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Nonparametric Test for Multivariate Location Translation Alternatives

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Abstract

In this paper we propose a nonparametric one sided test for location parameters in p-variate($p \geq 2$) location translation model. The exact null distributions of test statistics are calculated by permutation principle in the case of relatively small sample sizes and the asymptotic distributions are also considered. The powers of various tests are compared through computer simulation and the p-values with real data are also suggested through example.

Keywords : Nonparametric test, Multivariate location translation, Permutation principle

1. Introduction

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be p-variate two samples from X and Y populations with continuous distribution function F and G, respectively. We assume that for all $x \in R^p$, there is a $\theta \in R^p$ such that

$$G(x) = F(x - \theta)$$

i.e., the p-variate location translation model. Sometimes we are interested in testing the following hypotheses:

$$H_0: \theta_1 \leq \theta_{10},\ \theta_2 \leq \theta_{20},\ \cdots,\ \theta_p \leq \theta_{p0} \text{ v.s. } H_1: \textit{at least one of } \theta_i\textit{'s is larger than } \theta_{i0}$$

This is the so-called one sided testing problem for multivariate data. As an example, suppose that a laboratory has developed a medicine which may have effects on two symptoms simultaneously. One can draw a decision that this medicine is acceptable if it become effective for any one of two symptoms or for both. In this problem, the alternative under consideration can be formulated as

$$H_1: \textit{at least one symptom may be cured}.$$

In spite of those applicability of one sided test procedure, the developments have not been so

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fruitful. Bhattacharyya and Johnson (1970) considered the one-sided alternatives for the bivariate case based on the concept of two-dimensional layer ranks which were introduced by Barndorff-Nielsen and Sobel (1966). Boyett and Shuster (1977) proposed a nonparametric test procedure which is a maximal t-statistics and applied the permutation principle to obtain the null distribution function. However they did not provide the normal approximation for the large sample case. Wei and Knuiman (1987) considered the one- sided alternatives for censored data by specifying the alternatives based on the so-called stochastic ordering of the distribution functions. The test statistic was constructed by defining signum function for the pairs of observation vectors. Therefore the test statistic can be considered as an extension of Gehan test. However even for the small sample case, the exact null permutation distribution of the test statistic cannot be obtained. Therefore the derivation of the large sample approximation to the normal distribution becomes obvious. Up to now, the main obstruction has been the nonexistence of the table for the p-variate normal distribution functions. For the bivariate case, Owen (1962) published a book which contains the bivariate distribution functions for varying the values of correlation coefficient. However the tables are not sufficient since they can not contain all the values of the correlation coefficients. Therefore in this paper we propose a test procedure and consider the large sample approximation by obtaining the tail probability of the multivariate normal distribution.

2. Test Statistic and Small Sample Test

Let T_i be a univariate nonparametric test statistic for the i-th component for testing $H_0: \theta_i = \theta_{i0}$ for the two-sample problem. Since we are interested in dealing with the locally most powerful test procedures, T_i' s are not required to be the same type. For this problem, we use the maximum value among the p univariate test statistics. Therefore we will consider the standardized form for each component. Let $\mu_i(\theta_{i0}) = E_{H_0}(T_i)$ and $\sigma_i^2(\theta_{i0}) = V_{H_0}(T_i)$ be the mean and variance of T_i under H_0, respectively. Then we propose a test statistic for testing H_0 against H_1 in the following way:

$$Q = \max\left\{ \frac{T_1 - E_{H_0}(T_1)}{V_{H_0}^{1/2}(T_1)}, \frac{T_2 - E_{H_0}(T_2)}{V_{H_0}^{1/2}(T_2)}, \cdots, \frac{T_p - E_{H_0}(T_p)}{V_{H_0}^{1/2}(T_p)} \right\}$$

Then the testing rule would be to reject H_0 for large values of Q. For reasonable sample sizes, we may obtain the null distribution for Q based on the permutation principle. The procedure for obtaining the null distribution function for multivariate data based on the permutation principle is well summarized in Puri and Sen(1971). However, for large sample sizes, we have to consider the large sample approximation.

