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Bayesian Estimations of the Smaller and Larger for Two Pareto Scale Parameters

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Abstract

We shall derive Bayes estimators for the smaller and larger of two Pareto scale parameters with a common known shape parameter when the order of the scales is unknown and sample sizes are equal under squared error loss function. Also, we shall obtain biases and mean squared errors for proposed Bayes estimators, and compare numerically performances for the proposed Bayes estimators.

Keywords : Bayes estimator, Jeffreys's prior, joint uniform prior, squared error loss function

1. Introduction

The Pareto law in the shape-scale form is defined in terms of its density function by

$$f(x;\alpha,\beta)=\alpha\beta^{\alpha}x^{-(\alpha+1)}\quad x>\beta>0,\quad \alpha>0, \tag{1.1}$$

where α and β are referred as the shape and scale parameters, respectively, denoted by $PAR(\alpha,\beta)$. This distribution has been used in modeling a wide variety of socio-economic phenomena, such as the assets of firms, the sizes of cities, the sizes of business, insurance claims and personal incomes. Here, we shall consider Bayesian estimation for the smaller and larger of two Pareto scale parameters with a common known shape parameter.

Elfessi & Pal(1992) and Misra, Anand & Singh(1993) considered classical and Bayesian estimation of the smaller and larger of the two uniform scale parameters under the ordering among two scale parameters is unknown. Carpenter, Pal & Kushary(1992) and Carpenter & Pal(1995) studied estimations for the smaller and larger of two exponential location parameters with common known scale parameters and common unknown scale parameter. Misra & Dhariyal(1995) studied the problem of estimating ordered restricted scale parameters of k uniform distributions under the assumption that the correct ordering among the parameters is

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known apriori. Woo & Lee(1995) proposed classical estimators of the smaller and larger of two Pareto scale parameters with common known shape parameter. Carpenter & Hebert(1995) considered the estimation of the smaller and larger of two exponential location parameters.

In this paper, we shall derive Bayes estimators of the smaller and larger for two Pareto scale parameters with a common known shape parameter when the order of the scales is unknown and sample sizes are equal under squared error loss function by using a Jeffreys's prior and a uniform prior for the smaller and larger of two scale parameters. Also, we shall obtain the biases and mean squared errors(MSE) for the proposed Bayes estimators, and compare numerically performances for the proposed Bayes estimators.

2. Bayes estimators for smaller and larger scale parameters

Let $X_{i1},\cdots,X_{in}$, $i=1,2$, be independent simple random samples from two independent populations which are Pareto distributed with scale parameters β_i, $i=1,2$ and a common known shape parameter α. Define $\theta_1=\min(\beta_1,\beta_2)$ and $\theta_2=\max(\beta_1,\beta_2)$. It is known that complete and sufficient statistics for β_i, i=1,2, are $X_{i(n)}=\max\{X_{i1},\cdots,X_{in}\}$. So our goal is to propose Bayes estimators for θ_1 and θ_2 based on $X_{1(n)}$ and $X_{2(n)}$.

Define $Z_1=\min\{X_{1(n)},X_{2(n)}\}$ and $Z_2=\max\{X_{1(n)},X_{2(n)}\}$.

It is known that $X_{i(n)}$ follows $PAR(n\alpha,\beta_i)$ and $X_{1(n)}$ and $X_{2(n)}$ are independent. So, we can obtain the joint density function for (Z_1,Z_2) and marginal density functions for Z_1 and Z_2 as follows :

$$f(z_1,z_2)=\begin{cases} 2(n(\alpha+1))^2(\theta_1\cdot\theta_2)^{-n(\alpha+1)}(z_1\cdot z_2)^{n(\alpha+1)-1}, & 0\le z_1\le z_2\le\theta_1 \\ (n(\alpha+1))^2(\theta_1\cdot\theta_2)^{-n(\alpha+1)}(z_1\cdot z_2)^{n(\alpha+1)-1}, & 0\le z_1\le\theta_1\le z_2\le\theta_2. \end{cases}$$

