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R-Fuzzy δ-Closure and R-Fuzzy θ-Closure Sets

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ABSTRACT

We introduce r-fuzzy δ-cluster (θ-cluster) points and r-fuzzy δ-closure (θ-closure) sets in smooth fuzzy topological spaces in a view of the definition of A.P. Sostak [13]. We study some properties of them.

1. Introduction and preliminaries

A.P. Sostak [13] introduced the smooth fuzzy topology as an extension of Chang's fuzzy topology [3]. It has been developed in many directions [5,7-9,12]. S. Ganguly and S. Saha [6] introduced the concepts of fuzzy δ-cluster points and fuzzy θ-cluster points in fuzzy topological spaces in a view of [3].

In this paper, we define r-fuzzy δ-cluster (θ-cluster) points and r-fuzzy δ-closure (θ-closure) sets in smooth fuzzy topological spaces in a view of the definition of A.P. Sostak [13]. We study some properties of them. Every family of r-fuzzy θ-closure sets is a smooth fuzzy closure operator. But the family of the r-fuzzy δ-closure sets need not be a smooth fuzzy closure operator. Also, the r-fuzzy closure (resp. r-fuzzy δ-closure, r-fuzzy θ-closure) of an intersection of two fuzzy sets is not equal to the intersection of their r-fuzzy closures (resp. r-fuzzy δ-closures, r-fuzzy θ-closures). Furthermore, the r-fuzzy closure (resp. r-fuzzy δ-closure, r-fuzzy θ-closure) of a product of two fuzzy sets is not equal to the product of their r-fuzzy closures (resp. r-fuzzy δ-closures, r-fuzzy θ-closures).

Throughout this paper, let X be a nonempty set, $I=[0, 1]$ and $I_0=(0, 1]$. For $\alpha \in I$, $\bar{\alpha}(x)=\alpha$ for all $x \in X$. All the other notations and the other definitions are standard in the fuzzy set theory.

Definition 1.1 [13] A function $\tau : I^X \to I$ is called a *smooth fuzzy topology* on X if it satisfies the following conditions:

(O1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(O2) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,

(O3) $\tau(\vee_{i \in \Gamma} \mu_i) \geq \wedge_{i \in \Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i \in \Gamma} \subset I^X$.

The pair (X, τ) is called a *smooth fuzzy topological space*.

Definition 1.2 [12] A function $C : I^X \times I_0 \to I^X$ is called a *smooth fuzzy closure operator* on X if for $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(C1) $C(\bar{0}, r) = \bar{0}$.

(C2) $\lambda \leq C(\lambda, r)$.

(C3) $C(\lambda, r) \vee C(\mu, r) = C(\lambda \vee \mu, r)$.

(C4) $C(\lambda, r) \leq C(\lambda, s)$, if $r \leq s$.

The pair (X, C) is called a *smooth fuzzy closure space*.

A smooth fuzzy closure space (X, C) is *topological* if it satisfies for $\lambda \in I^X$ and $r \in I_0$,

$$C(C(\lambda, r), r) = C(\lambda, r).$$

Theorem 1.3 [12] Let (X, τ) be a smooth fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\lambda, r) = \wedge \{\mu \mid \lambda \leq \mu, \ \tau(\bar{1} - \mu) \geq r\}.$$

Then (X, C_τ) is a topological smooth fuzzy closure space.

We easily prove the following theorem from Theorem 1.3.

Theorem 1.4 Let (X, τ) be a smooth fuzzy topological space. For each $r \in I_0$, $\lambda \in I^X$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:

$$I_\tau(\lambda, r) = \vee \{\mu \mid \mu \leq \lambda, \ \tau(\mu) \geq r\}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r)$.

(2) If $I_\tau(C_\tau(\lambda, r), r) = \lambda$, then $C_\tau(I_\tau(\bar{1} - \lambda, r), r) = \bar{1} - \lambda$.

(3) $I_\tau(\bar{1}, r) = \bar{1}$.

(4) $I_\tau(\lambda, r) \leq \lambda$.

(5) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.

(6) $I_\tau(\lambda, r) \geq I_\tau(\lambda, s)$, if $r \leq s$.

(7) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Let τ_1 and τ_2 be smooth fuzzy topologies on X. We say τ_1 is *finer* than τ_2 (τ_2 is *coarser* than τ_1) if $\tau_2(\mu) \leq \tau_1(\mu)$ for all $\mu \in I^X$. Let (X, τ_1) and (Y, τ_2) be smooth fuzzy topological spaces and $f : X \to Y$ a function. f is called *fuzzy continuous* if $\tau_2(\mu) \leq \tau_1(f^{-1}(\mu))$ for all $\mu \in I^Y$.

Definition 1.5 [9] Let $\bar{0} \notin \Theta_X$ be a subset of I^X. A

function $\beta : \Theta_X \to I$ is called a *smooth fuzzy topological base* on X if it satisfies the following conditions:
(B1) $\beta(\bar{1}) = 1$.
(B2) $\beta(\mu_1 \wedge \mu_2) \geq \beta(\mu_1) \wedge \beta(\mu_2)$, for all $\mu_1, \mu_2 \in \Theta_X$.

