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Constrained Estimation of the Numbers of Trials in Several Binomial Populations

Myongsik Oh[1] and Eun-Kyoung Lee[2]

Abstract

The constrained maximum likelihood estimation of the number of trials in several binomial populations under order restriction, such as simple order, is discussed. The estimation procedure is based on, so called, pool adjacent violators algorithm. Three handy estimators are given and their performances are compared using an artificial example.

Keywords : Binomial Population, Constrained Maximum Likelihood Estimation, Pool Adjacent Violators Algorithm, Simple Order

1. Introduction

Many practical problems can be characterized as dichotomous phenomenon, whose outcomes are generically called success and failure. The most frequently used statistical model for this phenomenon is the binomial model. The binomial model is based on the binomial distribution which is characterized by two parameters; one is the number of trials, the other is the probability of success. In many practical situations where the binomial model is applied the very interest is focused on making statistical inference concerning the probability of success. On the other hand, it is rarely necessary to make inference on the number of trials. The problem, however, sometimes has practical motivation and studied by some researcher. Binet(1953) is among others.

In some practical situations we are only allowed to observe successes or failures. For instance, rape victims are very reluctant to report to police or other agents for various reasons. People who work for the preventive program of rape sometimes might need to know how many rape victims would be filed if all of them are reported. Since only the reported cases are counted the total number of rape victims should be statistically estimated. If we are aware of the rate of reported cases from prior or other studies we can easily estimate the

1) Assistant Professor, Department of Statistics, Pusan University of Foreign Studies, Pusan, 608-738, Korea. E-mail : moh@stat.pufs.ac.kr
2) Graduate Student, Graduate School of Education, Pusan University of Foreign Studies, Pusan 608-738, Korea.

total number of rape victims. Later we will discuss this estimation procedure in great details.

As discussed before it is not difficult to estimate the number of trials if the probability of success is known to us. It is possible to estimate the number of trials even if the probability of success is unknown to us provided that repeated observations are available and the number of trials is remained fixed for each observation. In this paper we study the estimation procedure of several numbers of trials when there appears an order restriction among parameters.

Suppose there are k populations. For ith population the binomial model with parameters N_i and p_i are assumed. Let $X_{ij}, i=1,\ldots,k, j=1,\ldots,n_i$ be the jth observation from ith population. Note that each n_i is the sample size from ith population and known. And assume that X_{ij}'s are independent and binomially distributed with N_i and p_i.

As discussed earlier the traditional problem is to estimate p_i's provided that N_i's are known. An order restriction would be imposed among p_i's, which is often studied in bioassay problem. Interested reader may refer Robertson, Wright and Dykstra (1988), Oh(1995) and many others.

We are, however, interested in estimating N_i's under an order restriction when p_i's are either known or unknown. Even though there are many types of orderings such as the simple tree order, the simple loop and so on, the simple order is of great interest since it is indicative of general behavior. Now we assume that

$$N_1 \le N_2 \le \cdots \le N_k. \tag{1.1}$$

In addition to this N_i's should be integers and must satisfy, for each i,

$$N_i \ge \max_{1 \le j \le n_i} x_{ij}. \tag{1.2}$$

Now the problem is to find N_i's which maximize

$$\prod_{i=1}^{k} \prod_{j=1}^{n_i} \binom{N_i}{x_{ij}} p_i^{x_{ij}} (1-p_i)^{N_i - x_{ij}} \tag{1.3}$$

subject to (1.1) and (1.2). We note that p_i's may or may not be known. For both cases the maximum likelihood estimation procedure can be implemented.

Since the estimation procedure may vary according to the fact that whether the probabilities of success are known to us or not, we need to consider the all possible cases which are;

Case 1. $p_1 = p_2 = \cdots = p_k = p$, p is known,

Case 2. $p_1, p_2, \cdots, p_k$, are completely known,

Case 3, $p_1 = p_2 = \cdots = p_k = p$, p is unknown,

Case 4. $p_1, p_2, \cdots, p_k$, are completely unknown.