Example

The following data is a part of the Actual Ordnance Survey (Mardia, 1980).

$$X = \begin{pmatrix} 257, & 529 \\ 279, & 149 \end{pmatrix} \quad Y = \begin{pmatrix} 292, & 259, & 508 \\ 90, & 665, & 433 \end{pmatrix}$$

Then the corresponding rank matrix of the combined sample is

$$R_5 = \begin{pmatrix} 1, & 5, & 3, & 2, & 4 \\ 3, & 2, & 1, & 5, & 4 \end{pmatrix}$$

Case I (Wilcoxon rank sum tests for both T_1 and T_2)

We deal this problem with Wilcoxon rank sum test. Then for each i, we have

$$T_1 = 3+2+4 = 9 \text{ and } T_2 = 1+5+4 = 10$$

Since $E_{H_0}(T_i) = n(m+n+1)/2 = 9$ and $V_{H_0}(T_i) = mn(m+n+1)/12 = 3$, we have

$$Q = \max\left\{ \frac{T_1 - E_{H_0}(T_1)}{\sqrt{V_{H_0}(T_1)}}, \frac{T_2 - E_{H_0}(T_2)}{\sqrt{V_{H_0}(T_2)}} \right\} = \max\{0, 1/\sqrt{3}\} = 1/\sqrt{3}$$

In order to perform the test procedure, we need the exact null distribution of Q. This can be obtained from the null permutation distribution of (T_1, T_2) Then by applying the permutation principle (cf. Puri and Sen, 1971), we obtain that under H_0,

$$P\{T_1=6, T_2=9\} = 1/10, P\{T_1=7, T_2=12\} = 1/10$$
$$P\{T_1=8, T_2=8\} = 1/10, P\{T_1=8, T_2=10\} = 1/10$$
$$P\{T_1=9, T_2=6\} = 1/10, P\{T_1=9, T_2=10\} = 1/10$$
$$P\{T_1=10, T_2=8\} = 1/10, P\{T_1=10, T_2=9\} = 1/10$$
$$P\{T_1=11, T_2=11\} = 1/10, P\{T_1=12, T_2=7\} = 1/10$$

Then some straightforward calculations show that under H_0,

$$P\{Q=-1/\sqrt{3}\} = 1/10, P\{Q=0\} = 2/10, P\{Q=1/\sqrt{3}\} = 4/10$$
$$P\{Q=2/\sqrt{3}\} = 1/10, P\{Q=\sqrt{3}\} = 2/10$$

Since the testing rule is to reject H_0 for large values of Q, the p-value would be $7/10$.

Case II (Median test for T_1 and Wilcoxon rank sum test for T_2)

For the same example, we consider median test for the first component and Wilcoxon rank sum test for the second component. The median test statistic is obtained in the following way: Let $M = [(m+n)/2]+1$ where $[\cdot]$ is the greatest integer. Then the median test statistic is the number of observations of Y sample whose values are greater than or equal to M. Then straightforward calculations give the following joint null permutation distribution of (T_1, T_2)

$$P\{T_1=1,\ T_2=9\} = 1/10 \quad P\{T_1=1,\ T_2=10\} = 1/10$$
$$P\{T_1=1,\ T_2=12\} = 1/10 \quad P\{T_1=2,\ T_2=6\} = 1/10$$
$$P\{T_1=2,\ T_2=8\} = 2/10 \quad P\{T_1=2,\ T_2=9\} = 1/10$$
$$P\{T_1=2,\ T_2=10\} = 1/10 \quad P\{T_1=2,\ T_2=11\} = 1/10$$
$$P\{T_1=3,\ T_2=7\} = 1/10$$

Thus $E_{H_0}(T_1) = 1.8$ and $V_{H_0}(T_1) = 0.36$ and

$$Q = \max\left\{\frac{T_1 - E_{H_0}(T_1)}{\sqrt{V_{H_0}(T_1)}}, \frac{T_2 - E_{H_0}(T_2)}{\sqrt{V_{H_0}(T_2)}}\right\} = \max\{1/3, 1/\sqrt{3}\} = 1/\sqrt{3}$$

Since, under H_0, the exact distribution of Q is

$$P\{Q=0\} = 1/10 \quad P\{Q=1/3\} = 4/10 \quad P\{Q=\sqrt{3}/3\} = 2/10$$
$$P\{Q=2\sqrt{3}/3\} = 1/10 \quad P\{Q=\sqrt{3}\} = 1/10 \quad P\{Q=2\} = 1/10$$

we see that the p-value for rejecting H_0 is $5/10$.