Theorem 1.6 [9] Let β be a smooth fuzzy topological base on X. Define the function $\tau_\beta : I^X \to I$ as follows: for each $\mu \in I^X$,

$$\tau_\beta(\mu) = \begin{cases} \vee \{ \wedge_{i \in j} \beta(\mu_i) \} & \text{if } \mu = \vee_{i \in J} \mu_i, \mu_i \in \Theta_X, \\ 1 & \text{if } \mu = \bar{0}, \\ 0 & \text{otherwise.} \end{cases}$$

where the first $\vee$ is taken over all families $\{\mu_i \in \Theta_X \mid \mu = \vee_{i \in J} \mu_i$. Then (X, τ_β) is a smooth fuzzy topological space.
If β is a smooth fuzzy topological base on X, then τ_β is called the smooth fuzzy topology *generated by* β.

Theorem 1.7 [9] Let $\{(X_i, \tau_i) \mid i \in \Gamma\}$ be a family of smooth fuzzy topological spaces and X a set and $f_i : X \to X_i$ a function, for each $i \in \Gamma$. Let $\Theta_X = \{\bar{0} \neq \mu = \wedge_{i \in F} f_i^{-1}(\nu_i) \mid \tau_i(\nu_i) > 0,\ i \in F\}$ be given, for every finite index set $F \subset \Gamma$. Define a function $\beta : \Theta_X \to I$ on X by

$$\beta(\mu) = \vee \{ \wedge_{i \in F} \tau_i(\nu_i) \mid \mu = \wedge_{i \in F} f_i^{-1}(\nu_i)\}$$

where the first $\vee$ is taken over all finite index subset F of Γ. Then:
(1) β is a smooth fuzzy topological base on X.
(2) The smooth fuzzy topology τ_β generated by β is the coarsest smooth fuzzy topology on X for which each $i \in \Gamma$, f_i is fuzzy continuous.
(3) A map $f : (Z, \tau_Z) \to (X, \tau_\beta)$ is fuzzy continuous iff for each $i \in \Gamma$, $f_i \circ f$ is fuzzy continuous.

Let X be the product $\Pi_{i \in \Gamma} X_i$ of the family $\{(X_i, \tau_i) \mid i \in \Gamma\}$ of smooth fuzzy topological spaces. The coarsest smooth fuzzy topology $\tau = \Pi_{i \in \Gamma} \tau_i$ on X for which each the projections $\pi_i : X \to X_i$ is fuzzy continuous is called the *product smooth fuzzy topology* of $\{\tau_i \mid i \in \Gamma\}$, and (X, τ) is called the *product smooth fuzzy topology space.*

2. r-fuzzy δ-cluster and r-fuzzy θ-cluster points

Definition 2.1 [5] Let (X, τ) be a smooth fuzzy topological space, $\mu \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$ where $Pt(X)$ is the family of all fuzzy points in X.
(1) μ is called a *r-fuzzy open Q-neighborhood* of x_t if $\tau(\mu) \geq r$ and $x_t\ q\ \mu$.
(2) μ is called a *r-fuzzy open R-neighborhood* of x_t if $x_t\ q\ \mu$ and $\mu = I_\tau(C_\tau(\mu, r), r)$.

We denote
$\mathcal{Q}_\tau(x_t, r) = \{\mu \in I^X \mid x_t\ q\ \mu,\ \tau(\mu) \geq r\}$,
$\mathcal{R}_\tau(x_t, r) = \{\mu \in I^X \mid x_t\ q\ \mu = I_\tau(C_\tau(\mu, r), r)\}$.

Definition 2.2 Let (X, τ) be a smooth fuzzy topological space, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$.
(1) x_t is called a *r-fuzzy cluster point* of λ if for every $\mu \in \mathcal{Q}_\tau(x_t, r)$, we have $\mu\ q\ \lambda$.
(2) x_t is called a *r-fuzzy δ-cluster point* of λ if for every $\mu \in \mathcal{R}_\tau(x_t, r)$, we have $\mu\ q\ \lambda$.
(3) x_t is called a *r-fuzzy θ-cluster point* of λ if for every $\mu \in \mathcal{Q}_\tau(x_t, r)$, we have $C_\tau(\mu, r)\ q\ \lambda$.
We denote
$cl_\tau(\lambda, r) = \vee \{x_t \in Pt(X) \mid x_t$ is a r-fuzzy cluster point of $\lambda\}$,
$D_\tau(\lambda, r) = \vee \{x_t \in Pt(X) \mid x_t$ is a r-fuzzy δ-cluster point of $\lambda\}$,
$T_\tau(\lambda, r) = \vee \{x_t \in Pt(X) \mid x_t$ is a r-fuzzy θ-cluster point of $\lambda\}$.
(4) $D_\tau(\lambda, r)$ is called a *r-fuzzy δ-closure* of λ.
(5) $T_\tau(\lambda, r)$ is called a *r-fuzzy θ-closure* of λ.

