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FOURIER-BESSEL TRANSFORMATION OF MEASURES WITH SEVERAL SPECIAL VARIABLES AND PROPERTIES OF SINGULAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to the investigation of mixed Fourier-Bessel transformation (see [2]) for $f \geq 0$:

$$\hat{f}(\xi,\eta) \stackrel{\text{def}}{=} \frac{1}{\prod_{l=1}^{m} \eta_l^{\nu_l}} \underbrace{\int_0^\infty \dots \int_0^\infty}_{m \text{ times}} \int_{\mathbf{R}^n} \prod_{l=1}^{m} y_l^{\nu_l+1} J_{\nu_l}(\eta_l y_l) \times$$
$$\times e^{-ix\cdot\xi} f(x,y) dx dy_1 \dots y_m$$

($\nu_l > -\dfrac{1}{2}$; $l = \overline{1,m}$). We apply the method of [6] which provides the estimate for weighted L_∞-norm of the spherical mean of $|\hat{f}|^2$ via its weighted L_1-norm (generally it is wrong without the requirement of the non-negativity of f). We prove that in the case of Fourier-Bessel transformation the mentioned method provides (in dependence on the relation between the dimension of the space of non-special variables n and the length of multiindex ν) similar estimates for weighted spherical means of $|\hat{f}|^2$; the allowed powers of weights are also defined by multiindex ν. Further those estimates are applied to partial differential equations with singular Bessel operators with respect to $y_1, \dots, y_m$ and we obtain the corresponding estimates for solutions of the mentioned equations.

0. Introduction

It is proved in [6] that if f is non-negative then for any $\alpha \in (0, \frac{n-1}{2}]$

$$\|r^\alpha \sigma(f)\|_\infty \leq C \|r^{\alpha-1}\sigma(f)\|_1, \tag{1}$$

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where $\sigma(f)(r)$ is the mean of $|\hat{f}|^2$ over the sphere of radius r with the center at the origin and C depends only on the dimension of the space.
Note that generally (1) does not hold: one can construct a sequence $\{f_m\}_{m=1}^{\infty}$ such that $\|r^{\alpha-1}\sigma(f_m)\|_1$ does not depend on m but at the same time $\sigma(f_m)(1)$ tends to infinity as $m \to \infty$. Thus the requirement of the non-negativity of f prohibits the above-mentioned behaviour. Actually it means a certain restriction for the shape of the graph of $\hat{f}$.

In this work we investigate the mixed Fourier-Bessel transformation which is applied in the theory of partial differential equations containing singular Bessel operators with respect to selected variables (they are called *special* variables). Those equations arise in models of mathematical physics with degenerative space heterogeneities.
It will be proved that the estimates of kind (1) are valid for weighted spherical means of $|\hat{f}|^2$ but the weights in both parts of the inequality and the weights of the means themselves are controlled by the relation between the dimension of the space of non-special variables and the length of multiindex formed by the indexes of Bessel functions from the kernel of the transformation (or by the parameters at the singularities of Bessel operators contained in the equation). More exactly the following statements are valid:
if $n > 1$ then for $(\alpha, \beta) = (\frac{n-1}{2}, \nu + \frac{1}{2})$ and for α from $(0, \frac{n-1}{2})$, β from $(0, \nu + \frac{1}{2})$

$$(2) \qquad \|r^{\alpha+|\beta|}\sigma^{p+\alpha, q+\beta}(f)\|_\infty \le C\|r^{\alpha+|\beta|-1}\sigma^{\alpha-n, \beta-1}(f)\|_1 \,;$$

if $n = 1$ then

$$(3) \qquad \|r^{|\nu|+\frac{m}{2}}\sigma^{p,q}(f)\|_\infty \le C\|y^{\frac{\nu}{2}-\frac{1}{4}}\hat{f}\Big|_{x=0}\|_2 \,;$$

if $n = 0$ then for $\beta = (\nu_1 + \frac{1}{2}, \dots, \nu_m + \frac{1}{2})$ and for β from $(0, \nu + \frac{1}{2})$

$$(4) \qquad \|r^{|\beta|}\sigma^{0, q+\beta}(f)\|_\infty \le C\|r^{|\beta|-1}\sigma^{0, \beta-1}(f)\|_1 .$$

Here q, β and ν are m-components multiindexes; $\beta - 1$ denotes the following multiindex: $(\beta_1 - 1, \dots, \beta_m - 1)$.

