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BOUNDARY REGULARITY TO THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Under the critical assumption that $\nabla u \in L^{\alpha,\beta}_{loc}$, $\frac{3}{\alpha}+\frac{2}{\beta}\le 2$ with $\alpha \ge \frac{3}{2}$, a boundary L^∞ estimate for the solution is derived if the pressure on the boundary is bounded. Here, our estimate is local.

1. Introduction and statement of the result

In this paper we study the boundary regularity of the weak solutions of the incompressible Navier-Stokes equations

$$u^i_t - \Delta u^i + (u\cdot\nabla)u^i + \nabla_i p = f, \qquad \nabla\cdot u = 0 \tag{1.1}$$

in $D = \Omega \times (0,\infty)$ with initial data $u(x,0) = u_0(x) \in L^2(\Omega)$ for $x \in \Omega$ and boundary data $u(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^3$ is an open bounded domain with smooth boundary. We let the initial data u_0 satisfy $\nabla\cdot u_0 = 0$ in Ω and $u_0 \cdot n = 0$ on $\partial\Omega$ in a weak sense, where n is the outward normal vector. We assume that any weak solution $u \in L^2(0,\infty; H^1(\Omega)) \cap L^\infty(0,\infty; L^2(\Omega))$ satisfies

$$\int u\cdot\phi_t - \nabla u\cdot\nabla\phi - (u\cdot\nabla)u\cdot\phi + p\nabla\cdot\phi dz = 0$$

for all $\phi \in C^\infty_0(D)$. The existence of weak solutions was proved by Leray [8] and Hopf [7], and the existence of suitably weak solutions was proved by Caffarelli, Kohn and Nirenberg [3]. Here, our definition of weak solution coincides with the definition of the suitably weak solution of [3]. For the simplicity we assume that f is a smooth function in $\overline{D}$.

It is well known that if the solution is bounded, then it lies in $L^\infty(0,\infty : H^1(\Omega)) \cap L^2(0,\infty : H^2(\Omega))$. We know that boundedness of u implies

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higher regularity of u in the interior and hence we can bound various higher norms in terms of L^∞-norm of u. From Sobolev's embedding theorem we know that the solution space of weak solution $L^2(0,\infty;H^1(\Omega))\cap L^\infty(0,\infty;L^2(\Omega))$ lies in $L^{\frac{10}{3}}_{loc}(D)$. But we do not know yet how to bound L^∞-norm of u in terms of $L^{\frac{10}{3}}$-norm of u. On the other hand, as far as interior is concerned, it was proved by Serrin [9] that any weak solution u of (1.1) on a cylinder $B\times(a,b)$ satisfying

$$\int_a^b\left(\int_B |u|^\alpha\,dx\right)^{\frac{\beta}{\alpha}}dt<\infty \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}<1, \alpha\geq 3$$

is necessarily L^∞ function on any compact subsets of the cylinder. Observe that when $\alpha = \beta = 5$, then u is in L^5 and 5 is the critical number for the homogeneous Lebesgue spaces. The limiting case $3/\alpha + 2/\beta = 1, \alpha > 3$ for the initial value problem was proved by Fabes-Jones-Riviere [5] and their method seems not applicable to local problems. Also Struwe [10] improved Serrin's method and proved the boundedness of weak solutions in interior for the critical case, that is, $\frac{3}{\alpha}+\frac{2}{\beta}=1, \alpha>3$. Takahashi [11] found some criterion for L^∞ regularity near boundary for the weak solution satisfying $u\in L^{\alpha,\beta}, \frac{3}{\alpha}+\frac{2}{\beta}\leq 1$. He imposed some integrability conditions on the velocity gradient and pressure in the domain D, that is,

$$\nabla u, p\in L^{\alpha,\beta} \quad \text{for all} \quad 1<\alpha,\beta<\infty \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta}=3.$$

Choe [4] proved the L^∞ regularity of u up to boundary for the limiting case that $u\in L^{\alpha,\beta}(D), \frac{3}{\alpha}+\frac{2}{\beta}\leq 1$ with $\alpha>3$, or $u\in L^{3,\infty}$ with $||u||_{L^{3,\infty}}\leq \varepsilon_0$ for some small ε_0 under the assumption that the boundary data of the pressure is bounded.