Theorem 2.3 Let (X, τ) be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following properties.
(1) $C_\tau(\lambda, r) = cl_\tau(\lambda, r)$.
(2) $D_\tau(\lambda, r) = \wedge \{\mu \mid \lambda \leq \mu,\ \mu = C_\tau(I_\tau(\mu, r), r)\}$.
(3) $T_\tau(\lambda, r) = \wedge \{\mu \mid \lambda \leq I_\tau(\mu, r),\ \tau(\bar{1} - \mu) \geq r\}$.
(4) x_t is a r-fuzzy δ-cluster (resp. r-fuzzy cluster, r-fuzzy θ-cluster) of λ iff $x_t \in D_\tau(\lambda, r)$ (resp. $x_t \in C_\tau(\lambda, r)$, $x_t \in T_\tau(\lambda, r)$).

Proof. (1) It is similarly proved as the following (2) and (3).
(2) Put $\rho = \wedge \{\mu \mid \lambda \leq \mu,\ \mu = C_\tau(I_\tau(\mu, r), r)\}$. Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that $D_\tau(\lambda, r) \not\geq \rho$. Then there exist $x \in X$ and $t \in I_0$ such that

$$D_\tau(\lambda, r)(x) < t < \rho(x).$$

Since x_t is not a r-fuzzy δ-cluster point of λ, there exists $\nu \in \mathcal{R}_\tau(x_t, r)$ such that $\lambda \leq \bar{1} - \nu$. Since $\nu = I_\tau(C_\tau(\nu, r), r)$, by Theorem 1.4(2), $\bar{1} - \nu = C_\tau(I_\tau(\bar{1} - \nu, r), r)$. Then

$$\lambda \leq \bar{1} - \nu = C_\tau(I_\tau(\bar{1} - \nu, r), r)$$

Thus, $\rho \leq \bar{1} - \nu$. Furthermore, $x_t\ q\ \nu$ implies $\rho(x) \leq (\bar{1} - \nu)(x) < t$. It is a contradiction. Hence $D_\tau(\lambda, r) \geq \rho$, for each $\lambda \in I^X$ and $r \in I_0$.
Suppose there exist $\lambda \in I^X$, $x \in X$ and $r, s \in I_0$ such that

$$D_\tau(\lambda, r)(x) > s > \rho(x).$$

Since $\rho(x) < s$, there exists $\mu \in I^X$ with $\lambda \leq \mu = C_\tau(I_\tau(\mu, r), r)$ such that

$$D_\tau(\lambda, r)(x) > s > \mu(x) \geq \rho(x).$$

Then $\overline{1} - \mu \in \mathcal{R}_\tau(x_s, r)$ such that

$$\lambda \, \overline{q} \, \overline{1} - \mu.$$

Hence x_s is not a r-fuzzy δ-cluster point of λ. It is a contradiction. Thus, $D_\tau(\lambda, r) \leq \rho$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) Put $\gamma = \wedge\{\mu \mid \lambda \leq I_\tau(\mu, r),\ \tau(\overline{1} - \mu) \geq r\}$. Suppose there exist $\lambda \in I^X$, $x \in X$ and $r, t \in I_0$ such that

$$T_\tau(\lambda, r)(x) < t < \gamma(x).$$

Since x_t is not a r-fuzzy θ-cluster point of λ, there exists $\mu \in \mathcal{Q}_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - C_\tau(\mu, r) = I_\tau(\overline{1} - \mu, r)$. It implies $\gamma \leq \overline{1} - \mu$. Then $\gamma(x) \leq (\overline{1} - \mu)(x) < t$. It is a contradiction. Hence $T_\tau(\lambda, r) \geq \gamma$, for each $\lambda \in I^X$ and $r \in I_0$.

Suppose there exist $\lambda \in I^X$, $x \in X$ and $r, s \in I_0$ such that

$$T_\tau(\lambda, r)(x) > s > \gamma(x).$$

Since $\gamma(x) < s$, there exists $\mu \in I^X$ with $\lambda \leq I_\tau(\mu, r)$ and $\tau(\overline{1} - \mu) \geq r$ such that

$$T_\tau(\lambda, r)(x) > s > \mu(x) \geq \rho(x).$$

Then $\overline{1} - \mu \in \mathcal{Q}_\tau(x_s, r)$ and $\lambda \leq I_\tau(\mu, r) = \overline{1} - C_\tau(\overline{1} - \mu, r)$ implies

$$\lambda \, \overline{q} \, C_\tau(\overline{1} - \mu, r).$$

Hence x_t is not a r-fuzzy θ-cluster point of λ. It is a contradiction. Thus, $T_\tau(\lambda, r) \leq \gamma$, for each $\lambda \in I^X$ and $r \in I_0$.

(4) ($\Rightarrow$) It is trivial.

($\Leftarrow$) Let x_t be not a r-fuzzy δ-cluster of λ. Then there exists $\nu \leq \mathcal{R}_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - \nu$. Since, by Theorem 1.4(2),

$$\overline{1} - \nu = C_\tau(I_\tau(\overline{1} - \nu, r), r),$$

we have $D_\tau(\lambda, r) \leq \overline{1} - \nu$. Furthermore, $x_t \, q \, \nu$ implies $D_\tau(\lambda, r)(x) \leq (\overline{1} - \nu)(x) < t$. Hence $x_t \notin D_\tau(\lambda, r)$.

Others are similarly proved. □

Definition 2.4 Let (X, τ) be a smooth fuzzy topological space, $\lambda \in I^X$ and $r \in I_0$.

