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ON THE BOUNDARY VALUE PROBLEMS FOR LOADED DIFFERENTIAL EQUATIONS

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ABSTRACT. The equations prescribed in $\Omega \subset R^n$ are called loaded, if they contain some operations of the traces of desired solution on manifolds (of dimension which is strongly less than n) from closure $\overline{\Omega}$. These equations result from approximations of nonlinear equations by linear ones, in the problems of optimal control when the control actions depends on a part of independent variables, in investigations of the inverse problems and so on. In present work we study the nonlocal boundary value problems for first-order loaded differential operator equations. Criterion of unique solvability is established. We illustrate the obtained results by examples.

1. Introduction

Boundary value problems for the loaded equations arise and find a wide applications in many applied problems [1], [5], [7], [9]. However they are not always stated correctly. The classical solution of Existence Questions of local boundary value problems for loaded equations are considered in work [1], [2] spectral problems — [3], generalized solvability — [6], [8], etc. Theses equations find a wide applications in many applied problems. In present work we study a nonlocal boundary value problems for loaded differential operator equations of first order. Criterion of unique solvability of considered problems is established. We illustrate the obtained results by examples.

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2. The Statement of Problem and Main Results

Let $(x \in)\ \Omega \subset R^n$ be a cube with a $2\pi-$ribs; $\mathcal{P}^\infty$ be a linear manifold of the smooth all over variables periodic complex-valued functions; $H \equiv L^2(\Omega)$. Let

$$A_k(s)=\sum_{0\le|\alpha^k|\le l_k} a_{\alpha^k}\cdot s^{\alpha^k},\ s^{\alpha^k}=s_1^{\alpha_1^k}...s_n^{\alpha_n^k},\ |\alpha^k|=\alpha_1^k+...+\alpha_n^k,\ k=0,1,...,m,$$

be the polynomials with constant coefficients. Further we define the linear differential operators $A_k(-iD),\ k=0,1,...,m,\ i=\sqrt{-1},\ D^\alpha = D_1^{\alpha_1}...D_n^{\alpha_n},\ D_j=\partial/\partial x_j,$ as the closure in H of the differential operations primary-defined on $\mathcal{P}^\infty$ such that

$$A_k(-iD)\exp\{is\cdot x\} = A_k(s)\exp\{is\cdot x\},\ s{\cdot}x=\sum_{j=1}^n s_j x_j,\ k=0,1,...,m.$$

The *operators* $A_k,\ k=0,1,...,m,$ will be called by Π-operators [4].

We denote

$$A(s)=\sum_{k=1}^m A_k(s),$$

$$B(s) = 2\pi A(s) + (\mu-1)[1+\sum_{k=1}^m A_k(s)t_k],$$

where $\mu \in \mathbb{C}$ (the complex number).

Let $\mathcal{S}=\{s=\{s_j\}_{j=1}^n | s_j = 0, \pm1, \pm2, ..., \forall j\},\ \mathcal{S}^0=\{s|s\in\mathcal{S},\ A_0(s)=0\},\ l_k = \text{degree}\ \{A_k(s)\},\ k=0,1,...,m,\ \bar{l}=\max\{l_k, k=1,...,m\},\ \mathcal{H}=L^2(0,2\pi;H),$ and

$$(2.1)\qquad \begin{cases} |\mu|<+\infty,\ l_k\le l_0,\ k=1,...,m, \\ \text{degree}\{B(s)\}=\text{degree}\{A(s)\}=\bar{l},\ B(s)|_{s=0}\ne 0, \\ 0<t_1<...<t_m<2\pi,\ \{t_k\}_{k=1}^m \text{ are a fixed points.} \end{cases}$$

We consider the boundary value problem with conditions (2.1):

(2.2)

$$\begin{cases} L_1 u \equiv (D_t^1 + A_0(-iD))u(t) + \sum\limits_{k=1}^{m} A_k(-iD)u(t_k) = f(t) \text{ in } (0, 2\pi), \\ \Gamma_\mu u \equiv \mu u(0) - u(2\pi) = \varphi. \end{cases}$$

The conditions (2.1) rule out the cases $\mu = \pm\infty$ that correspond to the Cauchy problem [7].

Definition 2.1. The function $u(x,t) \in \mathcal{H}$ is strong solution of the problem (2.2) if there is a sequence of the functions $\{u_j(x,t)\}_{j=1}^{\infty} \subset C^1(0, 2\pi; \mathcal{P}^\infty)$ such that $u_j(x,t) \to u(x,t)$, $L_1 u_j = f_j \to f$ in $\mathcal{H}$, and $\Gamma_\mu u_j \to \varphi$ in H.

The following theorem is established.

Theorem 2.1. *The problem* (2.2) *for* $\forall f \in \mathcal{H}$ *and* $\varphi \in H$ *admits a unique strong solution in* $\mathcal{H}$ *iff*

$$B(s) \neq 0 \ \forall s \in \mathcal{S}^0, \tag{2.3}$$

$$C(s) \equiv \mu - \exp\{-A_0(s)2\pi\} \neq 0 \quad \forall s \in \mathcal{S} \setminus \mathcal{S}^0, \tag{2.4}$$

$$\begin{aligned} D(s) \equiv 1 + [A_0(s)]^{-1} \sum_{k=1}^{m} A_k(s)\{1 \\ - [C(s)]^{-1}(\mu - 1)\exp\{-A_0(s)t_k\}\} \neq 0 \quad \forall s \in \mathcal{S} \setminus \mathcal{S}^0. \end{aligned} \tag{2.5}$$

For $\mu = 1$ we obtain from Theorem 2.1

Corollary 2.1. *The problem* (2.2) *for* $\forall f \in \mathcal{H}$ *and* $\varphi \in H$ *admits a unique strong solution in* $\mathcal{H}$ *iff*

$$A(s) + A_0(s) \neq 0 \ \forall s \in \mathcal{S}, \tag{2.6}$$

$$A_0(s) \neq iq \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0, \ q = \pm 1, \pm 2, \ldots. \tag{2.7}$$

Remark 2.1. The condition (2.6) follows from (2.3) and (2.5), the condition (2.7) follows from (2.4).