For Cases 3 and 4 the estimators for numbers of trials are trivial. So we only consider the Cases 1 and 2 in this paper.

In section 2 we propose three new easy-to-use estimators, which are maximum likelihood estimators, of the number of trials under no restriction but the limitation of being integer. In section 3 maximum likelihood estimators under restriction, which are (i) equality of two or more parameters (ii) the simple order. The algorithm for estimating parameters under order restriction is basically the same as PAVA (Pool Adjacent Violators Algorithm). We propose three estimation procedures and compare the performance of the estimators using artificial examples.

2. The Unrestricted Model

The rather details of estimation procedures under no restriction are given in Johnson and Kotz(1979). The method of moment estimates with ignoring the limitation that the number of trials must be integer are given by Fisher(1941, see Johnson and Kotz, 1969). Binet(1953, see also Johnson and Kotz, 1969) improve Fisher's estimation by taking into account the fact that the number of trials must be integer and must be greater than or equal to the maximum observation. Readers refer to Johnson and Kotz (1969) for full discussion of this procedures. In this section we are going to discuss the maximum likelihood estimation of the number of trials under no restriction other than being integer. The estimation procedure is quite intuitive and easy to use.

To find the maximum likelihood estimate of the number of trials we need to N_1 which maximizes

$$L(N_1)=\prod_{j=1}^{n_1}\binom{N_1}{x_{1j}}p_1^{x_{1j}}(1-p_1)^{N_1-x_{1j}}.$$

Consider the ratio $L(N_1)/L(N_1+1)$. We have

$$\frac{L(N_1)}{L(N_1+1)} = \prod_{j=1}^{n_1}\frac{\binom{N_1}{x_{1j}}p_1^{x_{1j}}(1-p_1)^{N_1-x_{1j}}}{\binom{N_1+1}{x_{1j}}p_1^{x_{1j}}(1-p_1)^{N_1+1-x_{1j}}}$$
$$= (1-p_1)^{-n_1}\prod_{j=1}^{n_1}\left(1-\frac{x_{1j}}{N_1+1}\right).$$

For fixed $x_{1j}, j=1,\dots,n_1$, we can easily show that $L(N_1)/L(N_1+1)$ is a strictly increasing function of N_1.

Suppose that $L(N_1)/L(N_1+1) > 1$ for $N_1 \ge \max\{x_{1j}, j=1,\dots,n_1\}$. Then $L(N_1)$ is maximized when $N_1=\max\{x_{1j}, j=1,\dots,n_1\}$ and hence the unrestricted maximum likelihood estimate, $\widehat{N}_1$, of N_1 is given by $\max\{x_{1j}, j=1,\dots,n_1\}$.

Suppose $0 \le L(N_1)/L(N_1+1) < \infty$ for $N_1 \ge \max\{x_{1j}, j=1,\dots,n_1\}$. To find an estimator we need to ignore the limitation that N_1 must be an integer temporarily. The estimator based on this temporary assumption provides us a good start to find the desired one, i.e., the estimator which meets the limitation. Since $L(N_1)/L(N_1+1)$ is continuous and strictly increasing with respect to N_1 there exists $\widetilde{N}_1$ which satisfies $L(\widetilde{N}_1)/L(\widetilde{N}_1+1)=1$ and hence maximizes $L(N_1)$. Then the maximum likelihood estimate of N_1 is determined around $\widetilde{N}_1$. If $\widetilde{N}_1$ is an integer then $L(\widetilde{N}_1)=L(\widetilde{N}_1+1)$ and hence $\widehat{N}_1$ is either $\widetilde{N}_1$ or $\widetilde{N}_1+1$. If is not an integer finding an exact maximum likelihood estimate becomes much more complicated. Let $[x]$ be the greatest integer which is less than or equal to x. Since $\widetilde{N}_1$ is not an integer we have

$$\frac{L(N_1)}{L(N_1+1)} < 1 \quad \text{for } N_1 \le [\widetilde{N}_1] \quad \text{and} \quad \frac{L(N_1)}{L(N_1+1)} > 1 \quad \text{for } N_1 \ge [\widetilde{N}_1]+1.$$

In other words $L(N_1)$ is maximized when N_1 is one of $\widetilde{N}_1$ or $\widetilde{N}_1+1$. Now we need to determine which one gives the maximum value of the likelihood function.