(1) λ is called a *r-fuzzy semi-open* if $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$.

(2) λ is called a *r-fuzzy pre-open* if $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$.

Lemma 2.5 Let (X, τ) be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following properties.

(1) $\mathcal{R}_\tau(x_t, r) \subset \mathcal{Q}_\tau(x_t, r)$.

(2) $C_\tau(I_\tau(\lambda, r), r) = C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r)$.

(3) $I_\tau(C_\tau(\mu, r), r) = I_\tau(C_\tau(I_\tau(C_\tau(\mu, r), r), r), r)$.

Proof. (1) Let $\mu \in \mathcal{R}_\tau(x_t, r)$. Then $\mu = I_\tau(C_\tau(\mu, r), r)$ and $\tau(\mu) \geq r$ from the definition of I_τ. Hence $\mu \in \mathcal{Q}_\tau(x_t, r)$.

(2) Since $I_\tau(C_\tau(I_\tau(\lambda, r), r), r) \leq C_\tau(I_\tau(\lambda, r), r)$, by (C3) of Definition 1.2, we have

$$C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r) \leq C_\tau(I_\tau(\lambda, r), r).$$

Furthermore, we have

$$I_\tau(\lambda, r) \leq C_\tau(I_\tau(\lambda, r), r)$$

(by (5,7) of Theorem 1.4)

$\Rightarrow I_\tau(\lambda, r) \leq I_\tau(C_\tau(I_\tau(\lambda, r), r), r)$

$\Rightarrow C_\tau(I_\tau(\lambda, r), r) \leq C_\tau(I_\tau(C_\tau(I_\tau(\lambda, r), r), r), r)$.

(3) It is easily proved from (2) and Theorem 1.4(1). □

Theorem 2.6 Let (X, τ) be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following properties.

(1) If $\rho = C_\tau(I_\tau(\rho, r), r)$, then $D_\tau(\rho, r) = \rho$.

(2) $C_\tau(\lambda, r) \leq D_\tau(\lambda, r) \leq T_\tau(\lambda, r)$.

(3) If λ is r-fuzzy semi-open, then $C_\tau(\lambda, r) = D_\tau(\lambda, r)$.

(4) If λ is r-fuzzy pre-open, then $C_\tau(\lambda, r) = D_\tau(\lambda, r) = T_\tau(\lambda, r)$.

(5) If $\tau(\lambda) \geq r$, then $C_\tau(\lambda, r) = D_\tau(\lambda, r) = T_\tau(\lambda, r)$.

(6) If λ is r-fuzzy pre-open and $\lambda = C_\tau(I_\tau(\lambda, r), r)$, then $T_\tau(\lambda, r) = \lambda$.

Proof. (1) By Lemma 2.5(2), $C_\tau(I_\tau(\rho, r), r) = \rho$. Hence $D_\tau(\rho, r) = \rho$ from Theorem 2.3 (2).

(2) Since $\mathcal{R}_\tau(x_t, r) \subset \mathcal{Q}_\tau(x_t, r)$ from Lemma 2.5 (1), then $x_t \in C_\tau(\lambda, r)$ implies $x_t \in D_\tau(\lambda, r)$. Hence $C_\tau(\lambda, r) \leq D_\tau(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_0$.

Suppose there exist $\lambda \in I^X$, $x \in X$ and $r, t \in I_0$ such that

$$D_\tau(\lambda, r)(x) > t > T_\tau(\lambda, r)(x).$$

Since x_t is not a r-fuzzy θ-cluster point of λ, there exists $\mu \in \mathcal{Q}_\tau(x_t, r)$ such that $\lambda \in \overline{1} - C_\tau(\mu, r)$. It implies

$$\lambda \leq \overline{1} - C_\tau(\mu, r) \leq \overline{1} - I_\tau(C_\tau(\mu, r), r).$$

Since $\tau(\mu) \geq r$ and $\mu \leq C_\tau(\mu, r)$, we have

$$\mu = I_\tau(\mu, r) \leq I_\tau(C_\tau(\mu, r), r).$$

Thus, $x_t \, q \, \mu$ implies $x_t \, q \, I_\tau(C_\tau(\mu, r), r)$. Since, by Lemma 2.5(3),

$$I_\tau(C_\tau(I_\tau(C_\tau(\mu, r), r), r), r) = I_\tau(C_\tau(\mu, r), r),$$

$I_\tau(C_\tau(\mu, r), r) \in \mathcal{R}_\tau(x_t, r)$ and $\lambda \leq \overline{1} - I_\tau(C_\tau(\mu, r), r)$. Hence x_t is not a r-fuzzy δ-cluster point of λ. By Theorem 2.3(4), $D_\tau(\lambda, r)(x) < t$. It is contradiction. Therefore, $D_\tau(\lambda, r) \leq T_\tau(\lambda, r)$, for each $\lambda \in I^X$ and $r \in I_0$.

(3) Let λ be r-fuzzy semi-open set. Since $\lambda \leq C_\tau(I_\tau(\lambda, r), r)$, then

$D_\tau(\lambda, r) \le C_\tau(I_\tau(\lambda, r), r)$ (By Lemma 2.5(2))
$\le C_\tau(\lambda, r)$
$\le D_\tau(\lambda, r)$.