$\sigma^{p,q}$ denotes the spherical mean of $|\cdot|^2$ with the weight $|x|^p \prod_{l=1}^{m} y_l^{q_l}$ and the allowed values of the parameters p, q in the inequalities above are ruled by the dimensions m, n and the value of ν.

Further we use (2)–(4) to derive estimates of solutions of the following singular equation:

$$P(-\Delta_B)u = f(x,y), \tag{5}$$

where $\Delta_B u \overset{\text{def}}{=} \sum_{l=1}^{n} \frac{\partial^2 u}{\partial x_l^2} + \sum_{j=1}^{m} \frac{1}{y_j^{k_j}} \frac{\partial}{\partial y_j}\Big(y_j^{k_j} \frac{\partial u}{\partial y_j}\Big)$, $k_j = 2\nu_j + 1$, $j = \overline{1,m}$; P is a polynomial with real coefficients.

More exactly, under the assumption of non-negativity and weighted summability of $\dfrac{\hat{f}(\xi,\eta)}{P(|\xi|^2 + |\eta|^2)}$ the following estimates for solutions of the equation (5) are valid.

(i) For $n > 1$:
if $p > -n$, $q_j > -1$; $j = \overline{1,m}$ then

$$\|r^{\alpha+|\beta|-p-|q|}\sigma^{\alpha,\beta}u\|_\infty \le C\|r^{\alpha+|\beta|-p-|q|-1}\sigma^{\alpha-n-p,\beta-q-1}u\|_1 \tag{6}$$

for $(\alpha,\beta) = (\alpha,\beta_1,\dots,\beta_m) = (p + \frac{n-1}{2}, q_1 + \frac{k_1}{2}, \dots, q_m + \frac{k_m}{2})$ and for any (α,β) such that $\alpha \in (p, p + \frac{n-1}{2})$, $\beta_j \in (q_j, q_j + \frac{k_j}{2})$; $j = \overline{1,m}$.

(ii) For $n = 1$:
if $p > -1$, $q_j > \frac{k_j}{2} - 1$; $j = \overline{1,m}$ then

$$\|r^{\frac{|k|}{2}}\sigma^{p,q}u\|_\infty \le C\|\prod_{j=1}^{m} y_j^{\frac{k_j}{2}-1}u^2(0,y)\|_1. \tag{7}$$

(iii) For $n = 0$, $m > 1$:
if $q_j > -1$; $j = \overline{1,m}$ then

$$\|r^{|\beta|-|q|}\sigma^{0,\beta}u\|_\infty \le C\|r^{|\beta|-|q|-1}\sigma^{0,\beta-q-1}u\|_1 \tag{8}$$

for $\beta = (q_1 + \frac{k_1}{2}, \dots, q_m + \frac{k_m}{2})$ and for any β such that β_j belongs to $(q_j, q_j + \frac{k_j}{2})$; $j = \overline{1,m}$.

All the constants in the inequalities (6)–(8) depend only on m, n, k, p, q.

1. Preliminaries

In this section the necessary notations and definitions are introduced; we also recall the necessary properties of Fourier-Bessel transformation.

$k \stackrel{\text{def}}{=} (k_1, \dots, k_m) \stackrel{\text{def}}{=} (2\nu_1 + 1, \dots, 2\nu_m + 1)$ - a positive multiindex;

$k_l > 0$ for any $l = \overline{1,m}$; $|k| \stackrel{\text{def}}{=} k_1 + \dots + k_m$ - the length of k.

$\mathbf{R}^m_{(+)} \stackrel{\text{def}}{=} \left\{ y = (y_1, \dots, y_m) \,\middle|\, y_l > 0 \text{ for any } l = \overline{1,m} \right\}$;

$\mathbf{R}^{n+m}_{+} \stackrel{\text{def}}{=} \left\{ (x, y) \,\middle|\, x \in \mathbf{R}^n, y \in \mathbf{R}^m_{(+)} \right\}$.

Hereafter all the absolute constants generally depend on k, m and n.

$S(r)$ denotes the following sphere: $\left\{ x \in \mathbf{R}^n \,\middle|\, |x| = r \right\}$.

$S^l_+(r)$ denotes the following spherical segment:

$\left\{ (x, y) \in \mathbf{R}^{n+l}_{+} \,\middle|\, |x|^2 + |y|^2 = r^2 \right\}$, where $l = \overline{1,m}$;

$S^+(r)$ denotes the following spherical segment: $\left\{ y \in \mathbf{R}^m_{(+)} \,\middle|\, |y| = r \right\}$.