3. Large Sample Test and Asymptotic Properties

3.1 Large Sample Test

For this section, we assume that

$$\lim_{N\to\infty} n/N = \lambda \quad 0<\lambda<1 \ (N=m+n)$$

and $T_i(i=1,\cdots,p)$s linear rank statistics. Puri and Sen (1971) showed with above condition that under H_0, the joint distribution of

$$Q' = \left\{\frac{T_1-E_{H_0}(T_1)}{V_{H_0}^{1/2}(T_1)}, \frac{T_2-E_{H_0}(T_2)}{V_{H_0}^{1/2}(T_2)}, \cdots, \frac{T_p-E_{H_0}(T_p)}{V_{H_0}^{1/2}(T_p)}\right\}$$

converges in distribution to a p-variate normal distribution with 0 mean vector and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 1 & \rho_{12} \cdots \rho_{1p} \\ \rho_{21} & 1 \cdots \rho_{2p} \\ & \cdots \\ \rho_{p1} & \rho_{p2} \cdots 1 \end{pmatrix}$$

with $\rho_{ij} = \rho_{H_0}(T_i, T_j)$ is the correlation coefficient between i-th and j-th components for each $i \neq j$. We note that for the asymptotic normality, we may use any consistent estimate $\hat{\rho}_{ij}$ instead of ρ_{ij}.

Since for any real number q,

$$P\{Q \leq q\} = P\left\{\frac{T_1 - E_{H_0}(T_1)}{V_{H_0}^{1/2}(T_1)} \leq q, \ldots, \frac{T_p - E_{H_0}(T_p)}{V_{H_0}^{1/2}(T_p)} \leq q\right\}$$

the limiting probability of $P\{Q \leq q\}$ is the tail probability of the p-variate normal distribution with 0 mean vector and covariance matrix Σ with the same values of all coordinates. Therefore in order to calculate the limiting probability of $P\{Q \leq q\}$, we need the values of $\rho_{H_0}(T_i, T_j)'s$ For this, we only consider the bivariate case since the extensions to the multivariate case become straightforward by considering all the possible pairs among the coordinates. Also we note that for the permutation correlation coefficient, $\rho_{H_0}(T_1, T_2)$ it is enough to consider the permutation covariance, $Cov_{H_0}(T_1, T_2)$ Let

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & N \\ r(1) & r(2) & r(3) & \ldots & r(N) \end{pmatrix}$$

be a rank matrix which is obtained from the rank matrix R_N by permuting its columns. Chatterjee and Sen (1964) provided the formula for $Cov_{H_0}(T_1, T_2)$ between Wilcoxon rank sum statistics as follows:

$$Cov_{H_0}(T_1, T_2) = \frac{mn}{N(N-1)} \sum_{i=1}^{N}\left(i - \frac{N+1}{2}\right)\left(r(i) - \frac{N+1}{2}\right)$$

For example, when the rank vector is $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $i = 4$ and $r(i) = 7$ in the above formula.

Especially, we note that in case of the independence among all the components,

$$P\{Q \leq q\} = \prod_{i=1}^{p} P\left\{\frac{T_i - E_{H_0}(T_1)}{V_{H_0}^{1/2}(T_1)}\right\} \approx \Phi^p(q)$$

where $\Phi(\cdot)$ stands for the cumulative standard normal distribution function. In this case, the determination of q becomes easy.

Example(continued)

Case I

We note that for small sample case, we do not need any covariance between T_1 and T_2. However for large sample case, we showed that Q' converges in distribution to a bivariate normal distribution with mean $(0,0)$ and variance $\Sigma = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{pmatrix}$ with

$$\rho_{12} = COV_{H_0}(T_1, T_2)/\sqrt{V_{H_0}(T_1)V_{H_0}(T_2)}$$

Therefore we need to calculate ρ_{12} with using the null distribution of (T_1, T_2) under H_0. For this we note that

$$E_{H_0}(T_1 T_2) = (54+84+64+80+54+90+80+90+121+84)/10 = 80.1$$

Thus we obtain that

$$COV_{H_0}(T_1, T_2) = E_{H_0}(T_1 T_2) - E_{H_0}(T_1)E_{H_0}(T_2) = 80.1-81 = -0.9$$

Then we obtain that

$$\rho_{12} = \frac{COV_{H_0}(T_1, T_2)}{\sqrt{V_{H_0}(T_1)V_{H_0}(T_2)}} = -\frac{0.9}{3} = -0.3$$