$$\begin{aligned} f(z_1)=\ & n(\alpha+1)(\theta_1^{-n(\alpha+1)}+\theta_2^{-n(\alpha+1)})z_1^{n(\alpha+1)-1} \\ & -2n(\alpha+1)(\theta_1\cdot\theta_2)^{-n(\alpha+1)}z_1^{2n(\alpha+1)-1}, \quad 0\le z_1\le\theta_1, \end{aligned} \tag{2.1}$$

$$f(z_2)=\begin{cases} 2n(\alpha+1)(\theta_1\cdot\theta_2)^{-n(\alpha+1)}\cdot z_2^{2n(\alpha+1)-1}, & 0\le z_2\le\theta_1 \\ n(\alpha+1)\theta_2^{-n(\alpha+1)}z_2^{n(\alpha+1)-1}, & \theta_1\le z_2\le\theta_2. \end{cases}$$

From the result (2.1), the k-moments for Z_1 and Z_2 are

$$E[Z_1^k]=\frac{n\alpha}{n\alpha-k}\theta_1^k-\delta^{n\alpha-k}\frac{k(n\alpha)}{(n\alpha-k)(2n\alpha-k)}\theta_1^k,$$

$$E[Z_2^k] = \frac{na}{na-k}\theta_2^k + \delta^{na}\frac{k(na)}{(na-k)(2na-k)}\theta_2^k, \tag{2.2}$$

where $\delta = \theta_1/\theta_2$ is the ratio between two scale parameters.

Here, we shall consider Bayesian estimation for the smaller and larger of two Pareto scale parameters with a common known shape parameter under squared error loss function by using a Jeffreys's prior and uniform prior on the smaller and larger of two scale parameters. This type estimation problems often arises in practice, for example, when we have the two independent wage(or income) distributions having the Pareto distribution with different scales which may be the minimum wage(or income) but a common known shape parameter.

Bayesian analysis with the noninformative prior is very common when little or no prior information is available. One of the widely used noninformative priors is Jeffreys prior which, up to a proportionality constant, is given by the square root of the determinant of Fisher information matrix. Then the noninformative prior for θ_1 and θ_2 is given by

$$\pi(\theta_1, \theta_2) \propto \theta_1 \cdot \theta_2, \quad 0 < \theta_1 \le \theta_2 < \infty.$$

According to the Bayes theorem, the joint posterior distribution and the marginal posterior distributions for θ_1 and θ_2 are obtained by

$$g(\theta_1, \theta_2 \mid z_1, z_2) = \begin{cases} (na)^2(\theta_1 \cdot \theta_2)^{na-1}(z_1 \cdot z_2)^{-na}, & \theta_1 \le z_1 \le \theta_2 \le z_2, \\ 2(na)^2(\theta_1 \cdot \theta_2)^{na-1}(z_1 \cdot z_2)^{-na}, & \theta_1 \le \theta_2 \le z_1 \le z_2, \end{cases}$$

$$g(\theta_1 \mid z_1, z_2) = na\theta_1^{na-1}(z_1^{-na} + z_2^{-na}) - 2na\theta_1^{2na-1}(z_1 \cdot z_2)^{-na}, \ 0 \le z_1 \le \theta_1, \tag{2.3}$$

$$g(\theta_2 \mid z_1, z_2) = \begin{cases} 2(na)\theta_2^{2na-1}(z_1 \cdot z_2)^{-na}, & 0 \le z_2 \le \theta_1 \\ (na)\theta_2^{na-1} z_2^{-na}, & \theta_1 \le z_2 \le \theta_2, \end{cases}$$

respectively. Therefore, the Bayes estimators for θ_1 and θ_2 under squared error loss function and a noninformative prior for θ_1 and θ_2 are

$$\hat{\theta}_1^N = \left[\frac{na}{na+1} - \hat{\delta}^{na}\frac{na}{(na+1)(2na+1)}\right]Z_1,$$

$$\hat{\theta}_2^N = \left[\frac{na}{na+1} + \hat{\delta}^{na+1}\frac{na}{(na+1)(2na+1)}\right]Z_2,$$

where $\hat{\delta} = Z_1/Z_2$ is an estimator of δ.

