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A Note on the Weak Negative Dependence Structure[1]

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Abstract

In this paper new results are obtained for multivariate processes which help us to identify weak negative orthant dependent(WNOD) structures among hitting times of the processes. Furthermore, an approach to derive dependence properties among the processes is proposed and a partial solution to the question that what kinds of the dependence properties, when they are imposed on processes, are reflected as analogous properties of corresponding hitting times is given. Examples are given to illustrate these concepts.

Keywords : WNOD, uo-cx(lo-cv), negatively associated, stochastically right tail decre- asing , extreme dependence function, hitting times.

1. Introduction

The theory of positive quadrant dependent(PQD) or negative quadrant dependent (NQD) random variables were initiated by the seminal paper of Lehman(1966). Stronger notions of bivariate positive or negative dependence were later developed by Esary and Proschan(1972) and Yanigimoto(1972). Also, Esary, Proschan and Walkup(1967) introduc ed a notion of associatedness which implies a strong form of positive dependence. Concepts of this dependence have subsequently been extended to stochastic processes in different directions by many authors and a number of aspects of dependence notions have been studied for several decades. For a bibliography of available results, see Friday(1981); recently Ebrahimi and Ramallingam(1989) introduced some dependence concepts in terms of the finite dimensional distributions of the hitting times of the components of a vector process.

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In this paper, we are concerned with weakly negative dependent structures.

The reliability, $\overline{F}(t)$, of a system(component) is the probability that the system will preserve its characteristics within specified limits during a specified time interval $[0, t]$. Suppose a system fails if at least one characteristic of the system shifts outside certain permissible limits. If T is the time to failure, then

$$\overline{F}(t) = P(T > t).$$

Suppose that the system reliability is determined by a finite number of characteristics. For $i = 1, \cdots, n$, denote the value of the ith characteristic at time t by $X_i(t)$ and assume that it is within permissible limits if $X_i(t) < a_i$, where $a_1, \cdots, a_n$ are fixed and known values. For example, we may look upon a_i as the breaking threshold of total damages $X_i(t)$ by time t. Let the random time $T_i(a_i)$, at which the ith characteristic first crosses its limit is given by

$$T_i(a_i) = \begin{cases} \inf\{t \in \Lambda \mid X_i(t) \geq a_i\} \\ \infty \text{ if } X_i(t) < a_i \text{ for } \textit{all } t \in \Lambda, \qquad i = 1, \cdots, n \end{cases} \tag{1.1}$$

where the index set Λ is a subset of $R_+ = [0, \infty)$. In this setting, the failure time of the system , T, is given by

$$T = \min(T_1(a_1), \cdots, T_n(a_n)). \tag{1.2}$$

In view of (1.1) and (1.2),

$$\overline{F}(t) = P(T_1(a_1) > t, \cdots, T_n(a_n) > t). \tag{1.3}$$

Formulation of system reliability by means of (1.1)-(1.3) is relevant to engineering disciplines relating to structural safety, variation of current and voltage, etc.

In general, it is possible to assess the system reliability $\overline{F}(t)$ provided that one can jointly model $X_1(t), \cdots, X_n(t)$ and one can also seek weak probability inequalities for system reliability. To obtain such weak probability inequalities, information about the dependence structure of $T_1(a_1), \cdots, T_n(a_n)$ is essential.

For example if we know that for some $s_1, \cdots, s_n$,

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} (P(\bigcap_{i=1}^{n} T_i(a_i) > x_i) - \prod_{i=1}^{n} P(T_i(a_i) > x_i))\, dx_n \cdots dx_1 \leq 0 \tag{1.4}$$

and

$$\int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} (P(\bigcap_{i=1}^{n} T_i(a_i) > x_i) - \prod_{i=1}^{n} P(T_i(a_i) > x_i))\, dx_n \cdots dx_1 \leq 0, \tag{1.5}$$

then we can assess $P(T_i(a_i) > t_i), i = 1, \cdots, n$ and derive a bound for $\overline{F}(t)$. Besides bounds, information about the dependence structure may bring forth new weak probability inequalities for stochastic processes.

The importance of this paper lies in the fact that the notion introduced is weaker than the notion of negative orthant dependence and also enjoys most of the properties and theoretical results of the latter notion. Certain kinds of dependence properties, when they are imposed on processes, are inflected as analogous properties of corresponding hitting times. These results are of value as they help us to understand in what ways for dependence structures of hitting times can be inherited from the corresponding processes.

In Section 2 of this paper, some notations and definitions are presented. In Section 3, we prove some theorems which help us to identify weak negative dependence structures both among processes and their corresponding hitting times. Finally, in Section 4, we give some examples of processes and hitting times.

2. Notations and definitions

In this section we present definitions, notations, and properties used throughout the paper. In what follows 'increasing' means 'non-decreasing' and 'decreasing' means 'non-increasing'. Suppose that $\{X(t)=(X_1(t), \cdots, X_n(t)) \mid t\in\Lambda\}$ is an n-dimensional stochastic processes, where the index set Λ is a subset of $R_+=[0, \infty]$. The state space of $\{X(t) \mid t\in\Lambda\}$ is the cartesian product $E=E_1\times E_2\times\cdots\times E_n$, which will be a subset of n-dimensional Euclidean space R^n. If the index set Λ is $\{0, 1, 2, \cdots\}$, then

$$P(\bigcap_{i=1}^{n} T_i(a_i) > t_i) = P(\max_{0\le j_i\le [t_i]} X_i(j_i) < a_i,\ i=1, \cdots, n), \tag{2.1}$$

where $[r]$ is the largest integer less than or equal to r.

