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A Family of Tests for Trend Change in Mean Residual Life with Known Change Point[1]

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Abstract

The mean residual life function is the expected remaining life of an item at age x. The problem of trend change in the mean residual life is great interest in the reliability and survival analysis. In this paper, we develop a family of test statistics for testing whether or not the mean residual life changes its trend. The asymptotic normality of the test statistics is established. Monte Carlo simulations are conducted to study the performance of our test statistics.

Keywords : mean residual life, trend change, test statistic

1. Introduction

Let F be a continuous life distribution(i. e., $F(x)=0$ for $x \le 0$) with the finite first moment and let X be a nonnegative random variable with distribution F. The mean residual life(MRL) function $e(x)$ is defined as

$$e(x)=E(X-x \mid X>x). \tag{1.1}$$

The MRL is the expected remaining lifetime, $X-x$, given that the item has survived to time x. The MRL function $e(x)$ in (1.1) can also be written as

$$e(x)=\frac{\int_x^{\infty}\overline{F}(u)\,du}{\overline{F}(x)},$$

where $\overline{F}(x)=1-F(x)$ is the reliability function.

The MRL function plays a very important role in the area of engineering, medical science, survival studies, social sciences, and many other fields. The MRL is used by engineers in

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burn-in studies, setting maintenance policies, and in comparison of life distributions of different systems. Social scientists use MRL, also called as inertia, in studies of lengths of wars, duration of strikes, job mobility etc. Medical researchers use MRL in lifetime experiments under various conditions. Actuaries apply MRL to setting rates and benefits for life insurance.

Guess and Proschan(1988) show that various families of life distributions defined in terms of the MRL(e. g. increasing MRL, decreasing MRL) have been used as models for lifetimes for which such prior information is available. One such family of distributions is called as "increasing initially then decreasing MRL(IDMRL)" distributions if there exists a change point $\tau\geq 0$ such that

$$e(s)\leq e(t) \quad \text{for } 0\leq s\leq t<\tau, \quad e(s)\geq e(t) \quad \text{for } \tau\leq s\leq t. \tag{1.2}$$

The dual class of "decreasing initially, then increasing MRL(DIMRL)" distributions is obtained by reversing inequalities on the MRL function in (1.2). See Guess and Proschan(1988) and the references therein for examples and applications of the IDMRL(DIMRL) class.

It is well known that $e(x)$ is constant for all $x\geq 0$ if and only if F is an exponential distribution (i. e., $F(x)=1-\exp(-x/\mu)$ for $x\geq 0$, $\mu>0$). Due to this "no-aging" property of the exponential distribution, it is of practical interest to know whether a given life distribution F is constant MRL or IDMRL. Therefore, we consider the problem of testing

H_0 : F is constant MRL,

against

H_1 : F is IDMRL (and not constant MRL),

based on random samples. When the dual model is proposed, we test H_0 against

H_1' : F is DIMRL (and not constant MRL).

This problem is noted by Guess, et al.(henceforth GHP, 1986), who obtain tests when the change point is known or when the proportion before the change point takes place is known. Aly(1990) suggests several tests for monotonicity of MRL. These tests consider the IDMRL alternative when either the change point or the proportion is known. Hawkins, Kochar and Loader (HKL, 1992) developed a test for exponentiality against IDMRL alternative when neither the change point nor the proportion is known. Lim and Park(1995) propose a test for the trend change in MRL when the proportion is known. Lim and Park (1998) studied a family of IDMRL tests when the proportion is known. Na et al.(henceforth NLK, 1998) propose a test for the trend change in MRL when the change point is known. They compare their test with GHP's(1986) test and Aly's(1990) test by considering the power of test.