$B^+(r)$ denotes the followding segment of the ball: $\left\{ y \in \mathbf{R}^m_{(+)} \,\middle|\, |y| \leq r \right\}$.

$L_{p,k}(\mathbf{R}^{n+m}_{+}) \stackrel{\text{def}}{=} \left\{ f \,\middle|\, \|f\| = \left(\int\limits_{\mathbf{R}^{n+m}_{+}} \prod\limits_{l=1}^{m} y_l^{k_l} |f(x,y)|^p dx dy \right)^{\frac{1}{p}} < \infty \right\}$ if p is finite;

$L_{\infty,k}(\mathbf{R}^{n+m}_{+}) \stackrel{\text{def}}{=} \left\{ f \,\middle|\, \|f\| = \sup \prod\limits_{l=1}^{m} y_l^{k_l} |f(x,y)| < \infty \right\}$.

The set of infinitely smooth functions with compact supports defined on $\mathbf{R}^{n+m}$ is denoted by $C_0^\infty(\mathbf{R}^{n+m})$.

The subset of $C_0^\infty(\mathbf{R}^{n+m})$ formed by functions which are even with respect to y is observed; the set of restrictions of elements of that subset to $\mathbf{R}^{n+m}_{+}$ is denoted by $C^\infty_{0,even}(\mathbf{R}^{n+m}_{+})$.

This $C^\infty_{0,even}(\mathbf{R}^{n+m}_{+})$ will be *the space of test functions.*

Distributions on $C^\infty_{0,even}(\mathbf{R}^{n+m}_{+})$ are introduced (following for instance [1]) with respect to the degenerative measure $\prod\limits_{l=1}^{m} y_l^{k_l} dx dy$: for any $\varphi \in C^\infty_{0,even}(\mathbf{R}^{n+m}_{+})$,

$$\langle f, \varphi \rangle \stackrel{\text{def}}{=} \int_{\mathbf{R}^{n+m}_{+}} \prod_{l=1}^{m} y_l^{k_l} f(x,y) \varphi(x,y) dx dy \tag{9}$$

Thus all linear continuous functionals on $C^{\infty}_{0,even}(\mathbf{R}^{n+m}_{+})$ which could be given by (9) $\Big($with $f \in L_{1,k,loc}(\mathbf{R}^{n+m}_{+})\Big)$ are called regular (and the corresponding function f is called *ordinary*).

Fourier-Bessel transformation is introduced following [2], [5]:

$$\hat{f}(\xi,\eta) \overset{\text{def}}{=} \mathcal{F}_b f \overset{\text{def}}{=} \int_{\mathbf{R}^m_{(+)}} \int_{\mathbf{R}^n} \prod_{l=1}^{m} y_l^{k_l} j_{\nu_l}(\eta_l y_l) e^{-ix\cdot\xi} f(x,y)\,dx\,dy,$$

where $j_\nu(z)$ is the normalized (in the uniform sense) Bessel function: $j_\nu(z) = \dfrac{J_\nu(z)}{z^\nu}$.

Note (see [2]) that

$$f(x,y) = C \int_{\mathbf{R}^m_{(+)}} \int_{\mathbf{R}^n} \prod_{l=1}^{m} \eta_l^{k_l} j_{\nu_l}(\eta_l y_l) e^{ix\cdot\xi} \hat{f}(\xi,\eta)\,d\xi\,d\eta.$$

The generalized convolution is introduced (see [4], [5]):

$$\begin{aligned}\hat{f}(\xi,\eta) &\overset{\text{def}}{=} (f * g)(\xi,\eta)\\ &\overset{\text{def}}{=} \int_{\mathbf{R}^m_{(+)}} \int_{\mathbf{R}^n} \prod_{l=1}^{m} y_l^{k_l} T_y^{\eta} f(x_1-\xi_1,\dots,x_m-\xi_m,y_1,\dots,y_m)\,dx\,dy,\end{aligned}$$

such that $\widehat{f{*}g} = \hat{f}\hat{g}$ (see also [3]).

