$\begin{cases} \dfrac{\pi_2^I(\mu_1, \mu_2)/(a\mu_1 + b\mu_2)}{\pi_1^I(a\mu_1 + b\mu_2)/(\mu_1\mu_2)} = B_2^*(\mu_1, \mu_2), \ \mu_1, \mu_2 > 0, \\ \dfrac{\pi_2^I(\mu, \mu)/\mu}{\pi_1^I(\mu)/\mu^2} = B_1^*(\mu), \ \mu > 0, \end{cases} \tag{3.4}$$

where π_i^I are intrinsic priors for $i = 1, 2$, and B_1^* and B_2^* are given respectively by (3.2) and (3.3).

Theorem 1. *For any proper density function $g(t)$, $t > 0$, the system of priors*

$$\begin{cases} \pi_1^I(\mu) = g(\mu), \ 0 < \mu < \infty, \\ \pi_2^I(\mu_1, \mu_2) = \dfrac{a\mu_1 + b\mu_2}{\mu_1\mu_2} B_2^*(\mu_1, \mu_2)\pi_1^I(a\mu_1 + b\mu_2), \ 0 < \mu_1, \mu_2 < \infty, \end{cases} \tag{3.5}$$

is a solution of (3.4), where B_2^* is given by (3.3). Furthermore, π_2^I is a proper density.

Proof: By Lemma 3.1, (3.5) is a solution of (3.4). To prove a propriety, let $s = \mu_1/\mu_2$, $t = \mu_2$. Then

$$\begin{aligned} \int_0^\infty \int_0^\infty \pi_2^I(\mu_1, \mu_2) d\mu_2 d\mu_1 &= B_1^* \int_0^\infty \int_0^\infty \frac{s^{2a-1}}{as + b} g(t(as + b)) dt ds \\ &= B_1^* \int_0^\infty \frac{s^{2a-1}}{(as + b)^2} ds \\ &= 1. \end{aligned}$$

Here, the last equality follows from the kernel of the Beta distribution by the appropriate transformation. □

Corollary 1. *When $g(\cdot)$ is the probability density function of Inverse Gamma(λ,η), the set of intrinsic priors is*

$$\begin{cases} \pi_1^I(\mu) = \dfrac{\eta^\lambda}{\Gamma(\lambda)\mu^{\lambda+1}} \exp\{-\dfrac{\eta}{\mu}\},\ 0<\mu<\infty, \\ \pi_2^I(\mu_1,\mu_2) = \dfrac{\eta^\lambda}{\Gamma(\lambda)} \dfrac{B_2^*(\mu_1,\mu_2)}{\mu_1\mu_2(a\mu_1+b\mu_2)^\lambda} \exp\{-\dfrac{\eta}{a\mu_1+b\mu_2}\},\ 0<\mu_1,\mu_2<\infty. \end{cases} \tag{3.6}$$

3.2 Test for H_1 versus H_3

It can be easily seen that the FBF of H_1 to H_3 is

$$B_{31}^F = B_{31}^N \cdot CFA_{13}(\frac{2}{N}), \tag{3.7}$$

where the Bayes factor for the full sample is

$$B_{31}^N = (x_{1\cdot}+x_{2\cdot})^N \int_0^1 \frac{r^{n_2-1}}{(x_{1\cdot}+x_{2\cdot}r)^N} dr,$$

and the correction factor is

$$CFA_{13}^F(\frac{2}{N}) = \frac{1}{(x_{1\cdot}+x_{2\cdot})^2} \left[\int_0^1 \frac{s^{2n_2/N-1}}{(x_{1\cdot}+x_{2\cdot}s)^2} ds\right]^{-1}.$$

Let us compute $B_i^*(\theta_i)$ for $i=1,3$.

Proposition 2. *The quantities $B_1^*(\mu)$ and $B_3^*(\mu_1,\mu_2)$ are given by*

$$B_1^*(\mu) \ = \ \left[\int_0^1 \frac{s^{2b-1}}{(a+bs)^2} ds\right]^{-1}, \tag{3.8}$$

and

$$B_3^*(\mu_1,\mu_2) = \frac{1}{(a\mu_1+b\mu_2)^2} \left[\int_0^1 \frac{s^{2b-1}}{(a\mu_1+b\mu_2 s)^2} ds\right]^{-1}. \tag{3.9}$$

Proof: The result immediately follows from the strong law of large numbers. □

For the simplicity we assume that $n_1 = n_2$. Then we have the following system of equations:

$$\begin{cases} \dfrac{2\pi_3^I(\mu_1,\mu_2)/(\mu_1+\mu_2)}{\pi_1^I((\mu_1+\mu_2)/2)/(\mu_1\mu_2)} = \dfrac{\mu_1}{\mu_1+\mu_2}, \ 0<\mu_1<\mu_2<\infty, \\ \dfrac{\pi_3^I(\mu,\mu)/\mu}{\pi_1^I(\mu)/\mu^2} = \dfrac{1}{2}, \ \mu>0, \end{cases} \tag{3.10}$$

where π_i^I are intrinsic priors for $i = 1, 3$.