Beirão da Veiga [1] showed that a weak solution u is regular if

$$\nabla u\in L^\beta(0,T;L^\alpha(\mathbb{R}^n))$$

for some β, $1<\beta\leq 2$, and

$$\frac{n}{\alpha}+\frac{2}{\beta}=2.$$

He [6] obtained similar results on bounded domains. We also showed the regularity under the assumption that any two components of u satisfy the Serrin condition in [2].

Here, we obtain the L^∞ regularity of u up to boundary for the case that $\nabla u \in L^{\alpha,\beta}(D), \frac{3}{\alpha}+\frac{2}{\beta}\leq 2$ with $\alpha>\frac{3}{2}$, or $\nabla u\in L^{\frac{3}{2},\infty}$ with

$||\nabla u||_{L^{\frac{3}{2},\infty}} \leq \varepsilon_0$ for some small ε_0 under the assumption that the boundary data of the pressure is bounded. For our proof, we follow the ways in [4].

We first show that $u \in L^5$, if $\nabla u \in L^{\alpha,\beta}(D)$, $\frac{3}{\alpha} + \frac{2}{\beta} \leq 2$ with $\alpha > \frac{3}{2}$, or $u \in L^{\frac{3}{2},\infty}$ with $\|\nabla u\|_{L^{\frac{3}{2},\infty}} \leq \varepsilon_0$ for some small ε_0 (Lemma 3.1). Lemma 3.1 corresponds to Lemma 15 of [4], which is assumed the Serrin condition. In our case we assume the Beirão da Veiga condition.

Then, we accept that $||u||_\infty$ can be bounded by $||u||_p$ for all $p > 5$ (Lemma 2.3), and the bound of $||u||_{5+\sigma}$ for some σ by $||u||_5$ (Lemma 2.4). The above two are Lemma 16 and Lemma17 in [4], which are obtained by employing Moser type iteration and by the reverse Hölder inequality, respectively.

Combining these two estimates we bound $||u||_\infty$ in terms of $||u||_5$. Finally, we remark that the weak solution is as regular as the boundary data of the pressure.

Set $x = (x_1, x_2, x_3)$ and $z = (x,t)$. We define $B_R(x_0) = \{x : |x - x_0| < R\}$, $B_R^+(x_0) = \{x \in B_R(x_0) : x_3 > 0\}$, $Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0)$ and $Q_R^+(x_0, t_0) = \{(x,t) \in Q_R(x_0, t_0) : x_3 > 0\}$. If there is no confusion in the local estimates, we assume $z_0 = (0,0)$ and drop z_0 in various expressions. We denote $L^m(Q), 1 \leq m < \infty$ the space of Lebesgue measurable functions with m-th power absolutely integrable. We define $\oint_Q f dz = \frac{1}{|Q|} \int_Q f dz$. We denote c a constant depending only on exterior data. Now we state our main result. Let $\lambda_0(\Omega) = \sup\{||v||_{L^6(\Omega)}/||v||_{H_0^1(\Omega)} : v \in H_0^1(\Omega)\}$ be the Sobolev constant. In the following two theorems we are interested in the boundary regularity near boundary and hence we assume that $D = Q_2^+$.

THEOREM 1.1. *Suppose (u,p) is a weak solution. There exists a positive constant σ such that if $\nabla u \in L^{\alpha,\beta}(D)$ for some (α, β) satisfying $\frac{3}{\alpha} + \frac{2}{\beta} \leq 2$ with $\alpha > \frac{3}{2}$, or $\nabla u \in L^{\frac{3}{2},\infty}(D)$ with $||\nabla u||_{L^{\frac{3}{2},\infty}} < \varepsilon_0$ for some small ε_0 depending only on the Sobolev constant $\lambda_0(\Omega)$, and $p \in L^\infty(\partial D \cap Q_{\frac{3}{2}})$, then*

$$\sup_{Q^-_{\frac{1}{8}}} |u| \leq c \left(\oint_{Q^-_1} |u|^3 \, dz \right)^{\frac{5-\sigma}{3\sigma}} + c$$

for some positive constant c depending on ε_0.