Here $T_y^h f(x,y) \overset{\text{def}}{=} T^{h_1,\dots,h_m}_{y_1,\dots,y_m} f(x,y) \overset{\text{def}}{=} T^{h_1}_{y_1} T^{h_2}_{y_2} \cdots T^{h_m}_{y_m} f(x,y)$, $T^{h_l}_{y_l}$ denotes (for $l = \overline{1,m}$) the generalized shift operator with respect to the corresponding special variable (see [4]):

$$T^{h_l}_{y_l} f(y) \overset{\text{def}}{=} C \int_0^{\pi} f\Big(\sqrt{y_l^2 + h_l^2 - 2y_l h_l \cos\theta}\Big) \sin^{k-1}\theta\,d\theta.$$

Note (see [4], [5]) that

$$\int_{\mathbf{R}^m_{(+)}} \prod_{l=1}^{m} \eta_l^{k_l} g(\eta) T_y^{\eta} f(y)\,d\eta = \int_{\mathbf{R}^m_{(+)}} \prod_{l=1}^{m} \eta_l^{k_l} f(\eta) T_y^{\eta} g(y)\,d\eta.$$

2. Estimates of Fourier-Bessel transforms of measures: the general case

We start our investigation from the case of several *non-special* variables: $n > 1$; the critical cases of $n = 1$ and $n = 0$ will be considered in Section 3 and Section 4 correspondingly. The case of a single special variable ($m = 1$) is investigated in [7], [8] so hereafter $m > 1$.

Thus let $m \geq 2$, $n \geq 2$.

Now (as in [8]) we have to define the weighted spherical mean and the corresponding distribution of weighted spherical averaging:

$$\begin{aligned}\sigma^{p,q}(f)(r) &\stackrel{\text{def}}{=} \sigma(f)(r)\\ &\stackrel{\text{def}}{=} \int_{S_+^m(1)} |x|^p y_1^{q_1} \cdots y_m^{q_m} |\hat{f}(rx, ry)|^2 dS_{x,y}\\ &\stackrel{\text{def}}{=} \langle \sigma_r, |\hat{f}|^2 \rangle \stackrel{\text{def}}{=} \langle \sigma_r^{p,q}, |\hat{f}|^2 \rangle,\end{aligned}$$

where $p > -n$; $q_l > -1, l = \overline{1, m}$.

Then we have to estimate

$$\begin{aligned}g_r(x, y) \stackrel{\text{def}}{=} \widehat{\sigma_r} &= \int_{S_+^m(1)} |\xi|^p e^{irx\cdot\xi} \prod_{l=1}^{m} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta}\\ &= \int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_{S_+^{m-1}(\sqrt{1-\eta_m^2})} |\xi|^p e^{irx\cdot\xi} \times\\ &\times \prod_{l=1}^{m-1} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta_1,\dots,\eta_{m-1}} \frac{d\eta_m}{\sqrt{1-\eta_m^2}}\\ &= \int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \times\\ &\times \int_{S_+^{m-2}(\sqrt{1-\eta_m^2-\eta_{m-1}^2})} |\xi|^p e^{irx\cdot\xi} \times\\ &\times \prod_{l=1}^{m-2} \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta_1,\dots,\eta_{m-2}} \frac{\sqrt{1-\eta_m^2}\, d\eta_{m-1}}{\sqrt{1-\eta_{m-1}^2-\eta_m^2}} \frac{d\eta_m}{\sqrt{1-\eta_m^2}}\end{aligned}$$

$$\dots$$

$$= \int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1} j_{\nu_1}(ry_1\eta_1)(1-|\eta|^2)^{\frac{p}{2}} \int_{S\left(\sqrt{1-|\eta|^2}\right)} e^{irx\cdot\xi} dS_\xi \times$$

$$\times \frac{1}{\sqrt{1-|\eta|^2}} d\eta_1 \dots d\eta_m$$

(see for instance [9], p. 155)

$$= \int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1} j_{\nu_1}(ry_1\eta_1) \frac{(1-|\eta|^2)^{\frac{n+p-1}{2}}}{\sqrt{1-|\eta|^2}} \times$$

$$\times \left(r\sqrt{1-|\eta|^2}|x|\right)^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(r\sqrt{1-|\eta|^2}|x|\right) d\eta$$

$$= r^{-|\nu|}|x|^{\frac{2-n}{2}} \prod_{l=1}^m y_l^{-\nu_l} \int_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(ry_m\eta_m) \times$$

$$\times \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots$$

$$\dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(ry_1\eta_1) \times$$

$$\times \frac{(1-|\eta|^2)^{\frac{n+p-1}{2}+\frac{2-n}{4}}}{\sqrt{1-|\eta|^2}} r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}\left(r\sqrt{1-|\eta|^2}|x|\right) d\eta$$

$$= r^{\frac{2-n}{2}-|\nu|}|x|^{\frac{2-n}{2}} \prod_{l=1}^m y_l^{-\nu_l} \int_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(ry_m\eta_m) \times$$

$$\times \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots$$

$$\cdots \int_0^{\sqrt{1-\eta_m^2-\cdots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(ry_1\eta_1)(1-|\eta|^2)^{\frac{n+2p-2}{4}} \times$$
$$\times J_{\frac{n-2}{2}}(r\sqrt{1-|\eta|^2}|x|)d\eta.$$