First we discuss about how to determine $\widetilde{N}_1$ with the limitation of being integer. By taking logarithm on the both sides of $L(N_1)/L(N_1+1) = 1$, we have

$$\frac{1}{n_1}\sum_{j=1}^{n_1}\ln\left(1-\frac{x_{1j}}{N_1+1}\right)=\ln(1-p_1).$$

Using Jensen's inequality we have

$$-\frac{1}{n_1}\sum_{j=1}^{n_1}\ln\left(1-\frac{x_{1j}}{N_1+1}\right)\ge -\ln\left(\frac{1}{n_1}\sum_{j=1}^{n_1}\left(1-\frac{x_{1j}}{N_1+1}\right)\right),$$

which is equivalent to

$$\frac{1}{n_1}\sum_{j=1}^{n_1}\ln\left(1-\frac{x_{1j}}{N_1+1}\right)\le\ln\left(1-\frac{\overline{x}_1}{N_1+1}\right),$$

where $\overline{x}_i=\sum_{j=1}^{n_i}x_{ij}/n_i$. Since the left side of the above equation is equal to $\ln(1-p_1)$ we have $\ln(1-p_1)\le\ln\left(1-\frac{\overline{x}_1}{N_1+1}\right)$ and hence $N_1\ge\frac{\overline{x}_1}{p_1}-1$. We now easily expect that maximum likelihood estimate is going to be determined at either $[\overline{x}_1/p_1]$ or $[\overline{x}_1/p_1]+1$. We are going to show later that the maximum of the likelihood function is attained at either $[\overline{x}_1/p_1]$ or $[\overline{x}_1/p_1]+1$ using an artificial example.

An approximate solution can be suggested as follows. By taking Taylor expansion for $\ln\left(1-\frac{x_{1j}}{N_1+1}\right)$ about p_1 up to the second degree, we have

$$\ln(1-\frac{x_{1j}}{N_1+1})\approx\ln(1-p_1)-\frac{1}{1-p_1}(\frac{x_{1j}}{N_1+1}-p_1)-\frac{1}{2(1-p_1)^2}(\frac{x_{1j}}{N_1+1}-p_1)^2$$

and

$$\frac{1}{n_1}\sum_{j=1}^{n_1}\ln(1-\frac{x_{1j}}{N_1+1})\approx\ln(1-p_1)$$

$$-\frac{1}{1-p_1}\left(\frac{\bar{x}_1}{N_1+1}-p_1\right)-\frac{1}{2(1-p_1)^2}\left(\frac{\sum_{j=1}^{n_1}x_{1j}^2/n_1}{(N_1+1)^2}-\frac{2\bar{x}_1p_1}{N_1+1}+p_1^2\right).$$

Equating the right side of the above equation to $\ln(1-p_1)$ we have

$$\frac{1}{1-p_1}\left(\frac{\bar{x}_1}{N_1+1}-p_1\right)+\frac{1}{2(1-p_1)^2}\left(\frac{\sum_{j=1}^{n_1}x_{1j}^2/n_1}{(N_1+1)^2}-\frac{2\bar{x}_1p_1}{N_1+1}+p_1^2\right)=0.$$

Solving the above equation for N_1 gives the solution, denoted by $N_1^{(a)}$,

$$-1+\frac{\bar{x}_1(2p_1-1)-\sqrt{\bar{x}_1^2(1-2p_1)^2-(3p_1^2-2p_1)\sum_{j=1}^{n_1}x_{1j}^2/n_1}}{3p_1^2-2p_1}.$$

Note that the solution must be positive. It has been verified by some simulation that the likelihood function is maximized when $N_1=[N_1^{(a)}+1]$. It is not straightforward to show analytically that $[N_1^{(a)}+1]$ gives actually the maximum value of the likelihood function.