In this paper, we develop a family of test statistics for testing H_0 against $H_1(H_1')$ alternative. We assume that when the change point is known. GHP(1986) provide excellent explaining that this assumption is very realistic in many interesting situations. We derive the asymptotic null distributions of our test statistics. To establish the asymptotic distribution of our test statistics, we use the differential statistical function approach. Monte Carlo simulations are conducted to compare the performance of our test statistics with those of GHP's(1986) test and Aly's(1990) test by the powers of tests.

Section 2 is devoted to develope a family of test statistics for testing H_0 against H_1 (H_1'). Results of simulations are presented in Section 3.

2. A Family of Test Statistics

In this section we propose a test statistic for testing exponentiality against IDMRL(DIMRL) alternative. We assume that the change point τ is known or has been specified by the user. Our test statistic is motivated by a simple observation. If $e(x)$ is differentiable and decreasing(increasing), then

$$\frac{de(x)}{dx} = \frac{f(x)v(x) - \overline{F}^2(x)}{\overline{F}^2(x)} \le (\ge) 0,$$

where $v(x) = \int_x^\infty \overline{F}(u)du$ and $f(x)$ denotes the probability density function corresponding to F. Thus $e(x)$ is decreasing(increasing) if and only if $f(x)v(x) \le (\ge) \overline{F}^2(x)$. Hence, as a measure of the deviation from the null hypothesis H_0 in favor of H_1, we propose the parameter

$$T_j(F) = \int_0^\tau \overline{F}^j(x)(f(x)v(x) - \overline{F}^2(x))dx + \int_\tau^\infty \overline{F}^j(x)(\overline{F}^2(x) - f(x)v(x))dx$$

where j is a integer with $j \ge -1$. This parameter coincides with that of Aly(1990) and NLK(1998) when $j = -1$ and $j - 0$, respectively. Note that $T_j(F)$ is zero for the exponential distribution F and strictly positive for the IDMRL F. Using integration by parts, we can rewrite $T_j(F)$ as

$$\begin{aligned} T_j(F) &= \frac{1}{j+1}\Big(\int_0^\infty \overline{F}(x)dx - (j+2)\int_0^\tau \overline{F}^{j+2}(x)dx \\ &\quad + (j+2)\int_\tau^\infty \overline{F}^{j+2}(x)dx - 2\overline{F}^{j+1}(\tau)\int_\tau^\infty \overline{F}(x)dx \Big). \end{aligned}$$

Let $F_n(x)$ be the empirical distribution formed by a random sample $X_1, \cdots, X_n$ from F and let $X_{(1)} < \cdots < X_{(n)}$ be the order statistics of the sample. Then we can estimate $T_j(F)$ by

$$\begin{aligned} T_j(F_n) &= \sum_{i=1}^{i^*} B_{1j}\left(\frac{n-i+1}{n}\right)(X_{(i)} - X_{(i-1)}) + B_{1j}\left(\frac{n-i^*}{n}\right)(\tau - X_{(i^*)}) \\ &\quad + B_{2j}\left(\frac{n-i^*}{n}\right)(X_{(i^*+1)} - \tau) + \sum_{i=i^*+2}^{n} B_{2j}\left(\frac{n-i+1}{n}\right)(X_{(i)} - X_{(i-1)}), \end{aligned}$$

where $0 = X_{(0)} < X_{(1)} < \cdots < X_{(i^*)} \le \tau < X_{(i^*+1)} < \cdots < X_{(n)}$,

$$B_{1j}(u) = \frac{1}{j+1}\{u-(j+2)u^{j+2}\} \text{ and}$$

$$B_{2j}(u) = \frac{1}{j+1}\{(1-2\ \overline{F}_n^{\,j+1}(\tau))u+(j+2)u^{j+2}\}.$$

To establish asymptotic distribution of $T_j(F_n)$, we use the differentiable statistical function approach of von Mises(1947) (cf. Boos and Serfling(1980) and Serfling(1980)). Also see NLK(1998) that obtain the asymptotic distribution of a test statistic $T_0(F_n)$. The asymptotic distribution of $T_j(F_n)$ is summarized in Theorem 2.1.