Since $|J_\mu(t)| \le \dfrac{C}{\sqrt{t}}$ for $t>0$ then

$$|g_r(x,y)| \le Cr^{\frac{m+2-n-|k|}{2}} |x|^{\frac{1-n}{2}} r^{-\frac{m+1}{2}} \prod_{l=1}^m y_l^{-\nu_l-\frac{1}{2}} \times$$
$$\times \int_{B^+(1)} (1-|\eta|^2)^{\frac{n+2p-3}{2}} \prod_{l=1}^m \eta_l^{q_l-\nu_l-\frac{1}{2}} d\eta$$
$$= Cr^{\frac{1-n-|k|}{2}} |x|^{\frac{1-n}{2}} \prod_{l=1}^m y_l^{-\frac{k_l}{2}}$$

(if $p > -\frac{n+1}{2}$; $q_l > \frac{k_l}{2}-1$, $l=\overline{1,m}$).
This means that Fourier-Bessel transform of σ_r is a regular distribution (while σ_r itself is a singular distribution).

Further similarly to [8] $\sigma(f)(r) = \langle f*g_r, f\rangle$, and for non-negative f the last expression is less than or equal to

$$\int_{\mathbf{R}_+^{n+m}} \prod_{l=1}^m \eta_l^{k_l} f(\xi,\eta) \int_{\mathbf{R}_+^{n+m}} \prod_{l=1}^m y_l^{k_l} |g_r(x,y)|\, T_y^\eta f(x-\xi,y)\, dxdyd\xi d\eta$$

(since the generalized shift operator preserves the sign).
Hence

$$\sigma(f)(r) \le Cr^{\frac{1-n-|k|}{2}} \Big\langle f, f * \Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^m y_l^{-\frac{k_l}{2}}\Big)\Big\rangle$$
$$= Cr^{\frac{1-n-|k|}{2}} \Big\langle \mathcal{F}_b\Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^m y_l^{-\frac{k_l}{2}}\Big), |\hat{f}|^2\Big\rangle.$$

On the other hand

$$\mathcal{F}_b\Big(|x|^{-\frac{n-1}{2}} \prod_{l=1}^m y_l^{-\frac{k_l}{2}}\Big) = C|\xi|^{-\frac{n+1}{2}} \prod_{l=1}^m \eta_l^{-\frac{k_l}{2}-1}$$

(see for instance [9], p. 155 and [7]).

Therefore for any non-negative f from $L_{1,k}(\mathbf{R}_+^{n+m}) \cap L_{2,k}(\mathbf{R}_+^{n+m})$

$$\sigma(f)(r) \leq C r^{\frac{1-n-|k|}{2}} \int_{\mathbf{R}^m_{(+)}} \int_{\mathbf{R}^n} |x|^{-\frac{n+1}{2}} \prod_{l=1}^{m} y_l^{\frac{k_l}{2}-1} |\hat{f}(x,y)|^2 dx dy$$

on $(0, +\infty)$.

This is actually the claimed estimate (2) for $(\alpha, \beta) = (\frac{n-1}{2}, \nu + \frac{1}{2})$.
In order to extend it to any α belonging to $(0, \frac{n-1}{2})$, β_1 belonging to $(0, \nu_1 + \frac{1}{2})$, $\ldots$, β_m belonging to $(0, \nu_m + \frac{1}{2})$ we have (as in [7], [8]) to introduce $f_{\gamma,\delta} \overset{\text{def}}{=} f * \left(|x|^{\gamma - n} \prod_{l=1}^{m} y_l^{\delta_l - k_l - 1} \right)$, where $\gamma \overset{\text{def}}{=} \frac{n-2\alpha-1}{4} > 0$, $\delta_l \overset{\text{def}}{=} \frac{2\nu_l - 2\beta_l + 1}{4} > 0$. Then we apply the last inequality to this new function $f_{\gamma,\delta}$.