We now present some concepts of weakly negative dependence, negatively associated, $\{X(t) \mid t\in\Lambda\}$ is smaller than $\{Y(t) \mid t\in\Lambda\}$ in the upper(lower) orthant convex(concave) order, and stochastically right tail decreasing for any n-dimensional stochastic process.

Definition 2.1. A one-dimensional process $\{X(t) \mid t\in\Lambda\}$ is weakly negative upper(lower) orthant dependent(WNUOD(WNLOD)) if it satisfes the following two conditions

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} (P(\bigcap_{i=1}^{n} X(s_i) > a_i) - \prod_{i=1}^{n} P(X(s_i) > a_i)) da_n \cdots da_1 \le 0 \quad \text{(WNUOD1)}$$

$$\left(\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} (P(\bigcap_{i=1}^{n} X(s_i) \le a_i) - \prod_{i=1}^{n} P(X(s_i) \le a_i)) da_n \cdots da_1 \le 0\right) \quad \text{(WNLOD1)}$$

and

$$\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n} (P(\bigcap_{i=1}^{n} X(s_i) > a_i) - \prod_{i=1}^{n} P(X(s_i) > a_i)) da_n \cdots da_1 \le 0 \quad \text{(WNUOD2)}$$

$$\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}(P(\bigcap_{i=1}^{n}X(s_i)\leq a_i)-\prod_{i=1}^{n}P(X(s_i)\leq a_i))da_n\cdots da_1\leq 0\right)\qquad\text{(WNLOD2)}$$

for any $0\leq s_1<s_2<\cdots<s_n$, $s_i\in\Lambda$, $x_i\in E_i$, $i=1,2,\cdots,n$, and for all finite n.

An n-dimensional stochastic process $\{X(t)|t\in\Lambda\}$ is WNUOD(WNLOD) if each comp- onent of it is WNUOD(WNLOD) and for all s_i in E_i and $t_i\in\Lambda, i=1,\ldots,n$, we have

$$\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}(P(\bigcap_{i=1}^{n}X_i(t_i)>a_i)-\prod_{i=1}^{n}P(X_i(t_i)>a_i))\,da_n\cdots da_1\leq 0\ \text{(WNUOD1)}$$

$$\left\{\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}(P(\bigcap_{i=1}^{n}X_i(t_i)\leq a_i)-\prod_{i=1}^{n}P(X_i(t_i)\leq a_i))\,da_n\cdots da_1\leq 0\right\}\text{(WNLOD1)}$$

and

$$\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}(P(\bigcap_{i=1}^{n}X_i(t_i)>a_i)-\prod_{i=1}^{n}P(X_i(t_i)>a_i))\,da_n\cdots da_1\leq 0\text{(WNUOD2)}$$

$$\left(\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}(P(\bigcap_{i=1}^{n}X_i(t_i)\leq a_i)-\prod_{i=1}^{n}P(X_i(t_i)\leq a_i))da_n\cdots da_1\leq 0\right)\text{(WNLOD2)}.$$

Moreover, an n-dimensional stochastic process $\{X(t)|t\in\Lambda\}$ is WNOD if they satisfy both WNUOD and WNLOD. Also, the hitting times $T_1(a_1),\cdots,T_n(a_n)$ are WNUOD if

$$\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}(P(\bigcap_{i=1}^{n}T_i(a_i)>t_i)-\prod_{i=1}^{n}P(T_i(a_i)>t_i))dt_n\cdots dt_1\leq 0\text{(WNUOD1)}$$

and

$$\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}(P(\bigcap_{i=1}^{n}T_i(a_i)>t_i)-\prod_{i=1}^{n}P(T_i(a_i)>t_i))dt_n\cdots dt_1\leq 0\text{(WNUOD2)}$$

for every $s_i\in E_i$ and $t_i\in\Lambda$, $i=1,2,\cdots,n$, and WNLOD can be defined a similar method by $T_i(a_i)$, $i=1,\cdots,n$.