THEOREM 1.2. *Suppose (u,p) is a weak solution and $\nabla u \in L^{\alpha,\beta}(D)$ for some (α,β) satisfying $\frac{3}{\alpha}+\frac{2}{\beta}\le 2$ with $\alpha > \frac{3}{2}$, or $\nabla u \in L^{\frac{3}{2},\infty}(D)$ with $\|\nabla u\|_{L^{\frac{3}{2},\infty}} < \varepsilon_0$ for some small ε_0 depending only on the Sobolev constant $\lambda_0(\Omega)$. Let k be a fixed nonnegative integer. Then, if the tangential derivatives of the pressure $\frac{\partial^{\beta_1-\beta_2}}{\partial x^{\beta_1}\partial t^{\beta_2}} p \in L^\infty(\partial D \cap Q_2)$ for each (β_1,β_2) with $|\beta_1|+\beta_2 \le k$, then $u \in C^{k,\lambda}(\overline{Q_1^+})$ for all $\lambda \in [0,1)$, where $\beta_1 = (k_1,k_2,0)$ is multi-index.*

2. Review of Known Results

In this section, we review some results shown in [4]. We assume that $p \in L^\infty(\partial D)$. Here, the meaning of the pressure on the boundary is defined in the sense of trace.

The following lemmas are given in [4].

LEMMA 2.1. *Suppose (u,p) is a weak solution. Let $\phi \in C_0^\infty(B_2)$ with $\phi \equiv 1$ in a neighborhood of B_1. Then, $1 \le r < 3$*

(2.1)

$$\|\phi p\|_{L^m} \le c\|\phi\|_{C^2}\left(\|u\|^2_{L^{2m}(\mathrm{supp}(\phi))} + \|p\|^2_{L^r(\mathrm{supp}(\phi))} + \|p\|_{L^\infty(\mathrm{supp}(\phi)\cap\partial D)}\right)$$

for all $m \in (1, \frac{3r}{3-r})$ and for some c depending on r.

Note that when $r=1$,

$$\|p\phi^2\|_{L^m(Q_2^-)} \le c\|u\phi\|^2_{L^{2m}(Q_2^-)} + c\|p\|_{L^1(Q_2^-)} + c\|p\|_{L^\infty(\partial D)} + c\|f\|_{L^2(Q_2^-)}$$

for all $m < \frac{3}{2}$. Iterating (2.1) and using Hölder's inequality we get for all $m \in (1,\infty)$

$$\left[\int_{Q_2^-} |p\phi^2|^m dz\right]^{\frac{1}{m}} \le c\left[\int_{Q_2^-} |u\phi|^{2m} dz\right]^{\frac{1}{m}} + c\|p\|_{L^1(Q_2^-)} + c\|p\|_{L^\infty(\partial D)} + c\|f\|_{L^2(Q_2^-)}$$

for some c.

LEMMA 2.2. *Suppose that (u,p) is a weak solution, then for all $m > 1$*

(2.2)

$$\|p\phi^2\|_{L^m(Q_2^-)} \le c\|u\phi\|^2_{L^{2m}(Q_2^-)} + c\|p\|_{L^1(Q_2^-)} + c\|p\|_{L^\infty(\partial D)} + c\|f\|_{L^2(Q_2^-)}$$

for some c depending on m.

LEMMA 2.3. *Suppose (u,p) is a weak solution. Then, for any positive constant σ,*

$$\sup_{Q^-_{\frac{1}{2}}} |u| \le c \left(\int_{Q^-_1} |u|^{5+\sigma}\, dz \right)^{\frac{1}{\sigma}} + c$$

for some constant c depending only on σ.

LEMMA 2.4. *Suppose (u,p) is a weak solution and $\int_{Q^-_1} |u|^5 dz < \infty$. Then there exists a constant c such that*

$$\left(\oint_{Q^-_{\frac{1}{2}}} |u|^{5+\sigma} dz \right)^{\frac{1}{5-\sigma}} \le c \left(\oint_{Q^-_1} |u|^5 dz \right)^{\frac{1}{5}} + c$$

for some $\sigma \in (0,1)$.

3. L^∞ estimate of velocity.

Since we are interested in boundary regularity locally, we assume that $D = Q^+_4$.