This yields (under the same assumptions about f):

$$\begin{aligned} &\sup_{\mathbf{R}_+} r^{\alpha+|\beta|} \sigma^{p-\frac{n-1}{2}+\alpha, q-\frac{k}{2}+\beta}(f)(r) \\ (10) \qquad &\leq C \int_{\mathbf{R}^m_{(+)}} \int_{\mathbf{R}^n} |x|^{\alpha-n} \prod_{l=1}^{m} y_l^{\beta_l - 1} |\hat{f}(x,y)|^2 dx dy. \end{aligned}$$

The right-hand side of (10) is equal to $C \int_0^\infty r^{\alpha+|\beta|-1} \sigma^{\alpha-n, \beta-1} f(r) dr$.

Thus the following statement is valid:

THEOREM 1. *Let $m \geq 2$, $n \geq 2$, k be a positive multiindex, $p > -n$, $q_l > -1$ $(l = \overline{1,m})$. Then there exists C such that for any non-negative $f \in L_{1,k}(\mathbf{R}_+^{n+m}) \cap L_{2,k}(\mathbf{R}_+^{n+m})$ for any $\alpha \in (0, \frac{n-1}{2})$, $\beta_l \in (0, \frac{k_l}{2})$ $(l = \overline{1,m})$ and for $(\alpha, \beta) = (\frac{n-1}{2}, \frac{k_1}{2}, \ldots, \frac{k_m}{2})$*

$$(11) \qquad \| r^{\alpha+|\beta|} \sigma^{p+\alpha, q+\beta}(f) \|_\infty \leq C \| r^{\alpha+|\beta|-1} \sigma^{\alpha-n, \beta-1}(f) \|_1.$$

3. The case of a single non-special variable

Let $n = 1$. Then similarly to Section 2

$$
\begin{aligned}
g_r(x,y) &= \int_{S_+^m(1)} |\xi|^p e^{irx\xi} \prod_{l=1}^m \eta_l^{q_l} j_{\nu_l}(ry_l\eta_l) dS_{\xi,\eta} \\
&= \int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots \\
&\dots \int_0^{\sqrt{1-|\eta|^2+\eta_1^2}} (1-|\eta|^2)^{\frac{p}{2}} \eta_1^{q_1} j_{\nu_1}(ry_1\eta_1) \times \\
&\times \left(e^{ix\sqrt{1-|\eta|^2}r} + e^{-ix\sqrt{1-|\eta|^2}r}\right) \frac{d\eta}{\sqrt{1-|\eta|^2}} \\
&= 2\int_0^1 \eta_m^{q_m} j_{\nu_m}(ry_m\eta_m) \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}} j_{\nu_{m-1}}(ry_{m-1}\eta_{m-1}) \dots \\
&\dots \int_0^{\sqrt{1-|\eta|^2+\eta_1^2}} \eta_1^{q_1} j_{\nu_1}(ry_1\eta_1)(1-|\eta|^2)^{\frac{p-1}{2}} \cos rx\sqrt{1-|\eta|^2} d\eta.
\end{aligned}
$$

Since $\left|\cos rx\sqrt{1-|\eta|^2}\right| \le 1$ then we obtain:

$$
|\widehat{\sigma_r}| = |g_r(x,y)| \le \frac{C}{r^{\frac{|k|}{2}} \prod_{l=1}^m y_l^{\frac{k_l}{2}}}
$$

for $p > -1$; $q_l > \frac{k_l}{2} - 1, l = \overline{1,m}$.
Therefore for positive r

$$
\begin{aligned}
\sigma^{p,q}(f)(r) &\le Cr^{-\frac{|k|}{2}} \left\langle \mathcal{F}_b\Big(\prod_{l=1}^m y_l^{\frac{k_l}{2}}\Big), |\hat{f}|^2 \right\rangle \\
&= Cr^{-\frac{|k|}{2}} \int_{\mathbf{R}_{(+)}^m} \prod_{l=1}^m y_l^{\frac{k_l}{2}-1} |\hat{f}(0,y)|^2 dy
\end{aligned} \tag{12}
$$

(cf. (1.3) of [8]).