Thus, $C_\tau(\lambda, r) = D_\tau(\lambda, r)$.

(4) Let λ be r-fuzzy pre-open set. From (2), we only show that $T_\tau(\lambda, r) \le C_\tau(\lambda, r)$. Since $\lambda \le I_\tau(C_\tau(\lambda, r), r)$, by Theorem 2.3(3),

$$T_\tau(\lambda, r) \le C_\tau(\lambda, r).$$

(5) Since $\tau(\lambda) \ge r$, we have

$$\lambda = I_\tau(\lambda, r) \le I_\tau(C_\tau(\lambda, r), r).$$

Hence λ is r-preopen set, by (4), it is trivial.

(6) It is easily proved from (1) and (4).

□

Theorem 2.7 Let (X, τ) be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r \in I_0$, it satisfies the following conditions:

(1) $D_\tau(\bar{0}, r) = \bar{0}$.
(2) $\lambda \le D_\tau(\lambda, r)$.
(3) $D_\tau(\lambda, r) \le D_\tau(\mu, r)$ if $\lambda \le \mu$.
(4) $D_\tau(\lambda, r) \vee D_\tau(\mu, r) = D_\tau(\lambda \vee \mu, r)$.
(5) $D_\tau(D_\tau(\lambda, r), r) = D_\tau(\lambda, r)$.

Proof. (1),(2) and (3) are easily proved from the definition of D_τ.

(4) From (3), $D_\tau(\lambda, r) \vee D_\tau(\mu, r) \le D_\tau(\lambda \vee \mu, r)$.

Suppose there exist $\lambda_1, \lambda_2 \in I^X$, $x \in X$ and $r, t \in I_0$ such that

$$D_\tau(\lambda_1, r)(x) \vee D_\tau(\lambda_2, r)(x) < t < D_\tau(\lambda_1 \vee \lambda_2, r)(x). \quad (1)$$

For each $i \in \{1, 2\}$, since $D_\tau(\lambda_i, r)(x) < t$, by Theorem 2.3(2), there exists $\nu_i \in I^X$ with $\lambda_i \le \nu_i = C_\tau(I_\tau(\nu_i, r), r)$ such that

$$D_\tau(\lambda_1, r)(x) \vee D_\tau(\lambda_2, r)(x) \le (\nu_1 \vee \nu_2)(x) < t. \quad (2)$$

Since $\nu_i = C_\tau(I_\tau(\nu_i, r), r)$, we have

$C_\tau(I_\tau(\nu_1 \vee \nu_2, r), r) \le C_\tau(\nu_1 \vee \nu_2, r)$
$= C_\tau(\nu_1, r) \vee C_\tau(\nu_2, r)$
$= \nu_1 \vee \nu_2$.

Moreover, since $I_\tau(\nu_1 \vee \nu_2, r) \ge I_\tau(\nu_1, r) \vee I_\tau(\nu_2, r)$, we have

$C_\tau(I_\tau(\nu_1 \vee \nu_2, r), r) \ge C_\tau(I_\tau(\nu_1, r), r) \vee C_\tau(I_\tau(\nu_2, r), r)$
$= \nu_1 \vee \nu_2$.

Thus,

$C_\tau(I_\tau(\nu_1 \vee \nu_2, r), r) = \nu_1 \vee \nu_2$.

Hence, by Theorem 2.3(2),

$D_\tau(\lambda_1 \vee \lambda_2, r) \le \nu_1 \vee \nu_2$.

It is a contradiction for (1) and (2).

(5) From (2), $D_\tau(D_\tau(\lambda, r), r) \ge D_\tau(\lambda, r)$. Suppose

$$D_\tau(D_\tau(\lambda, r), r)(x) > t > D_\tau(\lambda, r)(x).$$

Then there exists $\nu \in I^X$ with

$$\lambda \le \nu = C_\tau(I_\tau(\nu, r), r).$$

such that

$$D_\tau(D_\tau(\lambda, r), r)(x) > t > \nu(x) \ge D_\tau(\lambda, r)(x).$$

Since $\lambda \le \nu = C_\tau(I_\tau(\nu, r), r)$, we have

$D_\tau(\lambda, r) \le D_\tau(\nu, r)$ (by (3))
$= D_\tau(C_\tau(I_\tau(\nu, r), r), r)$
$= C_\tau(I_\tau(\nu, r), r) = \nu$.
(by Theorem 2.6(1))

Again, by the definition of D_τ from Theorem 2.3(2),

$$D_\tau(D_\tau(\lambda, r), r) \le \nu.$$

It is a contradiction.

□

From the following theorem, every family of r-fuzzy θ-closure sets is a smooth fuzzy closure operator.

Theorem 2.8 Let (X, τ) be a smooth fuzzy topological space. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $\bar{0} = T_\tau(\bar{0}, r)$.
(2) $\lambda \le T_\tau(\lambda, r)$.
(3) $T_\tau(\lambda, r) \le T_\tau(\mu, r)$ if $\lambda \le \mu$.
(4) $T_\tau(\lambda, r) \vee T_\tau(\mu, r) = T_\tau(\lambda \vee \mu, r)$.
(5) $T_\tau(\lambda, r) \le T_\tau(\mu, s)$ if $r \le s$.