Since inhomogeneous Lebesgue spaces are hard to handle, we show u lies in the homogeneous Lebesgue space L^5. This will greatly simplify our iterations.

LEMMA 3.1. *Suppose that p is bounded on $\partial\Omega$. Assume that $\nabla u \in L^{\alpha,\beta}(Q^+_1)$, $\frac{3}{\alpha} + \frac{2}{\beta} \le 2, \alpha > \frac{3}{2}$, or $\nabla u \in L^{\frac{3}{2},\infty}(Q^+_1)$ with $||\nabla u||_{L^{\frac{3}{2},\infty}(Q^-_1)} \le \varepsilon_0$ for some small ε_0, then $u \in L^5(Q^+_{\frac{1}{2}})$.*

Proof. Let $\frac{1}{2} < \rho < s \le 1$ and η a standard cut-off function such that

$$0 \le \eta \le 1, \qquad \eta = 0 \text{ on } \partial_p Q_s, \qquad \eta \equiv 1 \text{ in } Q_\rho,$$
$$|\nabla \eta| \le \frac{c}{s-\rho}, \qquad |\eta_t| \le \frac{c}{(s-\rho)^2}$$

for some positive constant c, where $\partial_p Q_s$ is the usual parabolic boundary of Q_s. We take $|u|u^i\eta^8$ as a test function to (1.1) and get

$$
\begin{aligned}
J &\stackrel{def}{=} \frac{1}{3}\sup_t \int_{B_s^-} |u|^3\eta^8(x,t)dx + \int_{Q_s^-} |u|\,|\nabla u|^2\eta^8 dz + \frac{1}{4}\int_{Q_s^-} |u|^{-1}|\nabla|u|^2|^2\eta^8 dz \\
&= \frac{8}{3}\int_{Q_s^-} |u|^3\eta^7\eta_t\, dz - \int (u\cdot\nabla)u\cdot|u|u\eta^8\, dz - 4\int_{Q_s^-} |u|\eta^7\nabla|u|^2\cdot\nabla\eta\, dz \\
&\quad + 8\int_{Q_s^-} p\eta^7|u|(u\cdot\nabla)\eta\, dz + \frac{1}{2}\int_{Q_s^-} p\eta^8|u|^{-1}(u\cdot\nabla)|u|^2 dz + \int_{Q_s^-} f\cdot u|u|\eta^8 dz.
\end{aligned} \tag{3.1}
$$

Notice that

$$
\begin{aligned}
&\frac{1}{2}\int_{Q_s^-} p\eta^8|u|^{-1}(u\cdot\nabla)|u|^2 dz \\
&\le \int \left(\int |\nabla u|^\alpha\, dx\right)^{\frac{1}{\alpha}} \left(\int |p|^{\frac{\alpha}{\alpha-1}}\,\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{\alpha}{\alpha-1}}\, dx\right)^{\frac{\alpha-1}{\alpha}} dt \\
&\le \left[\int \left(\int |\nabla u|^\alpha\, dx\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \left[\int \left(\int |p|^{\frac{\alpha}{\alpha-1}}\,\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{\alpha}{\alpha-1}}\, dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}} dt\right]^{\frac{\beta-1}{\beta}}.
\end{aligned}
$$

Hence, we have from (3.1) that

$$
\begin{aligned}
J &\le c\int_{Q_s^-} |u|^3\, dz + \left|\int (u\cdot\nabla)u\cdot|u|u\eta^8\, dz\right| + 8\int_{Q_s^-} p\eta^7|u|(u\cdot\nabla)\eta\, dz + c \\
&\quad + \frac{1}{4}J + \|\nabla u\|_{L^{\alpha,\beta}(Q_s^-)}\left[\int \left(\int |p|^{\frac{\alpha}{\alpha-1}}\,\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{\alpha}{\alpha-1}}\, dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}} dt\right]^{\frac{\beta-1}{\beta}}.
\end{aligned} \tag{3.2}
$$