4. The case of absence of non-special variables

Let $n = 0$. Then

$$\tilde{f}(\eta) \stackrel{\text{def}}{=} \int_{\mathbf{R}^m_{(+)}} y_1^{k_1} \dots y_m^{k_m} f(y_1, \dots, y_m) j_{\nu_1}(y_1\eta_1) \dots j_{\nu_m}(y_m\eta_m) dy.$$

Therefore

$$\begin{aligned} g_r(x,y) \stackrel{\text{def}}{=} \widetilde{\sigma_r} &= \int_{S^+(1)} \prod_{l=1}^m \eta_l^{q_l} j_{\nu_l}(r y_l \eta_l) dS_\eta \\ &= r^{-|\nu|} \prod_{l=1}^m y_l^{-\nu_l} \int_0^1 \eta_m^{q_m-\nu_m} J_{\nu_m}(r y_m \eta_m) \\ &\quad \int_0^{\sqrt{1-\eta_m^2}} \eta_{m-1}^{q_{m-1}-\nu_{m-1}} J_{\nu_{m-1}}(r y_{m-1} \eta_{m-1}) \dots \\ &\quad \dots \int_0^{\sqrt{1-\eta_m^2-\dots-\eta_2^2}} \eta_1^{q_1-\nu_1} J_{\nu_1}(r y_1 \eta_1) d\eta_1 \dots d\eta_m. \end{aligned}$$

Hence for $q_l > \frac{k_l}{2} - 1, l = \overline{1,m}$

$$|\widetilde{\sigma_r}| \leq C r^{-|\nu|} \prod_{l=1}^m y_l^{-\nu_l - \frac{1}{2}} r^{-\frac{m}{2}} \int_{S^+(1)} \prod_{l=1}^m \eta_l^{q_l - \frac{k_l}{2}} dS_\eta \stackrel{\text{def}}{=} C r^{-\frac{|k|}{2}} \prod_{l=1}^m y_l^{-\frac{k_l}{2}}.$$

So for any non-negative f from $L_{1,k}(\mathbf{R}^m_{(+)}) \cap L_{2,k}(\mathbf{R}^m_{(+)})$

$$\sigma^{0,q}(f)(r) \leq C r^{-\frac{|k|}{2}} \int_{\mathbf{R}^m_{(+)}} \prod_{l=1}^m y_l^{\frac{k_l}{2}-1} \tilde{f}^2(y) dy \quad \text{on } (0,+\infty)$$

and (similarly to Section 2) for any β_l from $(0, \frac{k_l}{2})$, $l = \overline{1,m}$

$$\sup_{\mathbf{R}_+} r^{|\beta|} \sigma^{0,q+\beta}(f)(r) \leq C \int_{\mathbf{R}^m_{(+)}} \prod_{l=1}^m y_l^{\beta_l - 1} \tilde{f}^2(y) dy. \tag{13}$$

The right-hand side of the inequality (13) is equal to

$$
\begin{aligned}
&C\int_0^\infty \int_{S^+(r)} \prod_{l=1}^m y_l^{\beta_l-1} \tilde{f}^2(y)dS_y dr \\
&= C\int_0^\infty \int_{S^+(1)} r^{|\beta|-m} \prod_{l=1}^m \eta_l^{\beta_l-1} \tilde{f}^2(r\eta) r^{m-1} dS_\eta dr \\
&= C\|r^{|\beta|-1}\sigma^{0,\beta-1}(f)\|_1.
\end{aligned}
$$

Thus the following statement is true:

THEOREM 2. *Let $m \geq 2$, $n = 0$, $q > -1$. Then there exists C such that for any non-negative f from $L_{1,k}(\mathbf{R}^m_{(+)}) \cap L_{2,k}(\mathbf{R}^m_{(+)})$ for $\beta = (\frac{k_1}{2}, \dots, \frac{k_m}{2})$ and for any β from $(0, \frac{k}{2})$*

$$\|r^{|\beta|}\sigma^{0,q+\beta}(f)\|_\infty \leq C\|r^{|\beta|-1}\sigma^{0,\beta-1}(f)\|_1.$$