Proof. (1) and (3) are easily proved from the definition of T_τ.

(2) Suppose there exist $\lambda \in I^X$, $x \in X$ and $r, t \in I_0$ such that

$$\lambda(x) > t > T_\tau(\lambda, r)(x).$$

Since, $T_\tau(\lambda, r)(x) < t$, by Theorem 2.3(4), x_t is not a r-fuzzy θ-cluster point of λ. Then there exists $\nu \in Q_\tau(x_t, r)$ such that

$$C_\tau(\nu, r) \le \bar{1} - \lambda.$$

It implies

$$\lambda \le \bar{1} - C_\tau(\nu, r) \le \bar{1} - \nu.$$

Thus, $\lambda(x) \le (\bar{1} - \nu)(x) < t$. It is a contradiction.

(4) From (3), $T_\tau(\lambda, r) \vee T_\tau(\mu, r) \le T_\tau(\lambda \vee \mu, r)$.

Suppose there exist $\lambda_1, \lambda_2 \in I^X$, $x \in X$ and $r, t \in I_0$ such that

$$T_\tau(\lambda_1, r)(x) \vee T_\tau(\lambda_2, r)(x) < t < T_\tau(\lambda_1 \vee \lambda_2, r)(x).$$

For each $i \in \{1, 2\}$, since, $T_\tau(\lambda_i, r)(x) < t$, by Theorem 2.3(3), there exists $\nu_i \in I^X$ and $\tau(\bar{1} - \nu_i) \ge r$

with $\lambda_i \leq I_\tau(\nu_i, r)$ such that

$T_\tau(\lambda_1, r)(x) \vee T_\tau(\lambda_2, r)(x) \leq (\nu_1 \vee \nu_2)(x) < t.$

Since $\lambda_i \leq I_\tau(\nu_i, r)$, we have

$$I_\tau(\nu_1 \vee \nu_2, r) \geq I_\tau(\nu_1, r) \vee I_\tau(\nu_2, r) \geq \lambda_1 \vee \lambda_2$$

and

$$\tau(\bar{1} - (\nu_1 \vee \nu_2)) \geq \tau(\bar{1} - \nu_1) \wedge \tau(\bar{1} - \nu_2) \geq r.$$

Hence,

$T_\tau(\lambda_1 \vee \lambda_2, r) \leq \nu_1 \vee \nu_2$.

It is a contradiction.

(5) We will show that $x_t \in T_\tau(\lambda, s)$ for each $x_t \in T_\tau(\lambda, r)$. Let $\mu \in \mathcal{Q}_\tau(x_t, r)$. Since $\tau(\mu) \geq s \geq r$, we have $\mu \in \mathcal{Q}_\tau(x_t, s)$. Since x_t is a r-fuzzy θ-cluster point of λ,

$C_\tau(\mu, r)\ q\ \lambda$.

Since $C_\tau(\mu, r) \leq C_\tau(\mu, s)$, we have

$C_\tau(\mu, s)\ q\ \lambda$.

Hence x_t is a s-fuzzy θ-cluster point of λ. □

In general, $C_\tau \neq D_\tau \neq T_\tau$ and $T_\tau(T_\tau(\lambda, r), r) \neq T_\tau(\lambda, r)$ from the following example.

Example 2.9 Let X be a nonempty set. We define a smooth fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{3}, & \text{if } \lambda = \overline{0.7}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.4}, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 2.3(1), we obtain $C_\tau : I^X \times I_0 \to I^X$ as follows:

$$C_\tau(\lambda, r) = \begin{cases} \bar{0}, & \text{if } \lambda = \bar{0},\ r \in I_0, \\ \overline{0.3}, & \text{if } \bar{0} \neq \lambda \leq \overline{0.3},\ 0 < r \leq \frac{1}{3}, \\ \overline{0.6}, & \text{if } \overline{0.3} \not\geq \lambda \leq \overline{0.6},\ 0 < r \leq \frac{1}{3}, \\ \overline{0.6}, & \text{if } \bar{0} \neq \lambda \leq \overline{0.6},\ \frac{1}{3} < r \leq \frac{2}{3}, \\ \bar{1}, & \text{otherwise.} \end{cases}$$

Since $C_\tau(I_\tau(\overline{0.6}, r), r) = \overline{0.6}$, for $0 < r < 2/3$, by Theorem 2.3(2,3), we obtain $D_\tau, T_\tau : I^X \times I_0 \to I^X$ as follows:

$$D_\tau(\lambda, r) = \begin{cases} \bar{0}, & \text{if } \lambda = \bar{0},\ r \in I_0, \\ \overline{0.6}, & \text{if } \bar{0} \neq \lambda \leq \overline{0.6},\ 0 < r \leq \frac{2}{3}, \\ \bar{1}, & \text{otherwise.} \end{cases}$$

$$T_\tau(\lambda, r) = \begin{cases} \bar{0}, & \text{if } \lambda = \bar{0},\ r \in I_0, \\ \overline{0.6}, & \text{if } \bar{0} \neq \lambda \leq \overline{0.4},\ 0 < r \leq \frac{2}{3}, \\ \bar{1}, & \text{otherwise.} \end{cases}$$