Definition 2.2. A one-dimensional process $\{X(t)|t\in\Lambda\}$ is smaller than $\{Y(t)|t\in\Lambda\}$ in the upper(lower) orthant-convex(concave) order $(X(t)\leq_{uo-cx}Y(t)\ (X(t)\leq_{lo-cv}Y(t)))$ if for any $0\leq s_1<s_2<\cdots<s_n$, $s_i\in\Lambda$, $x_i\in E_i$, $i=1,\cdots,n$, and any finite n,

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}P(\bigcap_{i=1}^{n}X(s_i)>a_i)\,da_n\cdots da_1\leq\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}P(\bigcap_{i=1}^{n}Y(s_i)>a_i)\,da_n\cdots da_1$$

$$\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}P(\bigcap_{i=1}^{n}X(s_i)\leq a_i)\,da_n\cdots da_1\geq\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}P(\bigcap_{i=1}^{n}Y(s_i)\leq a_i)\,da_n\cdots da_1\right).$$

An n-dimensional stochastic process $(X(t)\leq Y(t)_{uo-cx}(X(t)\leq_{lo-cv}Y(t)))$ if each component of it is $uo-cx(lo-cv)$ and for all x_i in E_i and $t_i\in\Lambda, i=1,\cdots,n$,

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}P(\bigcap_{i=1}^{n}X_i(t_i)>a_i)\,da_n\cdots da_1\leq\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}P(\bigcap_{i=1}^{n}Y_i(t_i)>a_i)\,da_n\cdots da_1$$

$$\left(\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}P(\bigcap_{i=1}^{n}X_i(t_i)\leq a_i)\,da_n\cdots da_1\geq\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}P(\bigcap_{i=1}^{n}Y_i(t_i)\leq a_i)\,da_n\cdots da_1\right).$$

Also, the hitting time $(T_1(a_1),\cdots,T_n(a_n))$ is smaller than $(S_1(a_1),\cdots,S_n(a_n))$ in the upper(lower) orthant-convex(concave) order if

$$\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}P(\bigcap_{i=1}^{n}T_i(a_i)>t_i)\,dt_n\cdots dt_1\le\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}P(\bigcap_{i=1}^{n}S_i(a_i)>t_i)\,dt_n\cdots dt_1$$

$$\left(\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}P(\bigcap_{i=1}^{n}T_i(a_i)\le t_i)\,dt_n\cdots dt_1\ge\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}P(\bigcap_{i=1}^{n}S_i(a_i)\le t_i)\,dt_n\cdots dt_1\right)$$

for every $s_i\in E_i$ and $t_i\in\Lambda$, $i=1,2,\cdots,n$.

Definition 2.3. A one-dimensional process $X(t)$ is negatively associated if for $0\le s_1<\cdots<s_n$ and any two disjoint subsets A_1 and A_2 of $\{1,\cdots,n\}$ and for any finite n,

$$Cov(f(X_j(s_i),\ i\in A_1),\ g(X_j(s_i),\ i\in A_2))\le 0,$$

for all increasing real valued functions f and g such that the covariance exists.

An n-dimensional stochastic process $\{X(t)|t\in\Lambda\}$ is negatively associated if each component of it is negatively associated and for every disjoint subsets B_1 and B_2 of $\{1,\cdots,n\}$,

$$Cov(f(X_i(t_i),\ i\in B_1),\ g(X_i(t_i),\ i\in B_2))\le 0$$

for all increasing real valued functions f and g such that the covariance exists and for all $t_i\in\Lambda$, $i=1,\cdots,n$. Also, we say that the hitting times $T_1(a_1),\cdots,T_n(a_n)$ are negati vely associated if every pair of disjoint subsets B_1 and B_2 of $\{1,\cdots,n\}$,

$$Cov(f(T_i(a_i),\ i\in B_1),\ g(T_i(a_i),\ i\in B_2))\le 0$$

for all increasing real valued functions f and g for which the covariance exists.

Definition 2.4. A one-dimensional process $\{X(t)|t\in\Lambda\}$ is stochastically right tail decreasing in $\{Y(t)|t\in\Lambda\}$ if for any $0\le s_1<\cdots<s_n$,

$$E(f(X(s_i))|\,Y(s_i)>y_i,\ i=1,\cdots,n)\text{ is decreasing in }y_1,y_2,\cdots,y_n$$

for every real valued increasing function f.

An n-dimensional stochastic process $\{X(t)\,|\,t\in\Lambda\}$ is stochastically right tail decreasing in $\{Y(t)\,|\,t\in\Lambda\}$ if each component of it is stochastically right tail decreasing in $\{Y(t)|t\in\Lambda\}$ and for all $t_i\in\Lambda$, $i=1,\cdots,n$,

$$E(f(X_i(t_i),i=1,\cdots,n)|\ Y_i(t_i)>y_i,\ i=1,\cdots,n)$$

is decreasing in $y_1,y_2,\cdots,y_n$ for every real valued increasing function f.

Also, we say that the hitting times $T_1(a_1),\cdots,T_n(a_n)$ are stochastically right tail decre asing in $T_1(b_1),\cdots,T_n(b_n)$ if $E(f(T_i(a_i))|\,T_i(b_i)>t_i,\ i=1,\cdots,n)$ is decreasing in $t_i\in\Lambda$ and $a_i,\ b_i\in E_i$ for all $i=1,\cdots,n$.

3. Theoretical Results

First, we prove some theorems which help us to identify weak negative orthant dependent structures among hitting times of the stochastic processes. In this section we will assume that the index set $\Lambda=\{1, 2, \cdots\}$.