THEOREM 2.1 Let F be the life distribution such that $0 < F(\tau) < 1$ and $\sigma^2(T_j, F) < \infty$. Then

$$\sqrt{n}(T_j(F_n) - T_j(F)) \xrightarrow{d} N(0, \sigma^2(T_j, F)).$$

Under H_0, (i.e. F is exponential with mean μ), we have that $\sqrt{n}T_j(F_n)$ is asymptotically normal distributed with mean 0 and variance $\mu^2/(2j+3)$. The distribution of $T(F_n)$ is not scale invariant. In order to make our test statistics scale invariant we use the test statistics

$$T_j^* = \frac{\sqrt{n}\,T_j(F_n)}{\overline{X}}$$

where $\overline{X}$ denote the sample mean. By Slusky's theorem, T_j^* is asymptotically normal distributed with mean 0 and variance $1/(2j+3)$, under H_0.

The IDMRL(τ) test procedures rejects H_0 in favor of H_1 at the approximation level α if $\sqrt{2j+3}\,T_j^* \geq z_\alpha$. Analogously, the DIMRL(τ) test rejects H_0 in favor of H_1' at the approximation level α if $\sqrt{2j+3}\,T_j^* \leq -z_\alpha$.

3. SIMULATION STUDY

In this section we perform a Monte Carlo simulation to investigate the speed of convergence of the proposed family of test statistics, for various τ and n, and to compare the performance of our test statistics with that of GHP's(1986) test and Aly's(1990) test by simulating the power of test. For Monte Carlo study we use the subroutine IMSL of the package FORTRAN.

To investigate the empirical test size, random numbers are generated from exponential distribution, $F(x) = 1-\exp(-x), x \geq 0$, since our test statistics are scale invariant. Table 3.1-3.3 present the empirical test size of IDMRL(τ) tests based on T_j^* for some j.

<Table 3.1> Empirical test size of IDMRL(τ) tests based on U_n^* and T_j^* for some j when $F(\tau)=0.1$

n	α	GHP	j =-1	j =0	j =1	j =2
20	.10	.125	.141	.160	.157	.149
	.05	.052	.044	.092	.089	.083
	.01	.008	.004	.022	.023	.014
40	.10	.125	.152	.136	.122	.009
	.05	.054	.057	.074	.068	.113
	.01	.004	.002	.019	.018	.061
60	.10	.129	.147	.140	.118	.108
	.05	.062	.068	.070	.059	.059
	.01	.004	.007	.013	.009	.006
80	.10	.133	.153	.128	.122	.104
	.05	.061	.066	.068	.060	.050
	.01	.013	.010	.014	.014	.014
100	.10	.121	.138	.109	.107	.105
	.05	.057	.067	.056	.050	.054
	.01	.016	.015	.017	.020	.015

<Table 3.2> Empirical test size of IDMRL(τ) tests based on U_n^* and T_j^* for some j when $F(\tau)=0.5$

n	α	GHP	j =-1	j =0	j =1	j =2
20	.10	.133	.120	.080	.050	.038
	.05	.060	.068	.040	.019	.012
	.01	.007	.012	.012	.002	.000
40	.10	.131	.145	.095	.070	.056
	.05	.060	.081	.051	.030	.016
	.01	.014	.025	.011	.002	.002
60	.10	.123	.134	.092	.074	.067
	.05	.057	.064	.048	.035	.030
	.01	.009	.016	.005	.001	.002
80	.10	.113	.145	.094	.075	.067
	.05	.057	.069	.050	.037	.028
	.01	.008	.022	.011	.003	.001
100	.10	.110	.140	.099	.088	.078
	.05	.058	.076	.059	.044	.037
	.01	.010	.017	.011	.006	.005