(We note that the above results for $COV_{H_0}(T_1, T_2)$ and ρ_{12} can be directly calculated from Chatterjee and Sen's formula described in this section. That is, for the given data, we obtain

$$Cov_{H_0}(T_1, T_2) = \frac{6}{5\times 4}\sum_{i=1}^{5}(i-3)(r(i)-3) = -9/10 \quad \text{and} \quad \rho_{12} = -0.3$$

These coincide the results calculated by exact null distribution of (T_1, T_2) Hereafter, including Section 4, we will use the Chatterjee and Sen's formula to obtain the exact null covariance of (T_1, T_2) because the permutation principle needs very tedious calculations for moderate or large sample size.)

Thus the distribution of $\left\{\frac{(T_1 - E_{H_0}(T_1))}{V^{1/2}_{H_0}(T_2)}, \frac{(T_2 - E_{H_0}(T_2))}{V^{1/2}_{H_0}(T_2)}\right\}$ converges in distribution to a bivariate normal distribution with $(0,0)$ mean vector and $\Sigma = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 1 \end{pmatrix}$ Then the p-value for Q can be calculated by $P\{Q \geq 1/\sqrt{3}\} = 0.5169$.

Case II

By applying similar method to Case I, we can obtain the followings. That is, $Q = \max\{1/3, 1/\sqrt{3}\} = 1/\sqrt{3}$, $E_{H_0}(T_1 T_2) = 15.6$ and $COV_{H_0}(T_1, T_2) = -0.6$. Therefore $\rho_{12} = -0.6/\sqrt{(0.36)3} = -0.6/(0.6\sqrt{3}) = -1/\sqrt{3}$ Thus the distribution of $\left\{\frac{(T_1 - E_{H_0}(T_1))}{V^{1/2}_{H_0}(T_2)}, \frac{(T_2 - E_{H_0}(T_2))}{V^{1/2}_{H_0}(T_2)}\right\}$ converges in distribution to a bivariate normal distribution with $(0,0)$ mean vector and $\Sigma = \begin{pmatrix} 1 & -1/\sqrt{3} \\ -1/\sqrt{3} & 1 \end{pmatrix}$ So the p-value for Q can be calculated by $P\{Q \geq 1/\sqrt{3}\} = 0.5439$.

The computations of the tail probability for the bivariate normal distribution may be carried out through computer program such as the *pmvnorm* function which is provided by S-Plus. For d-variate case with $d \geq 3$, we may use the M_X program (Neale, Xie, Hadady and Boker, 1998) to obatin the p-values. By using this program, we can computer the multiple integrals of the multivariate normal, up to dimension 10. The program and documentation can be downloaded from the website http://www.vipbg.vcu.edu/mxgui.

3.2 Asymptotic Properties

In order to deal with the asymptotic properties for our proposed tests, we note that the test statistic Q_N consists of d number of univariate nonparametric test statistics. Therefore some asymptotic properties of Q_N would be inherited from those of univariate nonparametric tests. We may take the consistency of tests as an example. For each i, let (T_{iN}) be a sequence of α-level tests of $H_0: \theta_i \leq \theta_{i0}$ which is consistent against the alternatives $H_1: \theta_i > \theta_{i0}$. Then the consistency of the tests based on the sequence (Q_N) follows immediately. Also the optimality property such as the locally most powerful test based on Q_N will follow naturally if a test for each component based on T_{iN} is locally most powerful.