Consider

$$
\int (u\cdot\nabla)u\cdot|u|u\eta^2\, dz
$$

of (3.2), which is estimated as follows:

$$\begin{aligned}
&\left|\int (u\cdot\nabla)u\cdot|u|u\eta^8\,dz\right| \le \int |u|^3\,|\nabla u|\eta^8\,dz\\
&\le \int \Big(\int |\nabla u|^\alpha\,dx\Big)^{\frac{1}{\alpha}}\Big(\int |u|^{\frac{3\alpha}{\alpha-1}}\eta^{\frac{8\alpha}{\alpha-1}}\,dx\Big)^{\frac{\alpha-1}{\alpha}}dt\\
&\le \Big[\int\Big(\int |\nabla u|^\alpha\,dx\Big)^{\frac{\beta}{\alpha}}dt\Big]^{\frac{1}{\beta}}\Big[\int\Big(\int |u|^{\frac{3\alpha}{\alpha-1}}\eta^{\frac{8\alpha}{\alpha-1}}\,dx\Big)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}\\
&= \|\nabla u\|_{L^{\alpha,\beta}(Q_r^-)}\Big[\int\Big(\int |u|^{\frac{3\alpha}{\alpha-1}}\eta^{\frac{8\alpha}{\alpha-1}}\,dx\Big)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}.
\end{aligned}$$

By Hölder's and Sobolev's inequalities, we have that for $\alpha > 3/2$,

$$\begin{aligned}
&\Big[\int\Big(\int \eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{3\alpha}{\alpha-1}}\,dx\Big)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}\\
&= \Big[\int\Big(\int \eta^{\frac{8(2\alpha-3)-24}{2(\alpha-1)}}|u|^{\frac{3(2\alpha-3)-9}{2(\alpha-1)}}\,dx\Big)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}\\
&\le \Big[\int\Big(\int \eta^8|u|^3\,dx\Big)^{\frac{(2\alpha-3)\beta}{2\alpha(\beta-1)}}\Big(\int \eta^{24}|u|^9\,dx\Big)^{\frac{\beta}{2\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}\\
&\le c\sup_t\Big(\int \eta^8|u|^3\,dx\Big)^{\frac{(2\alpha-3)}{2\alpha}}\Big[\int\Big(\int |\nabla(\eta^4|u|^{\frac{3}{2}})|^2\,dx\Big)^{\frac{3\beta}{2\alpha(\beta-1)}}dt\Big]^{\frac{\beta-1}{\beta}}\\
&\le c\sup_t\Big(\int \eta^8|u|^3\,dx\Big)^{\frac{(2\alpha-3)}{2\alpha}}\Big[\int |\nabla(\eta^4|u|^{\frac{3}{2}})|^2\,dz\Big]^{\frac{3}{2\alpha}}\\
&\le c\sup_t\int \eta^8|u|^3\,dx + c\int |u|\,|\nabla u|^2\eta^8\,dz + c\int \eta^6|\nabla\eta|^2\,|u|^3\,dz\\
&\le cJ + c\int_{Q_r^-}|u|^3\,dz \le cJ + c.
\end{aligned}$$

Here, the fact

$$\frac{3\beta}{2\alpha(\beta-1)} \le 1,$$

that is,

$$\frac{3}{\alpha}+\frac{2}{\beta}\leq 2,$$

is used. Hence, we have that

$$\left[\int\left(\int \eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{3\alpha}{\alpha-1}}\,dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\right]^{\frac{\beta-1}{\beta}}\leq cJ+c, \tag{3.3}$$

and that

$$\left|\int (u\cdot\nabla)u\cdot|u|u\eta^8\,dz\right|\leq c\|\nabla u\|_{L^{\alpha,\beta}(Q_r^-)}J+c. \tag{3.4}$$

Consider the term $8\int p\eta^7|u|(u\cdot\nabla)\eta\,dz$ of (3.2). Observe that

$$\begin{aligned}\int p\eta^7|u|(u\cdot\nabla)\eta\,dz &\leq \left(\int |p|^2\eta^7|\nabla\eta|\,dz\right)^{\frac{1}{2}}\left(\int |u|^4\eta^7|\nabla\eta|\,dz\right)^{\frac{1}{2}}\\ &\leq c\int |u|^4\eta^7\,dz+c,\end{aligned}$$

due to the estimates on p in Section 2. Consider $\int |u|^4\eta^7\,dz$;