Theorem 3.1.(a) Let a one-dimensional process $\{X_1(t)|t\in\Lambda\}$ be WNOD. Then $T(a_1), \cdots, T(a_n)$ are WNOD, where $T(a_i)=\inf\{n|X_1(n)\geq a_i\}$, $i=1, \cdots, n$.
(b) Let a one-dimensional process $\{X_1(t)|t\in\Lambda\}$ be WNOD and let f_i be increasing functions, $i=1, 2, \cdots, n$. Then $f_1(T(a_1)), \cdots, f_n(T(a_n))$ are WNOD, where $T(a_i)=\inf\{n|\ X_1(n)\geq a_i\}$, $i=1, \cdots, n$.

Proof of (a). We will prove this theorem for the case $n=2$ only. For the general $n>2$, the proof is similar. Suppose $X_1(t)$ is WNOD, then we need to show that for $a_1\leq a_2$,

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty}P(\bigcap_{i=1}^{2}T(a_i)>t_i)\,dt_2\,dt_1\leq\int_{x_1}^{\infty}\int_{x_2}^{\infty}\prod_{i=1}^{2}P(T(a_i)>t_i)\,dt_i$$

Now,

$$\begin{aligned}
&\int_{x_1}^{\infty}\int_{x_2}^{\infty}P(T(a_1)>t_1, T(a_2)>t_2)\,dt_2\,dt_1\\
&=\int_{x_1}^{\infty}\int_{x_2}^{\infty}P(\max{}_{0\leq j\leq[t_1]}X_1(j)<a_1,\ \max{}_{0\leq j\leq[t_2]}X_1(j)<a_2)\,dt_2\,dt_1\\
&=\int_{x_1}^{\infty}\int_{x_2}^{\infty}[P(X_1(j)<a_1,\ 0\leq j\leq[t_1],\ X_1(j)<a_2,\ [t_1]+1\leq j\leq[t_2])I(t_1\leq t_2)\\
&\qquad+P(X_1(j)<a_1,\ 0\leq j\leq[t_1])I(t_1>t_2)]\,dt_2\,dt_1\\
&\leq\int_{x_1}^{\infty}\int_{x_2}^{\infty}[P(X_1(j)<a_1,\ 0\leq j\leq[t_1])\,P(X_1(j)<a_2,\ [t_1]+1\leq j\leq[t_2])I(t_1\leq t_2)\\
&\qquad+P(X_1(j)<a_1,\ 0\leq j\leq[t_1])I(t_1>t_2)]\,dt_2\,dt_1\\
&\leq\int_{x_1}^{\infty}\int_{x_2}^{\infty}[P(T(a_1)>t_1)\,P(T(a_2)>t_2)I(t_1\leq t_2)\\
&\qquad+P(T(a_1)>t_1)\,P(T(a_2)>t_2)I(t_1>t_2)]\,dt_2\,dt_1\\
&=\int_{x_1}^{\infty}\int_{x_2}^{\infty}\prod_{i=1}^{2}P(T(a_i)>t_i)\,dt_i,
\end{aligned}$$

where I is the usual indicator function.

Proof of (b). $\int_{x_1}^{\infty}\int_{x_2}^{\infty}P(\bigcap_{i=1}^{2}f_i(T(a_i))>t_i)\,dt_2\,dt_1$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(f_1(\min\{n|X_1(n)\geq a_1\}>t_1,\ f_2(\min\{n|X_1(n)\geq a_2\}>t_2)\,dt_2dt_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(\min\{n|X_1(n)\geq a_1\}>f_1^{-1}(t_1),\ \min\{n|X_1(n)\geq a_2\}>f_2^{-1}(t_2))\,dt_2dt_1$$

$$\leq \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(T(a_1)>f_1^{-1}(t_1))P(T(a_2)>f_2^{-1}(t_2))\,dt_2dt_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} \prod_{i=1}^{2} P(f_i(T(a_i))>t_i)\,dt_2dt_1.$$

To prove Theorem 3.4 we need the following Lemma 3.2 and Theorem 3.3. In particular, Lemma 3.2 is a characterization of the orders $\leq_{uo-cx}$ and $\leq_{lo-cv}$.

Lemma 3.2. Let $\{X_1(t)|t\in\Lambda\},\cdots,\{X_n(t)|t\in\Lambda\}$ and $\{Y_1(t)|t\in\Lambda\},\cdots,\{\acute{Y}_n(t)|t\in\Lambda\}$ be stochastic processes. Then the corresponding hitting times $_{(lo-cv)}(S_1(a_1),\cdots,S_n(a_n))$ if and only if $E(\Pi_{i=1}^n f_i(T_i(a_i))\leq E(\Pi_{i=1}^n f_i(S_i(a_i))$ for all nonnegative(nonpositive) increasing convex(concave) functions $f_1,\ \cdots,\ f_n$.

Proof. The proof is similar to the proof of Theorem 5.A.14 in Shaked and Shantikumar(1994), p.160.

The following result provides relation between WNUOD1(WNLOD2) and the upper (lower) orthant convex(concave) order.