Theorem 2. *For any proper density function $g(t)$, $t > 0$, the system of priors*

$$\begin{cases} \pi_1^I(\mu) = g(\mu), \ 0<\mu<\infty, \\ \pi_3^I(\mu_1,\mu_2) = \dfrac{1}{2\mu_2}\pi_1^I(\dfrac{1}{2}(\mu_1+\mu_2)), \ 0<\mu_1<\mu_2<\infty, \end{cases} \tag{3.11}$$

is a solution of (3.10). Furthermore, π_3^I is a proper density. Here, the normalizing constant is $c = 1/\log 2$ with $c = \int\int \pi_3^I(\mu_1,\mu_2)d\mu_1 d\mu_2$.

Proof: By Lemma 3.2, (3.11) is a solution of (3.10). To prove the propriety, let $s = \mu_1/\mu_2$, $t = \mu_2$. Then

$$\begin{aligned} \int_0^\infty \int_0^{\mu_2} \pi_3^I(\mu_1,\mu_2)d\mu_1 d\mu_2 &= \int_0^1 \int_0^\infty \frac{1}{2} g(t(as+b))dtds \\ &= \log 2. \end{aligned}$$

□

Corollary 2. *When $g(\cdot)$ is the probability density function of Inverse Gamma(λ,η), the set of intrinsic priors is*

$$\begin{cases} \pi_1^I(\mu) = \dfrac{\eta^\lambda}{\Gamma(\lambda)\mu^{\lambda+1}}\exp\{-\dfrac{\eta}{\mu}\}, \ 0<\mu<\infty, \\ \pi_3^I(\mu_1,\mu_2) = \dfrac{1}{2\mu_2}\dfrac{\eta^\lambda}{\Gamma(\lambda)}\dfrac{1}{(\mu_1+\mu_2)^{\lambda+1}}\exp\{-\dfrac{\eta}{\mu_1+\mu_2}\}, \ 0<\mu_1,\mu_2<\infty. \end{cases} \tag{3.12}$$

4. NUMERICAL RESULTS

Example 1. The data in Table 1, given by Lawless (1982), are failure times (in minutes) for two types of electrical insulation in which the insulation was subjected to an increasing voltage stress. The original dataset is assumed to have two-parameter exponential distributions. We subtracted from the data to the MLE for the location parameter. So we may assume that the transformed data follow one parameter exponential distributions heuristically.

Table 1: The failure times for two types of electrical insulation.

Type A	9.5, 58.4, 12.1, 126.3, 139.6, 63.0, 83.2, 85.8, 30.9, 16.3, 34.6
Type B	200.8, 60.9, 67.5, 131.7, 3.2, 103.4, 22.0, 128.6, 16.6, 23.8, 30.2

Let μ_1 and μ_2 denote the mean failure times for Type A and Type B respectively. Suppose that we want to test $H_1 : \mu_1 = \mu_2$ versus $H_2 : \mu_1 \neq \mu_2$. Here $(n_1, n_2, \hat{\mu}_1, \hat{\mu}_2) = (11, 11, 55.0, 65.7)$, where $\hat{\mu}_i$ is the MLE of μ_i, for $i = 1, 2$. We computed the fractional Bayes factor and the Bayes factors using the set of intrinsic priors given by (3.6) with three choices of (λ, η). They are (0.01,0.01), (0.1,0.1), and (1.0,1.0). These computations were done using IMSL routines. The numerical values are reported in Table 2. The Bayes factors with intrinsic priors are quite close to the fractional Bayes factor. Since the Bayes factors are less than 1, one may conclude that the difference between the two types of electrical insulation is fairly small. Furthermore, there is not much difference between each value of (λ, η). Therefore, the Bayes factors using intrinsic priors are quite robust in terms of the hyperparameters (λ, η).

Table 2: Bayes factors for testing $H_1 : \mu_1 = \mu_2$ versus $H_2 : \mu_1 \neq \mu_2$

(λ, η)	$(0.01, 0.01)$	$(0.1, 0.1)$	$(1.0, 1.0)$
FBF	B_{21}^I	B_{21}^I	B_{21}^I
0.293	0.280	0.279	0.265

Example 2. We performed a simulation study for testing H_1 versus H_2. We examined the cases when $\mu_1 = \mu_2 = 1$, and $\mu_1 = 1$, $\mu_2 = 2$ for some choices of n_1 and n_2. We computed the average of the relative differences between the FBF and the Bayes factors with intrinsic priors for three choices of (λ, η) given by (3.6). We used three different replications to see the stability of numerical values. They are 100, 200, and 400. We also computed the standard deviations of relative differences based on each replication. The numerical values are reported in Table

3. The relative differences are quite small for each simulated dataset. Especially, as the sample size increases, the relative difference decreases. This is what we would expect from the theoretical results. We also note that the values are quite stable as the number of the replication increases.