$$\begin{aligned}&\int |u|^4\eta^7\,dz=\int |u|^{\frac{11}{8}+\frac{3}{32}+\frac{81}{32}}\eta^{\frac{1}{4}+\frac{27}{4}}\,dz\\ &\leq \int\left[\left(\int |u|^2\,dx\right)^{\frac{11}{16}}\left(\int |u|^3\eta^8\,dx\right)^{\frac{1}{32}}\left(\int |u|^9\eta^{24}\,dx\right)^{\frac{9}{32}}\right]dt\\ &\leq c\sup_t\left(\int |u|^3\eta^8\,dx\right)^{\frac{1}{32}}\int\left(\int |u|^9\eta^{24}\,dx\right)^{\frac{9}{32}}dt\\ &\leq c\sup_t\left(\int |u|^3\eta^8\,dx\right)^{\frac{1}{32}}\int\left(\int \left|\nabla(|u|^{\frac{3}{2}}\eta^4)\right|^2dx\right)^{\frac{27}{32}}dt\\ &\leq c\sup_t\left(\int |u|^3\eta^8\,dx\right)^{\frac{1}{32}}\left(\int \left|\nabla(|u|^{\frac{3}{2}}\eta^4)\right|^2dz\right)^{\frac{27}{32}}\\ &\leq c\sup_t\left(\int |u|^3\eta^8\,dx\right)^{\frac{1}{32}}\left(\int \left(|u|\,|\nabla u|^2\eta^8+c|u|^3\eta^6|\nabla\eta|^2\right)dz\right)^{\frac{27}{32}}\\ &\leq c\sup_t\left(\int |u|^3\eta^8\,dx\right)^{\frac{28}{32}}+\left(\int |u|\,|\nabla u|^2\eta^8\,dz+c\right)^{\frac{28}{32}}\\ &\leq cJ^{\frac{28}{32}}+c\leq\frac{1}{4}J+c.\end{aligned}$$

Hence, we have

$$\int p\eta^7|u|(u\cdot\nabla)\eta\,dz \le \frac{1}{4}J + c. \tag{3.5}$$

We now consider the last term of (3.2)

$$\begin{aligned}
&\left[\int\left(\int |p|^{\frac{\alpha}{\alpha-1}}\,\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{\alpha}{\alpha-1}}\,dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\right]^{\frac{\beta-1}{\beta}}\\
&\le\left[\int\left(\int |p|^{\frac{3\alpha}{2(\alpha-1)}}\,\eta^{\frac{8\alpha}{\alpha-1}}\,dx\right)^{\frac{2(\alpha-1)\beta}{3\alpha(\beta-1)}}\left(\int \eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{3\alpha}{\alpha-1}}\,dx\right)^{\frac{(\alpha-1)\beta}{3\alpha(\beta-1)}}dt\right]^{\frac{\beta-1}{\beta}}\\
&\le\left[\int\left(\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{3\alpha}{\alpha-1}}\,dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\right]^{\frac{\beta-1}{\beta}} + c.
\end{aligned}$$

By (3.3), we have

$$\left[\int\left(\int |p|^{\frac{\alpha}{\alpha-1}}\,\eta^{\frac{8\alpha}{\alpha-1}}|u|^{\frac{\alpha}{\alpha-1}}\,dx\right)^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}dt\right]^{\frac{\beta-1}{\beta}} \le cJ + c. \tag{3.6}$$

Therefore, combining (3.4)–(3.6), we have from (3.2) that

$$J \le c\|\nabla u\|_{L^{\alpha,\beta}(Q_s^-)}J + c + \frac{1}{2}J.$$

Choosing the domain of integration so small that $c\|\nabla u\|^2_{L^{\alpha,\beta}(Q_s^-)} < \frac{1}{4}$, we get $J<\infty$. Since

$$\begin{aligned}
\int |u|^5\eta^{\frac{16}{3}+8}\,dz &\le \int\Big(\int |u|^3\eta^8\,dx\Big)^{\frac{2}{3}}\Big(\int |u|^9\eta^{24}\,dx\Big)^{\frac{1}{3}}\,dt\\
&\le \sup_t\Big(\int |u|^3\eta^8\,dx\Big)^{\frac{2}{3}}\int \big|\nabla(|u|^{\frac{3}{2}}\eta^4)\big|^2\,dz\\
&\le \sup_t\Big(\int |u|^3\eta^8\,dx\Big)^{\frac{2}{3}}\int \Big(|u|\,|\nabla u|^2\eta^8 + 16|u|^3\eta^6|\nabla\eta|^2\Big)\,dz,
\end{aligned}$$