Hence $C_\tau \neq D_\tau \neq T_\tau$ and for $0 < r \leq 2/3$,

$$\bar{1} = T_\tau(T_\tau(\overline{0.4}, r), r) < T_\tau(\overline{0.4}, r) = \overline{0.6}.$$

In general, the family of r-fuzzy δ-closure sets need not be a smooth fuzzy closure operator, that is, it does not satisfy the condition (C4) of Definition 1.2. Furthermore, the r-fuzzy closure (resp. r-fuzzy δ-closure, r-fuzzy θ-closure) of an intersection of two fuzzy sets is not equal to the intersection of their r-fuzzy closures (resp. r-fuzzy δ-closures, r-fuzzy θ-closures) from the following example.

Example 2.10 Let $X = \{a, b\}$ be a set. Define $\mu, \rho \in I^X$ as follows:

$\mu(a)=0.3$, $\mu(b)=0.4$, $\rho(a)=0.6$, $\rho(b)=0.2$.

We define a smooth fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{2}, & \text{if } \lambda = \mu, \\ \frac{2}{3}, & \text{if } \lambda = \rho, \\ \frac{2}{3}, & \text{if } \lambda = \mu \wedge \rho, \\ \frac{1}{2}, & \text{if } \lambda = \mu \vee \rho, \\ 0, & \text{otherwise.} \end{cases}$$

We obtain the following:

$$\bar{1} = C_\tau(I_\tau(\bar{1}, r), r), \quad \forall r \in I_0,$$

$$\bar{1} - \mu = C_\tau(I_\tau(\bar{1} - \mu, r), r), \quad 0 < r \leq \frac{1}{2},$$

$$\bar{1} - (\mu \vee \rho) = C_\tau(I_\tau(\bar{1} - \rho, r), r), \quad 0 < r \leq \frac{1}{2},$$

$$\bar{1} - \rho = C_\tau(I_\tau(\bar{1} - \rho, r), r), \quad \frac{1}{2} < r \leq \frac{2}{3},$$

$$\bar{1} - \mu = C_\tau(I_\tau(\bar{1} - (\mu \wedge \rho), r), r), \quad 0 < r \leq \frac{1}{2},$$

$$\bar{1}-(\mu\wedge\rho) = C_\tau(I_\tau(\bar{1}-(\mu\wedge\rho),r),r), \quad \frac{1}{2}<r\le\frac{2}{3},$$

$$\bar{1}-(\mu\vee\rho) = C_\tau(I_\tau(\bar{1}-(\mu\vee\rho),r),r), \quad 0<r\le\frac{1}{2}.$$

From Theorem 2.3(2), we obtains $D_\tau: I^X\times I_0\to I^X$ as follows:

$$D_\tau(\lambda,r)=\begin{cases}\bar{0}, & \text{if } \lambda=\bar{0},\ r\in I_0,\\ \bar{1}-(\mu\vee\rho), & \text{if } \bar{0}\ne\lambda\le\bar{1}-(\mu\vee\rho),\ 0<r\le\frac{1}{2},\\ \bar{1}-\mu, & \text{if } \bar{1}-(\mu\vee\rho)\ngeq\lambda\le\bar{1}-\mu,\ 0<r\le\frac{1}{2},\\ \bar{1}-\rho, & \text{if } \bar{0}\ne\lambda\le\bar{1}-\rho,\ \frac{1}{2}<r\le\frac{2}{3},\\ \bar{1}-(\mu\wedge\rho), & \text{if } \bar{1}-\rho\ngeq\lambda\le\bar{1}-(\mu\wedge\rho),\ \frac{1}{2}<r\le\frac{2}{3},\\ \bar{1}, & \text{otherwise.}\end{cases}$$

In general, $D_\tau(\lambda, r)\nleq D_\tau(\lambda, s)$, if $r\le s$ from the following:

$$\bar{1} = D_\tau\left(\bar{1}-\rho,\frac{1}{2}\right)\ge D_\tau\left(\bar{1}-\rho,\frac{2}{3}\right) = \bar{1}-\rho.$$

Hence it does not satisfy the condition (C4) of Definition 1.2. Thus, the family of r-fuzzy δ-closure sets is not a smooth fuzzy closure operator.

Furthermore, for $0<r\le 1/2$,we have

$$\begin{aligned}\bar{1}-(\mu\vee\rho) &= D_\tau((\bar{1}-\mu)\wedge(\bar{1}-\rho),\ r)\\ &\ne D_\tau(\bar{1}-\mu,\ r)\wedge D_\tau(\bar{1}-\rho,\ r)\\ &= (\bar{1}-\mu)\wedge\bar{1}.\end{aligned}$$

From Theorem 2.3(3), we obtains $T_\tau: I^X\times I_0\to I^X$ as follows:

$$T_\tau(\lambda,r)=\begin{cases}\bar{0}, & \text{if } \lambda=\bar{0},\ r\in I_0,\\ \bar{1}-(\mu\vee\rho), & \text{if } \bar{0}\ne\lambda\le\mu,\ 0<r\le\frac{1}{2},\\ \bar{1}-\mu, & \text{if } \mu\ngeq\lambda\le\mu\vee\rho,\ 0<r\le\frac{1}{2},\\ \bar{1}-\rho, & \text{if } \bar{0}\ne\lambda\le\mu\wedge\rho,\ \frac{1}{2}<r\le\frac{2}{3},\\ \bar{1}-(\mu\wedge\rho), & \text{if } \mu\wedge\rho\ngeq\lambda\le\rho,\ \frac{1}{2}<r\le\frac{2}{3},\\ \bar{1}, & \text{otherwise.}\end{cases}$$

Thus, the family of r-fuzzy θ-closure sets is a smooth fuzzy closure operator.