For the limiting power of our test, we consider the following Pitman translation alternatives: For each N and for each i, $i=1,\ldots,p$ let

$$H_{1N}: \theta_{iN} = c_i/\sqrt{N},$$

where c_i is a fixed positive real number. We assume that all the univariate test statistics which we consider in this paper satisfy the assumptions and conditions in the section 3.8.3 (pp. 120-121) in Puri and Sen (1971). Then from some straightforward calculations, we have the limiting power of the test as follows:

$$\begin{aligned}\lim_{N\to\infty} P_{\theta_N}\{Q_N \geq C_N(\alpha)\} &= 1 - \lim_{N\to\infty} \Phi_{\Sigma_N}\Bigg(C_N(\alpha)\frac{\sigma_{1N}(\theta_{10})}{\sigma_{1N}(\theta_{1N})} - \frac{\mu_{1N}(\theta_{1N}) - \mu_{1N}(\theta_{10})}{\sigma_{1N}(\theta_{1N})}, \\ &\qquad \ldots, C_N(\alpha)\frac{\sigma_{pN}(\theta_{p0})}{\sigma_{pN}(\theta_{pN})} - \frac{\mu_{pN}(\theta_{pN}) - \mu_{pN}(\theta_{p0})}{\sigma_{pN}(\theta_{pN})}\Bigg) \\ &= 1 - \Phi_{\Sigma}(C(\alpha) - c_1 m_1, \ldots, C(\alpha) - c_p m_p)\end{aligned}$$

where Φ_Σ is the p-variate normal cumulative distribution function with 0 mean vector and covariance matrix Σ. $C(\alpha)$ is such that $\lim_{N\to\infty} P\{Q_N \geq C(\alpha)\} = \alpha$ and $m_i = \lim_{N\to\infty} \mu_{iN}'(\theta_{i0})/(\sqrt{N}\sigma_{iN}(\theta_{i0}))$ and $\mu_{iN}(\theta_{i0})$, $\sigma_{iN}(\theta_{i0})$ are the expectation and variance of T_{iN} under $H_0: \theta_i = \theta_{i0}$ for all i.

Bhattacharyya and Johnson obtained the efficacy for their test statistic L_N, under the

Pitman translation alternatives. We note that the limiting distribution of L_N is univariate normal. However, as we already have seen, the limiting distribution of Q_N is related with the multivariate normal distribution. Therefore the comparisons of the performances between two tests through ARE are not clear without any theoretical results concerned with asymptotic null distribution of Q_N. Therefore we compare the power through the computer simulations in the next section.

4. Simulation Results

4.1 Battacharyya and Johnson's Test

Let us briefly introduce the nonparametric test proposed by Bhattacharyya and Johnson(1970) for the one-sided bivariate location translation alternatives $K: \theta_1 \geq 0,\ \theta_2 \geq 0\ (\ \boldsymbol{\theta} \neq \mathbf{0}\ )$.Let $\{Z_1, Z_2, \cdots, Z_m, Z_{m+1}, \cdots, Z_{m+n}\}$ be the combined samples of m's X samples and n's Y samples. Let $N = m + n$ and $Z_i = (Z_{i1}, Z_{i2})$ Define

$$L(i,j) = \begin{cases} 1, & Z_{i1} \geq Z_{j1}, Z_{i2} \geq Z_{j2} \\ 0, & otherwise \end{cases}$$

and

$$L_i = \sum_{j=1}^{N} L(i,j),\ \ 1 \leq i \leq N,\ \ 1 \leq j \leq N, L = (L_1, \ldots, L_N).$$

Then L_i is called the 3rd quadrant layer rank of Z_i in the combined sample $\{Z_1, \ldots, Z_N\}$. Bhattacharyya and Johnson(1970) suggested a test statistic L_N by

$$L_N = N^{-2}[w \sum_{i=m+1}^{N} L_i - (1-w) \sum_{i=1}^{m} L_i], \qquad w = m/N.$$

They also showed that the null distribution of L_N is given by

$$L_N^* = L_N \cdot [\frac{mn}{N^5(N-1)} \sum_{i=1}^{N} (l_i - \bar{l})^2]^{-1/2} \quad \rightarrow \quad N(0,\ 1),$$

where l_i is the observed value of the 3rd quadrant layer ranks and $\bar{l} = \sum l_i / N.$

4.2 Power Comparisons

We compare the powers of Q_N with those of L_N, which is based on the layer ranks through simulation studies involving two different bivariate distributions, the bivariate normal distribution (Table 1) and the exponential distribution (Tables 2 and 3). In case of Q_N, we only consider the Wilcoxon rank sum statistics for both coordinates. For the normal distribution, we consider the cases of three different correlation coefficients, $\rho = 0$ 0.2 and 0.5 For the exponential distribution, we consider following two different cases: One is that two components are independent (Tables 2) and the other, so-called Marshall-Olkin type bivariate exponential distribution (cf. Barlow and Proschan 1975) (Tables 3). Also we consider two cases for each distribution that the location translation vector, $\theta = (\theta_1, \theta_2)$ varies with the same values of θ_1 and θ_2 and θ_1 varies while θ_2 is fixed as 0.