<Table 3.3> Empirical test size of IDMRL(τ) tests based on U_n^* and T_j^* for some j when $F(\tau)=0.9$

n	α	GHP	j =-1	j =0	j =1	j =2
20	.10	.070	.082	.052	.044	.035
	.05	.042	.038	.025	.012	.009
	.01	.010	.018	.004	.000	.000
40	.10	.079	.104	.073	.056	.048
	.05	.043	.069	.032	.028	.025
	.01	.009	.023	.004	.004	.005
60	.10	.099	.121	.079	.066	.058
	.05	.047	.072	.041	.030	.021
	.01	.015	.020	.006	.002	.002
80	.10	.099	.121	.079	.063	.064
	.05	.050	.072	.039	.027	.024
	.01	.015	.018	.009	.006	.002
100	.10	.086	.112	.081	.073	.064
	.05	.053	.065	.039	.027	.026
	.01	.012	.019	.009	.004	.003

The values in Tables are the fraction of times that H_0 is rejected in favor of H_1 when H_0 is true. The empirical test sizes are calculated based on 1000 replications for; α=0.10, 0.05, 0.01; $n=20,40,\cdots,100$; $F(\tau)=0.1$,0.5, 0.9.

From Table 3.1-3.3, we notice that the fastest convergence of T_j^* is obtained by using T_2^* when $F(\tau)=0.1$. The test size of T_2^* is close to the level of significance for $F(\tau)=0.1$ when $n\geq 40$. When $F(\tau)=0.5$, the fastest convergence of T_j^* is obtained by using T_0^*. The test size of T_0^* is close to the level of significance for $F(\tau)=0.5$ when $n\geq 40$. The fastest convergence of T_j^* is obtained by using T^*_{-1} when $F(\tau)=0.9$. The test size of T^*_{-1} is close to the level of significance for $F(\tau)=0.9$ when $n\geq 40$. The T^*_{-1} test overestimate test size the level of significance α for all case except $F(\tau)=0.9$. The T_j^* test for large value of j underestimate test size α for all case except $F(\tau)=0.1$.

To compare the power of our test based on T_j^* with that of GHP's(1986) test based on U_n^* and Aly's(1990) test based on T^*_{-1}, the random numbers are generated from

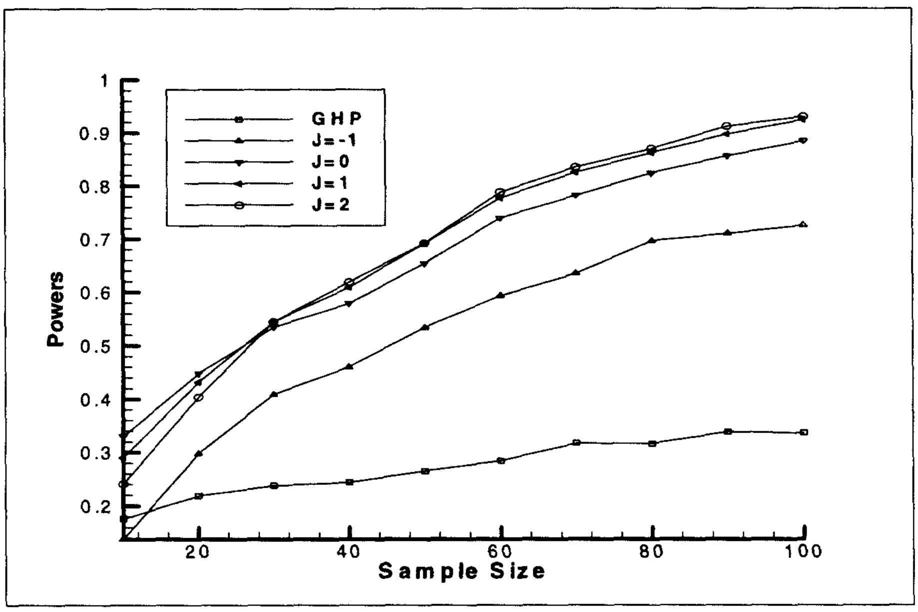
$$\overline{F}_{\alpha,\beta,\gamma}(x)=\left\{\frac{\beta}{\beta+\gamma\exp(-\alpha x)(1-\exp(-\alpha x))}\right\}\left\{\frac{[1+d]^2-c^2}{[\exp(\alpha x)+d]^2-c^2}\right\}^{1/2\alpha\beta}$$