Table 3: Relative difference $|B_{21}^I - B_{21}^F|/B_{21}^F$ for estimating the fractional Bayes factor. The relative difference (R.D.) is averaged over 100, 200, and 400 replications. The numbers in parentheses are the standard deviations of the relative differences, based on each replication (rep.).

		(λ, η)	$(.01, .01)$	$(.1, .1)$	$(1.0, 1.0)$
	(n_1, n_2)	rep.	R.D.	R.D.	R.D.
$\mu_1 = 1.0$	$(10, 10)$	100	0.0421(0.0067)	0.0422(0.0067)	0.0428(0.0111)
$\mu_2 = 1.0$		200	0.0423(0.0069)	0.0424(0.0068)	0.0436(0.0109)
		400	0.0423(0.0074)	0.0424(0.0074)	0.0433(0.0113)
	$(10, 20)$	100	0.0305(0.0070)	0.0306(0.0071)	0.0314(0.0089)
		200	0.0302(0.0071)	0.0302(0.0071)	0.0309(0.0088)
		400	0.0304(0.0077)	0.0305(0.0077)	0.0314(0.0091)
	$(20, 20)$	100	0.0225(0.0028)	0.0226(0.0028)	0.0231(0.0041)
		200	0.0229(0.0021)	0.0229(0.0022)	0.0232(0.0039)
		400	0.0227(0.0024)	0.0227(0.0024)	0.0230(0.0040)
	$(30, 30)$	100	0.0157(0.0009)	0.0157(0.0009)	0.0158(0.0021)
		200	0.0156(0.0012)	0.0156(0.0012)	0.0157(0.0021)
		400	0.0157(0.0010)	0.0157(0.0010)	0.0161(0.0022)
$\mu_1 = 1.0$	$(10, 10)$	100	0.0345(0.0115)	0.0356(0.0117)	0.0456(0.0145)
$\mu_2 = 2.0$		200	0.0299(0.0137)	0.0308(0.0141)	0.0402(0.0174)
		400	0.0304(0.0140)	0.0313(0.0143)	0.0404(0.0169)
	$(10, 20)$	100	0.0139(0.0097)	0.0145(0.0100)	0.0206(0.0127)
		200	0.0157(0.0094)	0.0163(0.0097)	0.0225(0.0125)
		400	0.0160(0.0092)	0.0166(0.0095)	0.0229(0.0123)
	$(20, 20)$	100	0.0152(0.0066)	0.0158(0.0067)	0.0209(0.0080)
		200	0.0153(0.0062)	0.0159(0.0063)	0.0212(0.0074)
		400	0.0161(0.0060)	0.0166(0.0061)	0.0219(0.0071)
	$(30, 30)$	100	0.0106(0.0036)	0.0110(0.0037)	0.0146(0.0042)
		200	0.0106(0.0035)	0.0110(0.0036)	0.0147(0.0041)
		400	0.0105(0.0036)	0.0109(0.0036)	0.0146(0.0042)

Example 3. Let μ_1 and μ_2 denote the mean failure times for Type A and Type B respectively in Table 1 of Example 1. Suppose that we want to test $H_1 : \mu_1 = \mu_2$ versus $H_3 : \mu_1 < \mu_2$. Here $(n_1, n_2, \hat{\mu}_1, \hat{\mu}_2) = (11, 11, 55.0, 65.7)$, where $\hat{\mu}_i$ is the MLE of μ_i, for $i = 1, 2$. We computed the fractional Bayes factor and the Bayes factors using the set of intrinsic priors given by (3.12) with four choices of (λ, η). They are (0.01,0.01), (0.1,0.1), (1.0,1.0), and (10,10). The numerical values are reported in Table 4. There is a little difference between the fractional Bayes factor and the Bayes factors using intrinsic priors. Since the normalizing constant for $\pi_3^I(\mu_1, \mu_2)$ is $1/\log 2$, this causes a calibration problem mentioned by Kim (2000). We can see from Table 4 that the Bayes factors computed by intrinsic priors are close to each other for $(\lambda, \eta) = (0.01, 0.01), (0.1, 0.1), (1.0, 1.0)$. Meanwhile, the Bayes factor using intrinsic priors approximates accurately the fractional Bayes factor for $(\lambda, \eta) = (10, 10)$.

Table 4: Bayes factors for testing $H_1 : \mu_1 = \mu_2$ versus $H_3 : \mu_1 < \mu_2$

(λ, η)	$(0.01, 0.01)$	$(0.1, 0.1)$	$(1.0, 1.0)$	$(10, 10)$
FBF	B_{21}^I	B_{21}^I	B_{21}^I	B_{21}^I
0.354	0.448	0.447	0.434	0.353

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