Theorem 3.3. Suppose that $\{X_1(t)|t\in\Lambda\},\cdots,\{X_n(t)|t\in\Lambda\}$are stochastic processes and $\{Y_1(t)|t\in\Lambda\},\cdots,\{Y_n(t)|t\in\Lambda\}$ are independent stochastic processes such that $P(\max_{0\leq j\leq[t_i]}X_i(j)<a_i)=P(\max_{0\leq j\leq[t_i]}Y_i(j)<a_i)$ for all $a_i\in E,\ t_i\in\Lambda,\ i=1,\cdots,n.$
Then $(T_1(a_1),\cdots,T_n(a_n))$ are WNUOD1(WNLOD2) if and only if $(T_1(a_1),\cdots,T_n(a_n))$ $\leq_{uo-cx(lo-cv)}(S_1(a_1),\cdots,\ S_n(a_n))$.

Proof. We only prove WNUOD1 case.

($\Rightarrow$) Assume $T_1(a_1),\cdots,T_n(a_n)$ are WNUOD1, then we have

$$\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} P(\bigcap_{i=1}^{n} T_i(a_i)>t_i)dt_n\cdots dt_1 \leq \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\Pi_{i=1}^n P(T_i(a_i)>t_i)\,dt_n\cdots dt_1$$

$$= \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\Pi_{i=1}^n P(\max_{0\leq j\leq[t_i]}X_i(j)<a_i)\,dt_n\cdots dt_1$$

$$= \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\Pi_{i=1}^n P(\max_{0\leq j\leq[t_i]}Y_i(j)<a_i)\,dt_n\cdots dt_1$$

$$= \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\Pi_{i=1}^n P(S_i(a_i)>t_i)\,dt_n\cdots dt_1$$

$$= \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} P(\bigcap_{i=1}^{n} S_i(a_i) > t_i)\, dt_n \cdots dt_1$$

Hence $(T_1(a_1), \cdots, T_n(a_n)) \le_{uo-cx} (S_1(a_1), \cdots, S_n(a_n))$.

($\Leftarrow$) It follows from assumptions $(T_1(a_1), \cdots, T_n(a_n)) \le_{uo-cx} (S_1(a_1), \cdots, S_n(a_n))$ and that $P(\max_{0 \le j \le [t_i]} X_i(j) < a_i) = P(\max_{0 \le j \le [t_i]} Y_i(j) < a_i)$

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} [P(\bigcap_{i=1}^{n} T_i(a_i) > t_i) - \Pi_{i=1}^{n} P(T_i(a_i) > t_i)]\, dt_n \cdots dt_1$$

$$\le \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} [P(\bigcap_{i=1}^{n} S_i(a_i) > t_i) - \Pi_{i=1}^{n} P(S_i(a_i) > t_i)]\, dt_n \cdots dt_1 = 0$$

The zero follows from the assumption that $\{Y_1(t)|t \in \Lambda\}, \cdots, \{Y_n(t)|t \in \Lambda\}$ are independent stochastic processes. Hence $(T_1(a_1), \cdots, T_n(a_n))$ are WNUOD1. Similarly, we can prove WNLOD2 case.

From Lemma 3.2 and Theorem 3.3 we obtain the following theorem.

Theorem 3.4. Suppose that $\{X_1(t)|t \in \Lambda\}, \cdots, \{X_n(t)|t \in \Lambda\}$ are stochastic processes. Then the hitting times $(T_1(a_1), \cdots, T_n(a_n))$ are WNUOD1(WNLOD2) if and only if $E(\Pi_{i=1}^{n} f_i(T_i(a_i))) \le \Pi_{i=1}^{n} E(f_i(T_i(a_i)))$ for all nonnegative(nonpositive) increasing convex(concave) functions $f_1, \cdots, f_n$.

To prove our next result we need to use the following notations. Let $X = (X_1, \cdots, X_k)$ be a k-dimensional vector with distribution function F and the marginal distribution functions F_j, $j = 1, 2, \cdots, k$. The dependence function of X (or of F) is defined by

$$D_F(u_1, \cdots, u_k) = P(F_j(X_j) \le u_j,\ j = 1, 2, \cdots, k). \tag{3.1}$$

It is clear that D_F is the distribution function on $[0, 1]^k$, and it has uniform marginal distributions if the F_j's are continuous. The marginal distributions together with the dependence function determine F, since $F(x_1, \cdots, x_k) = D_F(F_1(x_1), \cdots, F_k(x_k))$.

Furthermore, a dependence function D_F is said to be an extreme dependence function if all the marginals are non degenerative, and for each $n \ge 1$,

$$D_F^n(u_1, \cdots, u_k) = D_F(u_1^n, \cdots, u_k^n),\ (u_1, \cdots, u_k) \in [0, 1]^k.$$

Hsing(1989) showed that D_F is extreme dependence function if and only if

$$D_{F^n}(u_1, \cdots, u_k) = D_F(u_1, \cdots, u_k),\ (u_1, \cdots, u_k) \in [0, 1]^k. \tag{3.2}$$

It is clear that if $(X_{i1}, \cdots, X_{ik})$, $1 \le i \le n$ are independent random vectors all having distribution function F, then F^n is the distribution function of $(\max_{1\le i\le n} X_{i1}, \cdots, \max_{1\le i\le n} X_{ik})$ and hence (3.1) is equivalent to $D_{F^n}(u_1, \cdots, u_k) = P(G_j(\max_{1\le i\le n} X_{ij} \le u_j, j=1, 2, \cdots, k)$, where G_j is the distribution function of $\max_{1\le i\le n} X_{ij}$ which is F_j^n.