Furthermore, for $1/2<r\le 2/3$,

$$\begin{aligned}\bar{1}-\rho &= T_\tau(\mu\wedge\rho,\ r)\\ &\ne T_\tau(\mu,\ r)\wedge T_\tau(\rho,\ r)\\ &= \bar{1}\wedge(\bar{1}-(\mu\wedge\rho)).\end{aligned}$$

In general, the r-fuzzy closure (resp. r-fuzzy δ-closure, r-fuzzy θ-closure) of a product of two fuzzy sets is not equal to the product of their r-fuzzy closures (resp. r-fuzzy δ-closures, r-fuzzy θ-closures) from the following example.

Example 2.11 Let $X=\{a, b\}$ be a set. Let $\mu, \rho\in I^X$ as follows:

$\mu(a)=0.2,\ \mu(b)=0.4,\ \rho(a)=0.5,\ \rho(b)=0.1.$

We define smooth fuzzy topologies $\tau_1, \tau_2: I^X\to I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\mu,\\ 0, & \text{otherwise,}\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{2}{3}, & \text{if } \lambda=\rho,\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho)(a,b)=0.2,\ \pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho)(a,b)=0.1$$

$$\pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho)(b,a)=0.4,\ \pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho)(b,b)=0.1$$

$$\pi_1^{-1}(\mu)\vee\pi_2^{-1}(\rho)(a,a)=0.5,\ \pi_1^{-1}(\mu)\vee\pi_2^{-1}(\rho)(a,b)=0.2$$

$$\pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho)(b,a)=0.5,\ \pi_1^{-1}(\mu)\vee\pi_2^{-1}(\rho)(b,b)=0.4$$

Let $\lambda_i\in I^X$ for $i\in\{1, 2\}$ as follows:

$\lambda_1(a)=0.6,\ \lambda_1(b)=0.7,\ \lambda_2(a)=0.5,\ \lambda_2(b)=0.8.$

Then

$\lambda_1\times\lambda_2(a,\ a)=0.6,\ \lambda_1\times\lambda_2(a,\ b)=0.6,$
$\lambda_1\times\lambda_2(b,\ a)=0.5,\ \lambda_1\times\lambda_2(b,\ b)=0.7.$

From Theorem 1.7, we obtain the product smooth fuzzy topology $\tau_p: I^{X\times X}\to I$ on $X\times X$ of τ_1 and τ_2 as follows:

$$\tau_p(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\pi_1^{-1}(\mu),\\ \frac{2}{3}, & \text{if } \lambda=\pi_2^{-1}(\rho),\\ \frac{1}{2}, & \text{if } \lambda=\pi_1^{-1}(\mu)\wedge\pi_2^{-1}(\rho),\\ \frac{1}{2}, & \text{if } \lambda=\pi_1^{-1}(\mu)\vee\pi_2^{-1}(\rho),\\ 0, & \text{otherwise.}\end{cases}$$

For $0<r\le 1/2$, we have:

$$\pi_1^{-1}(\bar{1}-\mu)\vee\pi_2^{-1}(\bar{1}-\rho) = C_{\tau_p}(\lambda_1\times\lambda_2,r)$$

$$\neq C_{\tau_1}(\lambda_1, r) \times C_{\tau_2}(\lambda_2, r)$$
$$= \pi_2^{1}(\bar{1} - \rho)$$

$$\pi_2^{1}(\bar{1} - \rho) = D_{\tau_1}(\lambda_1, r) \times D_{\tau_2}(\lambda_2, r)$$
$$\neq D_{\tau_p}(\lambda_1 \times \lambda_2, r) = \bar{1}$$

Let $\rho_i \in I^X$ for $i \in \{1, 2\}$ as follows:

$\rho_1(a) = 0.2,\ \rho_1(b) = 0.3,$
$\rho_2(a) = 0.4,\ \rho_1(b) = 0.5.$

Then

$\rho_1 \times \rho_2(a,\ a) = 0.2,\ \rho_1 \times \rho_2(a,\ b) = 0.2,$
$\rho_1 \times \rho_2(b,\ a) = 0.3,\ \rho_1 \times \rho_2(b,\ b) = 0.3.$

For $0 < r \leq 1/2$, we have:

$$\pi_1^{1}(\bar{1} - \mu) \vee \pi_2^{1}(\bar{1} - \rho) = T_{\tau_p}(\rho_1 \times \rho_2, r)$$
$$\neq T_{\tau_1}(\rho_1, r) \times T_{\tau_2}(\rho_2, r)$$
$$= \pi_1^{1}(\bar{1} - \mu)$$

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