5. Estimates of solutions of singular equations

In this section we apply the above results to estimate norms of solutions of (5).
We will start from the general case of several special variables.
Let u from $L_{2,k}(\mathbf{R}^{n+m}_+)$ satisfy (5) at least in the sense of distributions.
Then $\hat{u}$ also belongs to $L_{2,k}(\mathbf{R}^{n+m}_+)$ (see [2]) and

$$P(|\xi|^2 + |\eta|^2)\hat{u}(\xi,\eta) = \hat{f}(\xi,\eta). \tag{14}$$

$P(|\xi|^2 + |\eta|^2) \in L_{2,k,loc}(\mathbf{R}^{n+m}_+)$, $\hat{u}(\xi,\eta) \in L_{2,k,loc}(\mathbf{R}^{n+m}_+)$ therefore $\hat{f}(\xi,\eta) \in L_{1,k,loc}(\mathbf{R}^{n+m}_+)$ that is $\hat{f}(\xi,\eta)$ is an ordinary function.
Thus (14) is an equality of ordinary functions and hence the following division is legible:

$$\hat{u}(\xi,\eta) = \frac{\hat{f}(\xi,\eta)}{P(|\xi|^2 + |\eta|^2)} \in L_{2,k}(\mathbf{R}^{n+m}_+).$$

Now we denote $\dfrac{\hat{f}(\xi,\eta)}{P(|\xi|^2+|\eta|^2)}$ by $g(\xi,\eta)$ and assume that g is non-negative ang belongs to $L_{1,k}(\mathbf{R}_+^{n+m})$. Then g satisfies the conditions of Theorem 1 and $u = \hat{g}$.

This implies the following statement:

THEOREM 3. *Let* $\dfrac{\hat{f}(\xi,\eta)}{P(|\xi|^2+|\eta|^2)}$ *be non-negative and belong to* $L_{1,k}(\mathbf{R}_+^{n+m})$, *u from* $L_{2,k}(\mathbf{R}_+^{n+m})$ *satisfy (at least in the sense of distributions) the equation* (5). *Let* $p > -n$, $q_j > -1$; $j = \overline{1,m}$. *Then there exists C such that*

$$\|r^{\alpha+|\beta|-p-|q|}\sigma^{\alpha,\beta}u\|_\infty \leq C\|r^{\alpha+|\beta|-p-|q|-1}\sigma^{\alpha-n-p,\beta-q-1}u\|_1$$

for $(\alpha,\beta) = (\alpha,\beta_1,\ldots,\beta_m) = (p+\frac{n-1}{2}, q_1+\frac{k_1}{2},\ldots,q_m+\frac{k_m}{2})$ *and for any* (α,β) *such that* $\alpha \in (p, p+\frac{n-1}{2})$, $\beta_j \in (q_j, q_j+\frac{k_j}{2})$; $j = \overline{1,m}$.

On the same way (12) yields the following statement:

THEOREM 4. *Let* $\dfrac{\hat{f}(\xi,\eta)}{P(\xi^2+|\eta|^2)}$ *be non-negative and belong to* $L_{1,k}(\mathbf{R}_+^{1+m})$, *u from* $L_{2,k}(\mathbf{R}_+^{1+m})$ *satisfy (at least in the sense of distributions) the equation* (5). *Let* $p > -1$, $q_j > \frac{k_j}{2}-1$; $j = \overline{1,m}$. *Then there exists C such that*

$$\|r^{\frac{|k|}{2}}\sigma^{p,q}u\|_\infty \leq C\|\prod_{j=1}^{m} y_j^{\frac{k_j}{2}-1}u^2(0,y)\|_1.$$

And, finally, Theorem 2 on the same way leads to

THEOREM 5. *Let* $\dfrac{\tilde{f}(\eta)}{P(|\eta|^2)} \in L_{1,k}(\mathbf{R}_{(+)}^m)$, $\dfrac{\tilde{f}(\eta)}{P(|\eta|^2)} \geq 0$; *let u from* $L_{2,k}(\mathbf{R}_{(+)}^m)$ *satisfy (at least in the sense of distributions) the equation* (5). *Let* $q_j > -1$; $j = \overline{1,m}$. *Then*

$$\|r^{|\beta|-|q|}\sigma^{0,\beta}u\|_\infty \leq C\|r^{|\beta|-|q|-1}\sigma^{0,\beta-q-1}u\|_1$$

for $\beta = (\beta_1, \ldots, \beta_m) = (q_1 + \frac{k_1}{2}, \ldots, q_m + \frac{k_m}{2})$ *and for any* β *such that* $\beta_j \in (q_j, q_j + \frac{k_j}{2})$; $j = \overline{1, m}$.

REMARK. In the inequalities (2)–(4) (correspondingly (6)–(8)) the constant C depends only on m, n, k, p, q.

REMARK. Under the assumptions of Theorems 3–5 the right-hand sides of the corresponding inequalities converge.

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