Now we will define a concept for a one-dimensional process. For a one-dimensional process $\{X_1(t) \mid t\in \Lambda\}$, $\{W(t_1, \cdots, t_m) \mid t_1, \cdots, t_m \in \Lambda, m\in \{0, 1, \cdots\}\}$ is said to be a dependen nce function if for a given m, and $0\le t_1 \le \cdots \le t_m$, $W(t_1, \cdots, t_m) = D_{F_{X_1(t_1),\cdots,X_1(t_m)}}$. Here from (3.1) $D_{F_{X_1(t_1),\cdots,X_1(t_m)}}(u_1, \cdots, u_m) = P(F_i(X_1(t_i)) \le u_i,\ i=1, \cdots, m)$. Where $F_i(x) = P(X_1(t_i) \le x)$. Then we obtain the following theorem.

Theorem 3.5. Let (a) $\{X_1(t) \mid t\in \Lambda\}$be a one-dimensional process such that $\{X_1(t) \mid t\in \Lambda\}$ is WNOD, (b) $X_1(t)$ is strictly stationary(The process $X_1(t)$ is said to be strictly station any if for $0\le t_1 < t_2 < \cdots < t_k$ and $h>0, (X_1(t_1+h), \cdots, X_1(t_k+h) =^d (X_1(t_1), \cdots, X_1(t_k))$, (c) $D_G(u_1, \cdots, u_k) = D_F(u_1, \cdots, u_k)$, $(u_1, \cdots, u_k) \in [0, 1]^k$, this condition is equivalent to condition (3.2) for the case that X_{ij}'s are not i.i.d, where $D_G(u_1, \cdots, u_k) = P(G_1(\max_{1\le j\le n_i} X_{ij}(j) \le u_i, i=1, \cdots, k)$and $D_F(u_1, \cdots, u_k) = P(F_i(X_1(0) \le u_i,\ i=1, \cdots, k)$.
Then $(T(a_1), \cdots, T(a_k))$ is WNOD.

Proof. We will prove this theorem for $k=2$ only. For the general $n>2$, the proof is similar.

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty} P(\bigcap_{i=1}^{2} T(a_i) > n_i) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(\max_{0\le j\le n_1} X_1(j) \le a_1,\ \max_{0\le j\le n_2} X_1(j) \le a_2) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} P(G_1(\max_{0\le j\le n_1} X_1(j)) < G_1(a_1),\ G_1(\max_{0\le j\le n_2} X_1(j)) < G_1(a_2)) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} D_G(G_1(a_1), G_1(a_k)) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} D_F(G_1(a_1), G_1(a_k)) dn_2 dn_1$$

$$\le \int_{x_1}^{\infty}\int_{x_2}^{\infty} D_{F_1}(G_1(a_1)) D_{F_1}(G_1(a_2)) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} D_{G_1}(G_1(a_1)) D_{G_1}(G_1(a_2)) dn_2 dn_1$$

$$= \int_{x_1}^{\infty}\int_{x_2}^{\infty} \prod_{i=1}^{2} P(T_1(a_i) > n_i) dn_2 dn_1.$$

Here $G_1$and G_1 are distribution functions of $\max_{1\le j\le n_1} X_1(j)$and $\max_{1\le j\le n_2} X_1(j)$, respec tively and G is the joint distribution functions of $\max_{1\le j\le n_1} X_1(j)$ and $\max_{1\le j\le n_2} X_1(j)$, $F(x_1, x_2) = P(X_1(0)\le x_1, X_1(0)\le x_2)$, and $F_1(x_i) = P(X_1(0)\le x_i)$, $i=1,2$.

To prove this Theorem 3.7, we will need the following lemma for the preservation of the WNOD under mixture.

Lemma 3.6. Let (a) $\{Y(t)|t\in\Lambda\}$ be WNOD, (b) $\{X(t)|t\in\Lambda\}$, given $Y(t)$, be conditionally WNOD, (c) $X(t)$ is stochastically right tail decreasing in $Y(t)$. Then $(X(t), Y(t))$ is WNOD and $X(t)$ is WNOD. Furthermore, the corresponding hitting times $(T_1(a), T_2(b))$ are WNOD, $T_1(a)$ is WNOD, here $T_1(a) = \inf\{n|X(n)\ge a\}$, $T_2(b) = \inf\{n|Y(n)\ge b\}$.

Next, we show that the property of WNOD structure can be created and preseved through suitable combinations.

Theorem 3.7. Suppose $\{X(t)|t\in\Lambda\}$ and $\{Y(t)|t\in\Lambda\}$ satisfy a linear regression relationship of the form $X(t) = aY(t) + Z(t)$, where $a>0$, $Z(t)$ is stochastic process independent of $Y(t)$ and $Y(t)$ is WNOD. Then $(X(t), Y(t))$ is WNOD. Furthermore, the hitting times $(T_1(a), T_2(b))$ are WNOD, here $T_1(a) = \inf\{n|X(n)\ge a\}$, $T_2(b) = \inf\{n|Y(n)\ge b\}$.