$$\times\left\{\frac{\exp(\alpha x)+d-c}{\exp(\alpha x)+d+c}\frac{1+d+c}{1+d-c}\right\}^{\gamma/4\alpha\beta^2 c}, \quad x\geq 0$$

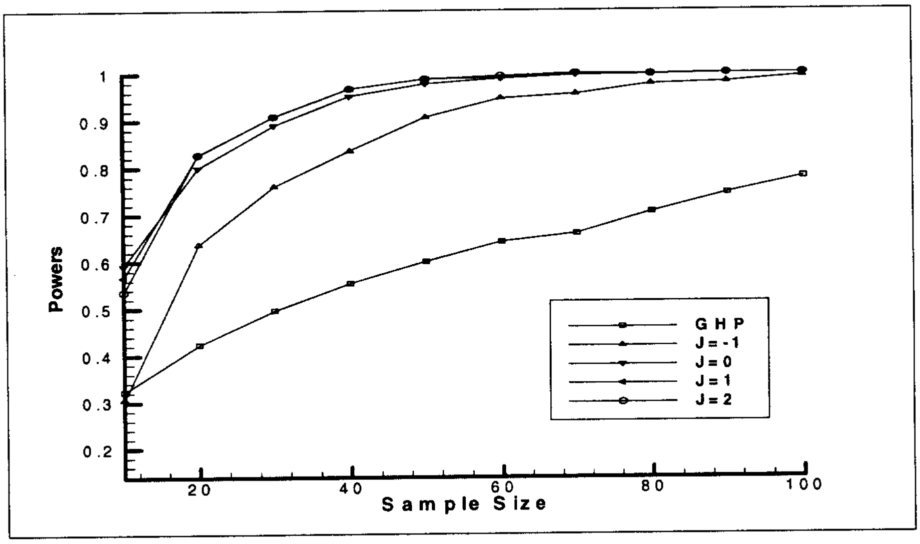
where $d=\gamma/2\beta$, $c^2=[4(\beta/\gamma)+1]/[4(\beta/\gamma)^2]$. This distribution has MRL function $e_{\alpha,\beta,\gamma}(x)=\beta+\gamma\exp(-\alpha x)(1-\exp(-\alpha x))$, $x\geq 0$. The motivation for choosing $\overline{F}_{\alpha,\beta,\gamma}$ is that $\overline{F}_{\alpha,\beta,\gamma}$ has IDMRL structure with the change point $\tau=\ln 2/\alpha$ for any choice of (α,β,γ) and $\overline{F}_{\alpha,\beta,\gamma}$ is exponential distribution if $\gamma=0$.

Figures 3.1~3.4 contain Monte Carlo estimated powers based on 1000 replications of sample size $n=10,20,\cdots,100$ from $\overline{F}_{\alpha,\beta,\gamma}$ for $\beta=1$ and a selection of (α,γ) when the level of significance is 0.05.