So far we have proposed three estimation procedures. We are going to compare these three procedures using an artificial example. For various numbers of trials and probabilities of success artificial data is generated. For N_1 we choose four values which are 5, 10, 25 and 50, for p_1 the values 0.1, 0.25, 0.5 and 0.75 are used. Four different values of sample sizes are used. These are 5, 10, 25 and 50. The maximum likelihood estimate is determined by searching the maximum point numerically, which is also a method for finding maximum likelihood estimator but quite tedious. At this moment we need to point out that we are not interested in whether these estimators result in over-estimation or under-estimation. A simulation study based on huge amount of repeated experiments is needed to have an answer to this question. We are not going to deal with this problem in this paper. Our main focus is placed on which estimator produce the estimates close to real maximum likelihood estimate. As is shown in Table 1 the approximate estimator gives the better performance. The use of $\bar{x}/p+1$ is not recommended.

Table 1. Comparison of Three Estimators for p_1=0.1 Various Values of N_1 , p_1 and Sample Size - One sample, No Restriction

		p_1=0.1				p_1=0.25			
N	Size	$\bar{x}/p$	$\bar{x}/p+1$	Approx.	Real	$\bar{x}/p$	$\bar{x}/p+1$	Approx.	Real
5	5	0	1	0	0	6	7	6	6
	10	3	4	4	5	2	3	3	3
	25	4	5	5	5	4	5	4	4
	50	4	5	5	5	4	5	5	5
10	5	15	16	16	16	5	6	6	6
	10	11	12	12	12	10	11	11	12
	25	9	13	10	10	9	10	9	10
	50	8	9	8	8	10	11	11	11
25	5	23	24	24	24	27	28	27	27
	10	27	28	28	28	30	31	30	30
	25	29	30	29	29	24	25	24	24
	50	23	24	23	23	25	26	25	25
50	5	45	46	46	46	52	53	52	52
	10	50	51	51	51	58	59	58	58
	25	45	46	45	45	52	53	52	52
	50	51	52	51	51	47	48	48	48
		p_1=0.5				p_1=0.75			
5	5	6	7	6	6	5	6	5	5
	10	4	5	4	4	4	5	5	5
	25	5	6	6	6	5	6	5	5
	50	4	5	4	4	5	6	5	5
10	5	10	11	10	10	9	10	11	11
	10	11	12	11	11	10	11	10	10
	25	10	11	11	11	10	11	11	11
	50	9	10	10	10	10	11	10	10
25	5	20	21	21	22	24	25	25	25
	10	25	26	25	25	24	25	24	24
	25	23	24	24	24	24	25	24	24
	50	24	25	24	24	24	25	25	25
50	5	47	48	47	47	54	55	54	54
	10	48	49	48	48	49	50	49	49
	25	50	51	51	51	50	51	51	51
	50	50	51	50	50	51	52	52	52

Now we close this section with stating the estimation procedure for cases 3 and 4. Since the estimation procedure for case 3 is basically the same as for the case 4, we discuss here case 4 only. It suffices to consider the one sample case. To maximize the likelihood $L(N_1)$

the following conditions must be satisfied.

$$\frac{\partial \ln L}{\partial p_1} = 0 \Rightarrow p_1 N_1 = \bar{x}_1,$$
$$\frac{\partial \ln L}{\partial N_1} = 0 \Rightarrow \sum_{j=1}^{n_1} \sum_{\ell=0}^{x_{1j}-1} \frac{1}{N_1 - \ell} = -n_1 \ln(1-p_1).$$

Solve the following system of equations with respect to $\tilde{p}_1$ and $\tilde{N}_1$

$$\tilde{p}_1 \tilde{N}_1 = \bar{x}_1,$$
$$\sum_{j=1}^{n_1} \sum_{\ell=0}^{x_{1j}-1} \frac{1}{\tilde{N}_1 - \ell} = -n_1 \ln(1-\tilde{p}_1).$$

Note that we need to take into account the fact that (i) N_1 must be an integer (ii) $N_1 \geq \max\{x_{1j}, j=1,\dots,n_1\}$. If $\tilde{N}_1 > \max\{x_{1j}, j=1,\dots,n_1\}$, then take the nearest integer to $\tilde{N}_1$ and if $\tilde{N}_1 \leq \max\{x_{1j}, j=1,\dots,n_1\}$ then take $\tilde{N}_1 = \max\{x_{1j}, j=1,\dots,n_1\}$. For more details we refer readers to page 57-58 of Johnson and Kotz (1969).