Proof. Since $X(t) = aY(t) + Z(t)$ is stochastically right tail decreasing in $Z(t)$, $X(t)$ given $Z(t)$, is WNOD, by Lemma 3.6, $(X(t), Y(t))$ is WNOD. Furthermore, we can show that the corresponding hitting times $(T_1(a), T_2(b))$ are WNOD.

Finally, we show that weak negative orthant dependent structures both among stocha stic processes and their corresponding hitting times present a sampling of a useful examples of the theory developed in Section 3.

4. Examples

Example 4.1. Consider the uniformly modulated model (see Priestly(1988)) such that non-stationary process $X(t)$ given by

$$X(t) = \alpha(t)\,Y(t),\ t\ge 0,$$

where $\alpha(t)$ is a deterministic continuous function such that $\alpha(t)\ge 0$ and $Y(t)$ is non -negative stationary process. If $Y(t)$ is WNOD, we can show that $X(t)$ is WNOD. Using the Theorem 3.1(b), for increasing functions, f_i, $i=1,\cdots,n$, the corresponding hitting times $f_1(T(a_1)), \cdots, f_n(T(a_n))$are WNOD, here $T(a_i) = \inf\{n|X(n)\ge a_i\}$, $i=1,\cdots,n$.

Example 4.2. Leslie(1969) has considered the waiting time until the occurrence of a 'cluster of size k' $k\ge 2$,, in a homogeneous Poisson process. Marshall and Shaked(1983) argue that such a waiting time is the hitting time $T(k-\frac{1}{2})$ of a new better than used(NBU) process $Z(t)$ given in Example 2.4 of their paper. One can verify that $Z(t)$ is WNOD with itself in Definition 2.1. Consequently, given $2\le k_1 < \cdots < k_n$, a bound for the joint distribution of the waiting times for clusters of sizes $k_1, \cdots, k_n$ is provided by

$$\int_{s_1}^{\infty}\cdots\int_{s_n}^{\infty}[P(\bigcap_{i=1}^{n} T(k_i-\frac{1}{2})>x_i)-\prod_{i=1}^{n}P(T(k_i-\frac{1}{2})>x_i)]dx_n\cdots dx_1\le 0, \qquad (4.1)$$

and

$$\int_{-\infty}^{s_1}\cdots\int_{-\infty}^{s_n}[P(\bigcap_{i=1}^{n} T(k_i-\frac{1}{2})>x_i)-\prod_{i=1}^{n}P(T(k_i-\frac{1}{2})>x_i)]dx_n\cdots dx_1\le 0.$$

It should be noted that the terms in the left-hand side of the inequality (4.1) can be computed by the methods of Leslie(1969), Section 4.

Example 4.3. Consider the following stress-strength model for two systems. Let $Z_i(t)$, $i=1, 2$, be the strength of system i at time t. We will assume that the two systems receive shocks from a common source. Using a cumulative damage shock model (see Barlow and Proschan (1975)), we now let $N(t)$ denote the number of shocks occurring by time t and let U_i be i.i.d. positive random variables denoting the damage to each system due to the ith shock ($i=1,2,\cdots$). Hence, the stress experienced by each system at time t is given by the process $W(t) = \sum_{i=1}^{N(t)} U_i$. Then, we can show that $W(t)$ is WNOD with itself in Definition 2.1. Assuming that $Z_1(t)$ and $Z_2(t)$ are independent processes with non-increasing sample paths and that $Z_1(t)$ and $Z_2(t)$ are independent of $W(t)$ and $X(t)=W(t)-Z_1(t)$ and $Y(t)=W(t)-Z_2(t)$, we obtain using Theorem 3.7 that the bivariate processes $\{(X(t), Y(t)) : t\in\{0, 1, \cdots\}\}$ is WNOD processes. Furthermore, the corresponding hitting times $T_1(a)$ and $T_2(b)$ are WNOD.

Example 4.4. Following Barlow and Proschan(1976), we define a process $\{X(t)| t\ge 0\}$which denotes the accumulated repair time since the last failure of a system. In other words, if A_i, $i=1, 2, \cdots$, is a sequence of inter-failure times and B_i, $i=1, 2, \cdots$, is another sequence of inter-repair times, then

$$X(t)=\sum_{n=1}^{\infty}(t-(V_n+W_{n-1}))\,I[V_n+W_{n-1}\le t\le V_n+W_n],$$

where $V_n=\sum_{i=1}^{n}A_i$, $W_n=\sum_{i=1}^{n}B_i$, $n=1, 2, \cdots$, and $I(D)$ is the indicator variable of the event D. If (A_i, B_i), $i=1, 2, \cdots$, are independent and if A_i, B_i are associated random variables for each i, then it can be easily shown that the $X(t)$ process is WNOD with itself. It may be noted that the results of Barlow and Proschan(1981) imply, under further conditions on A_i, B_i, that, for all $s_i \in E$, $t_i \in \Lambda$, $i=1,2$,

$$\int_{s_1}^{\infty} \cdots \int_{s_n}^{\infty} (P(\bigcap_{i=1}^{n} X(t_i) > a_i) - \prod_{i=1}^{n} P(X(t_i) > a_i))\, da_n \cdots da_1 \le 0$$

and

$$\int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_n} (P(\bigcap_{i=1}^{n} X(t_i) > a_i) - \prod_{i=1}^{n} P(X(t_i) > a_i))\, da_n \cdots da_1 \le 0,$$

whereas our results show that the hitting times of $X(t)$ are WNOD random variables.