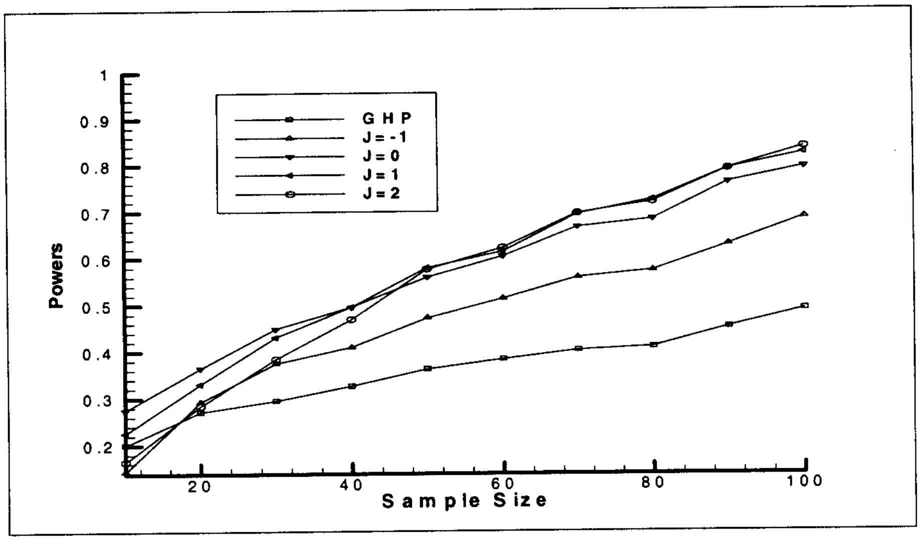
From figures, we notice that the powers of all tests increase rapidly as γ increases when α is fixed and also as α increases (i.e., the change point τ decreases) when γ is fixed. It is further better to increase γ than α. This is generally to be expected since the width of $e(x)$ increases as γ increases. Figures also show that our tests generally dominates the other tests except small α and small γ. But the power of our tests increase more rapidly than those of the other tests as n increases for any α and γ.



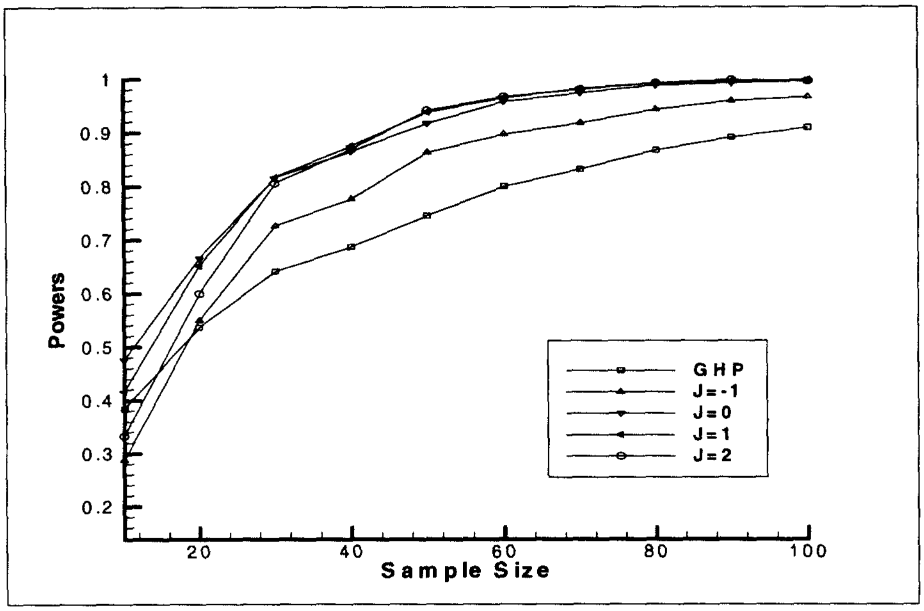
<Figure 3.1> Empirical power of U_n^* and T_j^* tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with parameter $\alpha=5$, $\beta=1$ and $\gamma=0.5$.



<Figure 3.2> Empirical power of U_n^* and T_j^* tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with parameter $\alpha=5$, $\beta=1$ and $\gamma=1$.



<Figure 3.3> Empirical power of U_n^* and T_j^* tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with parameter $\alpha=3$, $\beta=1$ and $\gamma=0.5$.



<Figure 3.4> Empirical power of U_n^* and T_j^* tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with parameter $\alpha=3$, $\beta=1$ and $\gamma=1$.

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