From result (2.2), we can obtain absolute biases and MSEs for $\hat{\theta}_1^N$ and $\hat{\theta}_2^N$ as follows :

$$|BIAS(\hat{\theta}_1^N)|=\left|\frac{1}{(na)^2-1}-\frac{(na)^2}{2(na-1)(2na-1)}\delta^{na-1}+\frac{(na)^2}{2(na+1)(2na+1)}\delta^{na}\right|\cdot\theta_1,$$

$$|BIAS(\hat{\theta}_2^N)|=\left|\frac{1}{(na)^2-1}-\frac{(na)^2}{2(na-1)(2na-1)}\delta^{na-1}+\frac{(na)^2}{2(na+1)(2na+1)}\delta^{na}\right|\cdot\theta_2,$$

$$\begin{aligned} MSE[\hat{\theta}_1^N]=\Big[& \frac{(na)^2+na+2}{(na+1)^2(na-1)(na-2)}+\frac{(na)^2}{(na-1)(na-2)}\delta^{na-1} \\ & -\frac{(na)^2(na+2)}{2(na+1)^2(2na+1)}\delta^{na}-\frac{(na)^3}{3(na)^2(2na+1)^2(na+2)}\delta^{2na} \\ & +\frac{(na)^3\{3(na)^3+15(na)^2+16(na)+4\}}{6(na+1)^2(na-1)(na-2)(na+2)(2na+1)}\delta^{na-2}\Big]\cdot\theta_1^2, \end{aligned} \tag{2.4}$$

$$\begin{aligned} MSE[\hat{\theta}_2^N]=\Big[& \frac{(na)^2+na+2}{(na+1)^2(na-1)(na-2)}+\frac{(na)^2(na-1)}{(na+1)^2(4(na)^2-1)}\delta^{na-1} \\ & +\frac{(na)^2\{9(na)^4-14(na)^3-73(na)^2-44(na)-12\}}{3(na+1)^2(na-2)(na-1)(2na-1)(na+2)(2na+1)}\delta^{na} \\ & -\frac{(na)^3}{3(na+1)^2(2na+1)^2(na+2)}\delta^{2na+2}\Big]\cdot\theta_2^2. \end{aligned}$$

Since the noninformative prior is an improper, we shall consider the joint uniform prior distribution for θ_1 and θ_2 as follows :

$$\pi(\theta_1,\theta_2)=\frac{2}{\beta^2},\quad 0<\theta_1\le\theta_2<\beta,$$

According to the Bayes theorem, the joint posterior distribution and the marginal posterior distributions for θ_1 and θ_2 are obtained by

$$g(\theta_1,\theta_2\,|\,z_1,z_2)=\begin{cases}(na+1)(\theta_1\cdot\theta_2)^{na}(z_1\cdot z_2)^{-(na+1)}, & \theta_1\le z_1\le\theta_2\le z_2,\\ 2(na+1)(\theta_1\cdot\theta_2)^{na}(z_1\cdot z_2)^{-(na+1)}, & \theta_1\le\theta_2\le z_1\le z_2,\end{cases}$$

$$\begin{aligned} g(\theta_1\,|\,z_1,z_2)= & \,(na+1)\theta_1^{na}(z_1^{-(na+1)}+z_2^{-(na+1)}) \\ & -2(na+1)\theta_1^{2na+1}(z_1\cdot z_2)^{-(na+1)},\quad 0\le z_1\le\theta_1, \end{aligned} \tag{2.5}$$

$$g(\theta_2\,|\,z_1,z_2)=\begin{cases}2(na+1)\theta_2^{2na}z_2^{-(na+1)}, & 0\le z_2\le\theta_1\\ (na+1)\theta_2^{na}z_2^{-(na+1)}, & \theta_1\le z_2\le\theta_2,\end{cases}$$

respectively. Therefore, the Bayes estimators for θ_1 and θ_2 under squared error loss function and a joint uniform prior for θ_1 and θ_2 are

$$\widehat{\theta}_1^{\,U}=\left[\frac{na+1}{na+2}-\widehat{\delta}^{\,na+1}\frac{na+1}{(na+2)(2na+3)}\right]\cdot Z_1,$$

$$\widehat{\theta}_2^{\,U}=\left[\frac{na+1}{na+2}+\widehat{\delta}^{\,na+1}\frac{na+1}{(na+2)(2na+3)}\right]\cdot Z_2.$$