3. The Restricted Model

In this section we consider the estimation procedure when there appears an order restriction among N_i's. First we consider the case that the equality of the numbers of trials is assumed. Suppose that $N_1 = N_2$ but $p_1 \neq p_2$. Let $N = N_1 = N_2$. Assume that N is positive real number temporarily. As we did for the one sample case we begin with finding $L(N)/L(N+1)$ which is given by

$$(1-p_1)^{-n_1}(1-p_2)^{-n_2} \prod_{j=1}^{n_1}\left(1-\frac{x_{1j}}{N+1}\right) \prod_{j=1}^{n_2}\left(1-\frac{x_{2j}}{N+1}\right).$$

If $L(N)/L(N+1) > 1$ for $N \geq \max\{x_{ij}, j=1,\dots,n_i, i=1,2\}$ then the common estimate, N^*, of N_1 and N_2 is given by $\max\{x_{ij}, j=1,\dots,n_i, i=1,2\}$.

Next suppose $L(N)/L(N+1) > 0$ and solve the equation $L(N)/L(N+1) = 1$. By taking logarithm on both sides of the equation we have

$$\sum_{j=1}^{n_1} \ln\left(1-\frac{x_{1j}}{N+1}\right) + \sum_{j=1}^{n_2} \ln\left(1-\frac{x_{2j}}{N+1}\right) = n_1 \ln(1-p_1) + n_2 \ln(1-p_2)$$

Applying Jensen's inequality and after some algebra we have

$$n_1 \ln\left(1-\frac{\bar{x}_1}{N+1}\right) + n_2 \ln\left(1-\frac{\bar{x}_2}{N+1}\right) \geq n_1 \ln(1-p_1) + n_2 \ln(1-p_2).$$

Note that $a^x b^y < \left(\frac{xa+yb}{x+y}\right)^{x+y}$. See page 17 of Hardy, Littlewood and Polya(1952). Applying this well-known theorem on the left side of the above inequality we have

$$\ln\left(1-\frac{\alpha \bar{x}_1+(1-\alpha)\bar{x}_2}{N+1}\right)\geq \alpha\ln(1-p_1)+(1-\alpha)\ln(1-p_2),$$

where $\alpha=n_1/(n_1+n_2)$ and hence

$$N\geq \frac{\alpha \bar{x}_1+(1-\alpha)\bar{x}_2}{1-(1-p_1)^{\alpha}(1-p_2)^{1-\alpha}}-1. \tag{3.1}$$

Note that N must satisfy that $N\geq \max\{x_{ij}, j=1,\ldots,n_i, i=1,2\}$

An approximate solution is suggested next. By the similar manner that we used for one sample case we have the following equation.

$$A_2(N+1)^2+B_2(N+1)+C_2=0 \tag{3.2}$$

where

$$\begin{aligned} A_2 &= \frac{n_1(3p_1^2-2p_1)}{2(1-p_1)^2}+\frac{n_2(3p_2^2-2p_2)}{2(1-p_2)^2}, \\ B_2 &= \frac{n_1\bar{x}_1(1-2p_1)}{(1-p_1)^2}+\frac{n_2\bar{x}_2(1-2p_2)}{(1-p_2)^2}, \\ C_2 &= \frac{\sum_{j=1}^{n_1}x_{1j}^2}{2(1-p_1)^2}+\frac{\sum_{j=1}^{n_2}x_{2j}^2}{2(1-p_2)^2}. \end{aligned}$$

Solving the above quadratic equation gives a good start for finding the maximum likelihood estimate N^*.