Example 4.5. Consider a simple form of an econometrical model of investment and capital gain in Theorem 3.7. Let $\{X(t) | t \in \Lambda\}$ and $\{Y(t) | t \in \Lambda\}$ denote the investment and capital gain at time t, respectively. The model is

$$Y(t) = aX(t-1) + Z_1(t) \tag{4.2}$$
$$X(t) = bY(t) + Z_2(t),$$

where $a, b > 0$, $Z_1(t)$ and $Z_2(t)$ are both stochastic processes and $(Z_1(t), Z_2(t))$ are a sequence of independent random vectors. Then, from (4.2) we can write that

$$Y(n) = \sum_{i=0}^{n} (ab)^{n-i} Z_1(i) + \sum_{i=0}^{n-1} a^{n-i} b^{n-i-1} Z_2(i), \quad n \ge 1$$

$$Y(0) = Z_1(0),$$

and it can be easily shown that the $\{Y(n) \mid n \in \{0, 1, 2, \cdots\}\}$ is WNOD
Similarly, from the (4.2) we obtain that

$$X(n) = \sum_{i=0}^{n} a^{n-i} Z_1(i) + \sum_{i=0}^{n-1} a^{n-i-1} Z_2(i)$$
$$X(0) = Z_2(0),$$

and consequently $\{X(n) \mid n \in \{0, 1, 2, \cdots\}\}$ process is also WNOD.
Now, for $0 < n_1 < \cdots < n_k$, then it can be easily shown that

$$\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} P(\bigcap_{i=1}^{n} Y(n_i) > a, \bigcap_{j=1}^{n} X(n_j) > b)\, da \cdots da db \cdots db$$
$$\le \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} \prod_{i=1}^{n} P(Y(n_i) > a) \prod_{j=1}^{n} P(X(n_j) > a)\, da \cdots da db \cdots db$$

and

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \int_{-\infty}^{y_1} \cdots \int_{\infty}^{y_n} P(\bigcap_{i=1}^{n} Y(n_i) \le a, \bigcap_{j=1}^{n} X(n_j) \le b)\, da \cdots da db \cdots db$$
$$\le \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \int_{-\infty}^{y_1} \cdots \int_{\infty}^{y_n} \prod_{i=1}^{n} P(Y(n_i) \le a) \prod_{j=1}^{n} P(X(n_j) \le b)\, da \cdots da db \cdots db.$$

Furthermore, we can show that the corresponding hitting times $(T_1(a), T_2(b))$ are WNOD.

Example 4.6. Block et al.(1988) proposed a bivariate exponential autoregressive model of order m, BEAR(m),

$$X(n) = \begin{cases} E(n), & n=0,\cdots,m-1 \\ \sum_{q=1}^{m} A(n,q)X(n-q) + B(n)E(n), & n=m, m+1,\cdots \end{cases} \qquad (4.3)$$

where $E(n)=(E_1(n), E_2(n))$is a sequence of independent bivariate exponential random vectors with mean $(\lambda_1^{-1}, \lambda_2^{-1})'$, $\lambda_1, \lambda_2 > 0$, $B(n)$ is a 2×2 diagonal matrix with $B(n)=\mathrm{diag}\{\pi_1(n), \pi_2(n)\}$, $0<\pi_1(n), \pi_2(n)<1$, e_j is an $m-$dimensional vector with component j equal to one and the other component equal to zero, $j= 1,\cdots, n$, 0 is the m-dimensional zero vector, $I'(n)=(I_1(n,1),\cdots, I_1(n,m), I_2(n,1),\cdots, I_2(n,m))$ is a sequence of $2m$-dimensional independent random vectors with components assuming values one or zero independent of all $E(n)$, and finally $A(n,q)$ is a 2×2 random diagonal matrix with $A(n,q)=\mathrm{diag}(I_1(n,q), I_2(n,q))$, $q=1,\cdots,m$. It is assumed that

$$\sum_{j=1}^{m} P\{(I_1(n,1),\cdots, I_1(n,m))=e'_j\}=1-\pi_i(n)$$

and

$$P\{(I_1(n,1),\cdots, I_1(n,m))=0'\}=\pi_i(n),\ l=1,2.$$

The following theorem gives the result about dependence structure of the bivariate process BEAR (m), $X(n)=(X_1(n), X_2(n))$.

Lemma 1. Suppose for $j=0,1,\cdots,m-1$, the random variables $X_1(j), X_2(j)$ in (4.3) are WNOD. Then $X(n)$ is WNOD. Furthermore, the corresponding hitting times are also WNOD.

Proof. We can obtain the result using a method similar to that used in the proof of Lemma 4.8, p.147, Barlow and Proschan(1981) and $WP_0 \sim WP_3$, Baek(1997). The proof of the second part of the Lemma is similar to the proof of Example 4.5.

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