From result (2.2), we can obtain absolute biases and MSEs for $\widehat{\theta}_1^{\,U}$ and $\widehat{\theta}_2^{\,U}$ as follows :

$$\left|BIAS(\widehat{\theta}_1^{\,U})\right|=\left|\frac{2}{(na-1)(na+2)}-\frac{na(na+1)^2}{2(na-1)(2na-1)(2na+1)}\delta^{na-1}\right.$$
$$\left.+\frac{(na)^2(na+1)}{2(na+2)(2na+1)(2na+3)}\delta^{na+1}\right|\cdot\theta_1,$$

$$\left|BIAS(\widehat{\theta}_2^{\,U})\right|=\left|\frac{2}{(na-1)(na+2)}-\frac{na(na+1)^2}{2(na-1)(2na-1)(2na+1)}\delta^{na}\right.$$
$$\left.-\frac{(na)^2(na+1)}{2(na+2)(2na+1)(2na+3)}\delta^{na+2}\right|\cdot\theta_2,$$

$$\begin{aligned}MSE[\widehat{\theta}_1^{\,U}]=\Bigg[&\frac{(na)(na+1)^2}{(na-2)(na+2)^2}-\frac{2na(na+1)}{(na-1)(na+2)}+1\\&+\frac{na(na+1)^2}{(na-1)(na+2)}\bigg(\frac{na}{(na+2)(na+4)(2na+3)(3na+2)}\\&\quad-\frac{1}{(na-2)(na+2)}-\frac{2na}{3(2na+1)(2na+3)}\bigg)\delta^{na-2}\\&-\frac{na(na+1)^2}{(na-1)(2na-1)(2na+1)}\delta^{na-1}\\&+\frac{(na)^2(na+1)(5na+8)}{3(na+2)^2(2na+1)(2na+3)}\delta^{na+1}\\&+\frac{(na)^2(na+1)^2}{(na+2)^2(na+4)(2na+3)(3na+2)}\delta^{2na+2}\Bigg]\cdot\theta_1^2,\end{aligned}\tag{2.6}$$

$$\begin{aligned}MSE[\widehat{\theta}_2^{\,U}]=\Bigg[&\frac{(na)(na+1)^2}{(na-2)(na+2)^2}-\frac{2na(na+1)}{(na-1)(na+2)}+1\\&+\bigg\{\frac{(na)^2(na+1)^2}{(na-1)(na+2)(2na+3)}\left(\frac{na+5}{2(na-2)}+\frac{1}{(na+4)(3na+2)}\right)\\&\quad-\frac{na(na+1)^2}{(na-1)(2na-1)(2na+1)}\bigg\}\delta^{na}\end{aligned}$$

$$+\frac{n\alpha(n\alpha-1)(n\alpha+1)}{(n\alpha+2)^2(2n\alpha+1)(2n\alpha+3)}\delta^{n\alpha-2}$$
$$\left.-\frac{(n\alpha)^2(n\alpha+1)^2}{(n\alpha+2)^2(n\alpha+4)(2n\alpha+3)^2(3n\alpha+2)}\delta^{2n\alpha+4}\right]\cdot\theta_2^2.$$

From results (2.4) and (2.6), Bayes estimators $\widehat{\theta_1}^N$, $\widehat{\theta_1}^U$, $\widehat{\theta_2}^N$ and $\widehat{\theta_2}^U$ with respect to a Jeffreys prior and uniform prior for θ_1 and θ_2 are simple consistent, respectively.