we have

$$\int |u|^5\eta^{\frac{40}{3}}\,dz \le J^{\frac{5}{3}} + c < \infty,$$

which completes the proof. For the case $\alpha=\frac{3}{2}$ and $\beta=\infty$, we can do in a similar way. □

Proof of Theorem 1.1. We define a sequence $\{R_i\}_{i=0}^{\infty}$ by

$$R_i = \left(1 - 2^{-i-1}\right) R_0, \quad i = 0, 1, 2, \cdots .$$

As in Lemma 2.4, we obtain

$$\left(\int_{Q^-_{R_i}} |u|^5 dz \right)^{\frac{1}{5}} \le c\varepsilon_0^{\frac{2}{15}} \left(\int_{Q^-_{R_{i-1}}} |u|^5 dz \right)^{\frac{1}{5}} + c^i \left(\int_{Q^-_{R_{i-1}}} |u|^3 dz \right)^{\frac{1}{3}} + c^i.$$

We set $c_0 = c\varepsilon_0^{\frac{2}{15}}$. Hence defining

$$\Phi_i = \left(\int_{Q^-_{R_i}} |u|^5 dz \right)^{\frac{1}{5}} \quad \text{and} \quad \Psi_i = \left(\int_{Q^+_{R_i}} |u|^5 dz \right)^{\frac{1}{3}},$$

we obtain a recurrence relation

$$\Phi_i \le c_0 \Phi_{i+1} + c^i \Psi_{i+1} + c^i. \tag{3.7}$$

Iterating (3.7) we obtain

$$\Phi_0 \le c_0^k \Phi_k + 2^5 \sum_{j=0}^{k} c_0^j c^j (\Psi_{j+1} + 1).$$

Consequently, if we choose R_0 so that $c_0 c \ll 1$, then we

$$\left(\int_{Q^-_{\frac{1}{2}R_0}} |u|^5 dz \right)^{\frac{1}{5}} \le c \left(\int_{Q^-_{R_0}} |u|^3 dz \right)^{\frac{1}{3}} + c. \tag{3.8}$$

Therefore, we obtain

$$\begin{aligned}\sup_{Q^-_{\frac{1}{8}R_0}} |u| &\le c\left(\int_{Q^-_{\frac{1}{4}R_0}} |u|^{5+\sigma} dz\right)^{\frac{1}{\sigma}} + c && \text{[by Lemma 2.3]}\\ &\le c\left(\int_{Q^-_{\frac{1}{2}R_0}} |u|^{5} dz\right)^{\frac{5-\sigma}{5\sigma}} + c && \text{[by Lemma 2.4]}\\ &\le c\left(\int_{Q^-_{R_0}} |u|^{3} dz\right)^{\frac{5-\sigma}{3\sigma}} + c && \text{[by (3.8)].}\end{aligned}$$

The proof is completed. □

We now consider the higher regularity in Theorem 1.2. We can also obtain that the velocity is as regular as the boundary data of the pressure if $\nabla u \in L^{\alpha,\beta}(D)$ with $\frac{3}{\alpha} + \frac{2}{\beta} \le 2, r > \frac{3}{2}$, or $\nabla u \in L^{\frac{3}{2},\infty}(D)$ with $\|\nabla u\|_{L^{\frac{3}{2},\infty}} \le \varepsilon_0$ for some small ε_0. The proof is similar to that in Choe [4]. Choe [4] first showed that if the velocity is bounded near boundary, then it is Hölder continuous. The boundedness of the pressure at the boundary is necessary in the proof to show that the pressure p lies in higher L^m space for sufficiently large m. He employed a perturbation method with isomorphism theorem between Campanato space and Hölder space. The Navier-Stokes solutions are simply compared with the solutions of heat equations. Then from a bootstrap argument, the higher regularity theorem is obtained.

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