For each distribution, simulations have been carried out under the nominal significance level $\alpha = 0.05$ The results are based on 1000 simulations with sample sizes $m = 15$ and $n = 20$ for each distribution. For the language, we used S-Plus 4. Especially, for obtaining the quantile points required when we determine the critical values, we used *pmvnorm* function of S-Plus.

From Tables 1, 2 and 3, we see that the two tests reveal little difference in powers in case of the same values of θ_1 and θ_2 . However, we note that for the case that one is fixed while the other varies, our procedure is much superior to the that based on L_N for all distributions. Also we note that in case of both types of the exponential distributions, the powers based on Q_N and L_N are much better than those of normal distribution since the both procedures are nonparametric. Therefore our proposed test can be a good alternative to the test by Bhattacharyya and Johnson.

Table 1. Bivariate normal distribution

Test Statistics	ρ	(θ_1, θ_2) Location Translation								
		(0,0)	(0.3,0)	(0.3,0.3)	(0.6,0)	(0.6,0.6)	(0.9,0)	(0.9,0.9)	(1.2,0)	(1.2,1.2)
Q_N	0	0.047	0.194	0.266	0.486	0.688	0.781	0.957	0.943	0.997
L_N	0	0.055	0.116	0.293	0.280	0.671	0.499	0.948	0.687	0.996
Q_N	0.2	0.048	0.182	0.240	0.480	0.649	0.778	0.945	0.943	0.999
L_N	0.2	0.055	0.107	0.271	0.275	0.652	0.442	0.898	0.618	0.995
Q_N	0.5	0.047	0.167	0.266	0.465	0.689	0.781	0.958	0.951	0.997
L_N	0.5	0.053	0.106	0.233	0.231	0.588	0.374	0.882	0.608	0.979

Table 2. Independent exponential distribution

Test Statistic	(θ_1, θ_2) Location Translation								
	(0,0)	(0.3,0)	(0.3,0.3)	(0.6,0)	(0.6,0.6)	(0.9,0)	(0.9,0.9)	(1.2,0)	(1.2,1.2)
Q_N	0.048	0.274	0.457	0.661	0.868	0.889	0.987	0.973	0.998
L_N	0.048	0.219	0.523	0.455	0.916	0.635	0.991	0.785	1

Table 3. Marshall-Olkin's bivariate exponential distribution

Test Statistic	(θ_1, θ_2) Location Translation								
	(0,0)	(0.3,0)	(0.3,0.3)	(0.6,0)	(0.6,0.6)	(0.9,0)	(0.9,0.9)	(1.2,0)	(1.2,1.2)
Q_N	0.051	0.662	0.843	0.976	0.996	1	1	1	1
L_N	0.051	0.384	0.873	0.746	0.997	0.898	1	0.962	1

5. Concluding Remarks

In this section, first of all, we consider some variants of the one-sided test and briefly discuss to modify the test procedures for the variants. For the simplicity of arguments, we confine our discussion to the bivariate case. The extensions to the multivariate cases are only notational matters and straightforward. Now we consider the following hypotheses:

(i) $H_0: \theta_1 \geq \theta_{10}, \theta_2 \geq \theta_{20}$ v.s. H_1: *at least one of θ_i's is strictly smaller than θ_{i0}*

(ii) $H_0: \theta_1 \leq \theta_{10}, \theta_2 \geq \theta_{20}$ v.s. $H_1: \theta_1 > \theta_{10}$ *or* $\theta_2 < \theta_{20}$ *or both*.

For case (i), by switching the roles of two samples, still we may use the maximum between two univariate test statistics. Or if the roles of two samples are maintained, then we may use the minimum. In any case, we note that we may draw the same conclusions. For case (ii), we may use the reversed rank system such as we assign 1 for the largest observation and N, the smallest one when we use the Wilcoxon rank sum statistic. Or we may count the observations which are less than or equal to a median from the combined sample in case of the median test for the second part of hypotheses. Then the rest of the test procedures remains unchanged.

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