Table 1 and Table 2 show exact numerical values of absolute standardized biases ($|E(\widehat{\theta}/\theta-1)|$) and standardized risks ($|E(\widehat{\theta}/\theta-1)^2|$) for Bayes estimators for the smaller and larger of the two Pareto scale parameters with a common known shape parameter for the sample size n=10 and 25, the shape parameter $\alpha=3$, and the ratio of the smaller and larger of two Pareto scale parameters $\delta=0.90(0.01)0.99$. From Table 1 and Table 2, in the case of the smaller of two Pareto scale parameters, the Bayes estimator $\widehat{\theta_1}^U$ with respect to an uniform prior is more efficient than $\widehat{\theta_1}^N$ for the smaller of two Pareto scale parameters in the sence of MSE. In the case of the larger of the two Pareto scale parameters, the Bayes estimator $\widehat{\theta_2}^N$ with respect to a noninformative prior is more efficient than $\widehat{\theta_2}^U$ as δ close to *1* and sample size is small. But the Bayes estimator $\widehat{\theta_2}^U$ with respect to an uniform prior is more efficient than $\widehat{\theta_2}^N$ as sample size increase.

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Table 1. Biases and MSE's of the Bayes estimators for the smaller of two Pareto scale parameters ($\alpha=3$).

n	δ	$\hat{\theta}_1^N$		$\hat{\theta}_1^U$	
		BIAS	MSE	BIAS	MSE
10	0.90	0.00118	0.00319	0.00002	0.00087
	0.91	0.00190	0.00375	0.00071	0.00290
	0.92	0.00281	0.00453	0.00159	0.00075
	0.93	0.00397	0.00546	0.00272	0.00068
	0.94	0.00542	0.00661	0.00414	0.00061
	0.95	0.00723	0.00799	0.00591	0.00055
	0.96	0.00946	0.00966	0.00812	0.00051
	0.97	0.01218	0.01160	0.01081	0.00050
	0.98	0.01547	0.01737	0.01408	0.00053
	0.99	0.01937	0.01632	0.01796	0.00063
25	0.90	0.00020	0.00019	0.00035	0.00018
	0.91	0.00014	0.00020	0.00032	0.00017
	0.92	0.00011	0.00023	0.00029	0.00017
	0.93	0.00005	0.00029	0.00022	0.00017
	0.94	0.00007	0.00040	0.00010	0.00016
	0.95	0.00032	0.00061	0.00013	0.00015
	0.96	0.00078	0.00100	0.00059	0.00013
	0.97	0.00164	0.00168	0.00143	0.00011
	0.98	0.00316	0.00283	0.00294	0.00009
	0.99	0.00574	0.00462	0.00551	0.00008

Table 2. Biases and MSE's of the Bayes estimators for the larger of two Pareto scale parameters $(\alpha=3)$.

n	δ	$\widehat{\theta}_2^{\,N}$		$\widehat{\theta}_2^{\,U}$	
		BIAS	MSE	BIAS	MSE
10	0.90	0.00318	0.00242	0.00411	0.00241
	0.91	0.00385	0.00291	0.00476	0.00290
	0.92	0.00472	0.00359	0.00560	0.00357
	0.93	0.00583	0.00451	0.00669	0.00450
	0.94	0.00725	0.00578	0.00807	0.00576
	0.95	0.00903	0.00751	0.00982	0.00750
	0.96	0.01126	0.00987	0.01201	0.00986
	0.97	0.01401	0.01306	0.01473	0.01307
	0.98	0.01736	0.01737	0.01806	0.01741
	0.99	0.02139	0.02318	0.02207	0.02328
25	0.90	0.00019	0.00018	0.00035	0.00018
	0.91	0.00020	0.00019	0.00037	0.00019
	0.92	0.00023	0.00020	0.00040	0.00020
	0.93	0.00029	0.00023	0.00046	0.00023
	0.94	0.00041	0.00030	0.00058	0.00029
	0.95	0.00065	0.00045	0.00081	0.00042
	0.96	0.00110	0.00077	0.00125	0.00072
	0.97	0.00194	0.00147	0.00208	0.00136
	0.98	0.00345	0.00297	0.00358	0.00273
	0.99	0.00603	0.00617	0.00615	0.00568