Table 2 shows the performance of three estimators for two sample case. The probabilities of success considered in this study are $p_1=0.25$ and $p_2=0.25$. The other values are considered but the results are not shown in this paper. The approximate estimator outperforms the other two estimators as one-sample case.

Next we consider the case that the numbers of trials are the same for more than two populations. Assume that $N=N_{i_1}=\cdots=N_{i_2}$, where $1\leq i_1<\cdots<i_2\leq k$. Analogues of (3.1) and (3.2) can be obtained easily, respectively. First we have

$$N\geq \frac{\sum_{i=i_1}^{i_2}\alpha_i\bar{x}_i}{1-\prod_{i=i_1}^{i_2}(1-p_i)^{\alpha_i}}-1, \tag{3.3}$$

where $\alpha_i=n_i/(\sum_{i=i_1}^{i_2}n_i)$. An approximate solution is the solution to the following quadratic equation;

$$A_{i_1:i_2}(N+1)^2+B_{i_1:i_2}(N+1)+C_{i_1:i_2}=0,$$

where

$$A_{i_1 i_2} = \sum_{i=i_1}^{i_2} \frac{n_i(3p_i^2 - 2p_i)}{2(1-p_i)^2},$$
$$B_{i_1 i_2} = \sum_{i=i_1}^{i_2} \frac{n_i \bar{x}_i(1-2p_i)}{(1-p_i)^2},$$
$$C_{i_1 i_2} = \sum_{i=i_1}^{i_2} \frac{\sum_{j=1}^{n_i} x_{ij}^2}{2(1-p_i)^2}.$$

Table 2. Comparison of Three Estimators for Various Values of Sample Size - Two sample, Equality Restriction

		n_2			
n_1	Estimator	5	10	25	50
5	$\bar{x}/p$	26	31	27	26
	$\bar{x}/p+1$	27	32	28	27
	Approx.	26	31	27	26
	Real	26	31	27	26
10	$\bar{x}/p$	26	23	25	25
	$\bar{x}/p+1$	27	24	26	26
	Approx.	26	24	25	26
	Real	27	24	25	26
25	$\bar{x}/p$	25	29	23	25
	$\bar{x}/p+1$	26	30	24	26
	Approx.	26	30	24	26
	Real	25	30	24	26
50	$\bar{x}/p$	25	28	26	25
	$\bar{x}/p+1$	26	29	27	26
	Approx.	26	28	26	26
	Real	25	28	26	26

The above result will be used extensively in the following restricted estimation procedure.

Now we begin to discuss about the main result of this paper. First we consider the case of $k=2$. Suppose $N_1 \leq N_2$, $p=p_1=p_2$ and p is known. The problem is to find N_1 and N_2 which maximize

$$\prod_{j=1}^{n_1} \binom{N_1}{x_{1j}} \cdot \prod_{j=1}^{n_2} \binom{N_2}{x_{2j}} \cdot p^{x_{..}}(1-p)^{n_1 N_1 + n_2 N_2 - x_{..}} \tag{3.4}$$

subject to $N_1 \leq N_2$, N_i's are integer and $N_i \geq \max\{x_{ij}, j=1,\ldots,n_i\}$. Note that $x_{..} = \sum_{i=1}^{2} \sum_{j=1}^{n_i} x_{ij}$. If $\widehat{N}_1 \leq \widehat{N}_2$, then the restricted estimates, N_1^* and N_2^*, are given as

$N_1^* = \widehat{N}_1$ and $N_2^* = \widehat{N}_2$, respectively. Suppose that $\widehat{N}_1 > \widehat{N}_2$. To find the restricted maximum likelihood estimate we need the following lemma.

Lemma: Let f and g be the two nonnegative real-valued unimodal functions with peaks at a and b, respectively, and $a \le b$. That is $f(x) \le f(x')$ for $x \le x' \le a$ and $f(x) \ge f(x')$ for $a \le x \le x'$. Similarly for g. Then the maximum of $f(x)g(y)$ subject to $x \ge y$ is attained when $x = y$.

Proof: Suppose $y \ge a$. Then $x \ge a$ and hence $f(x) \le f(y)$. Hence we have $f(x)g(y) \le f(y)g(y)$. Suppose $x \le b$. Then $y \le b$ and hence $g(y) \le g(x)$. Therefore we have $f(x)g(y) \le f(x)g(x)$. This completes the proof.

Consider the two likelihood functions in (3.4). Note that $L(N_1)/L(N_1+1)$ is strictly increasing in N_1 with $L(N_1)/L(N_1+1) > 0$. This means that $L(N_1)$ is unimodal with respect to N_1. So $L(N_2)$ is. It is clear that the peaks of these functions are observed at $\widehat{N}_1$ and $\widehat{N}_2$, respectively. Note that $\widehat{N}_1 > \widehat{N}_2$. Since we need to maximize (3.4) subject to $N_1 \le N_2$ the maximum value of (3.4) is obtained when $N_1 = N_2$.

Now we rewrite (3.4) in terms of new variables. Let $N = N_1 = N_2$. And let $y_j = x_{1j}$ for $j = 1, \dots, n_1$ and $y_{j+n_1} = x_{2j}$ for $j = 1, \dots, n_2$. Then we have

$$\prod_{j=1}^{n_1+n_2} \binom{N}{y_j} \cdot p^{y.} (1-p)^{n_1 N + n_2 N - y.},$$

where $y_. = \sum_{j=1}^{n_1+n_2} y_j/(n_1+n_2)$. This is just a one sample problem. We have discussed about estimation procedure extensively in section 2 and earlier part of this section. The maximum likelihood estimate, N_1^* and N_2^*, is given by either $[\bar{y}/p]$ or $[\bar{y}/p]+1$. An approximate solution is the solution to the following equation;

$$\frac{(n_1+n_2)(3p^2-2p)}{2(1-p)^2}(N+1)^2 + \frac{y_.(1-2p)}{(1-p)^2}(N+1) + \frac{\sum_{j=1}^{n_1+n_2} y_j^2}{2(1-p)^2} = 0.$$

It is quite straightforward to extend this to the case of unequal probabilities of success. For estimates use (3.1) and (3.2).

Finally we discuss the estimation procedure for the general ordering. Let $\prec$ be a partial order on an index set $I = \{1, 2, \dots, k\}$. A real valued function defined on I is said to be isotonic with respect to the partial ordering $\prec$ on I if $x, y \in I$ and $x \prec y$ implies $f(x) \le f(y)$. A subset B of I is a level set if and only if there exists an isotonic function f on I and a real number a such that $B = [f = a]$. Let us revisit the case $k = 2$. The index set is

$I=\{1,2\}$ and partial order is defined as $1<2$. The restriction is $N_1 \le N_2$, i.e., N_i's are isotonic with respect to $<$. The problem is to find an isotonic function N_i^* with respect to $<$. If $\widehat{N}_1 < \widehat{N}_2$ then $N_1^* = \widehat{N}_1$ and $N_2^* = \widehat{N}_2$. Note that $N_1^* < N_2^*$ and hence we have two level sets which are $\{1\}$ and $\{2\}$. If $\widehat{N}_1 > \widehat{N}_2$ then $N_1^* = N_2^*$ and we have one level set which is $\{1,2\}$. To find the estimate we pool two sets of observations. This is the key concept in finding an isotonic function, which we call isotonic regression. A well known algorithm for finding an isotonic regression is so called PAVA (Pool Adjacent Violators Algorithm). See Robertson, Wright and Dykstra (1988) for full description of isotonic regression.

4. Concluding Remarks

In this paper we did not discuss about testing problem. We can implement the likelihood ratio test procedures for testing for and against an order restriction among the numbers of trials. Since we were not able to give the explicit form, which is practically intractable, of maximum likelihood estimate the derivation of testing procedure would be very difficult. We also did not give the full simulation results. As we discussed earlier we need to check whether the proposed estimators produce over-estimation or under-estimation.

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