3. Auxiliary results

We shall find a solution of the problem (2.2) by Fourier method:

$$u = \sum_{s \in \mathcal{S}} u_s(t) e^{is \cdot x}, \ f = \sum_{s \in \mathcal{S}} f_s(t) e^{is \cdot x}. \tag{3.1}$$

For the coefficients of the series (3.1) we have

$$\begin{cases} D_t^1 u_s(t) + \sum\limits_{k=1}^{m} A_k(s) u_s(t_k) = f_s(t) \text{ in } (0, 2\pi), \\ \mu u_s(0) - u_s(2\pi) = \varphi_s, \ s \in \mathcal{S}^0; \end{cases} \tag{3.2}$$

$$\begin{cases} (D_t^1 + A_0(s)) u_s(t) + \sum\limits_{k=1}^{m} A_k(s) u_s(t_k) = f_s(t) \text{ in } (0, 2\pi), \\ \mu u_s(0) - u_s(2\pi) = \varphi_s, \ s \in \mathcal{S} \setminus \mathcal{S}^0; \end{cases} \tag{3.3}$$

According to the ([4], p.118) we have:

Lemma 3.1. *Boundary value problem* (2.2) *has a unique strong solution in the space $\mathcal{H}$ for $\forall f \in \mathcal{H}, \forall \varphi \in {}^{\cdot}H$ iff each of the problems* (3.2), (3.3) *admit a unique solution and there is a constant $C > 0$ that doesn't depend on s and such that*

$$\|u_s(t)\|_{L_2(0,2\pi)} \leq C[\|f_s(t)\|_{L_2(0,2\pi)} + |\varphi_s|] \ \forall s \in \mathcal{S}. \tag{3.4}$$

By the Lemma 3.1 Theorem 2.1 will follow immediately from the following Lemmas.

Lemma 3.2. *For all $s \in \mathcal{S}^0$ and each $f_s \in C(0, 2\pi)$ the* Problem (3.2) *has a unique $u_s \in C^1(0, 2\pi) \bigcap C[0, 2\pi]$ and the estimate* (3.4) *is true iff the condition* (2.3) *is held.*

Lemma 3.3. *For all $s \in \mathcal{S} \setminus \mathcal{S}^0$ and each $f_s \in C(0, 2\pi)$ the* Problem (3.3) *has a unique $u_s \in C^1(0, 2\pi) \bigcap C[0, 2\pi]$ and the estimate* (3.4) *is true iff the conditions* (2.4) *and* (2.5) *are held.*

Proof of the Lemma 3.2. At the conditions (2.3) only the Problem (3.2) has a unique solution

(3.5)

$$u_s(t) = \int_0^t f_s(\tau)\,d\tau + [2\pi A(s)]^{-1}\Big[1 + \sum_{k=1}^m A_k(s)(t_k - t)\Big]$$

$$\cdot \Big[\int_0^{2\pi} f_s(\tau)d\tau + \varphi_s\Big] - [A(s)]^{-1} \sum_{k=1}^m A_k(s) \int_0^{t_k} f_s(\tau)\,d\tau, \text{ if } \mu = 1,$$

(3.6)

$$u_s(t) = \int_0^t f_s(\tau)\,d\tau - [B(s)]^{-1}\Big[t(\mu - 1) + 2\pi\Big] \sum_{k=1}^m A_k(s) \int_0^{t_k} f_s(\tau)\,d\tau$$

$$+ \Big\{(\mu - 1)^{-1} - [B(s)]^{-1}A(s)\Big[t + (\mu - 1)^{-1}2\pi\Big]\Big\}$$

$$\cdot \Big[\varphi_s + \int_0^{2\pi} f_s(\tau)\,d\tau\Big], \quad \text{if } \ \mu \neq 1,$$

where $A(s),\ B(s)$ are defined in section 2.

These representations (3.5), (3.6) are obtained by the following way. Firstly we integrate the equation (3.2) and obtain

$$u_s(t) = u_s(0) + \int_0^t f_s(\tau)d\tau - t\sum_{k=1}^m A_k(s)u_s(t_k). \tag{3.7}$$

Secondly we have for $t = 2\pi$ from (3.7)

$$(\mu - 1)u_s(0) = \varphi_s + \int_0^{2\pi} f_s(\tau)d\tau - 2\pi\sum_{k=1}^m A_k(s)u_s(t_k). \tag{3.8}$$

Further we have for $t = t_k, k = 1, ..., m$, from (3.7)

(3.9)

$$\Big[1 + \sum_{k=1}^m A_k(s)t_k\Big]\Big[\sum_{k=1}^m A_k(s)u_s(t_k)\Big] = A(s)u_s(0) + \sum_{k=1}^m A_k(s) \int_0^{t_k} f_s(\tau)d\tau.$$

Specifically, from here we have for $\mu = 1$

$$\sum_{k=1}^{m} A_k(s)u_s(t_k) = (2\pi)^{-1}\left[\varphi_s + \int_0^{2\pi} f_s(\tau)d\tau\right]; \tag{3.10}$$

$$\begin{aligned} u_s(0) = {} & [2\pi A(s)]^{-1}\left[1 + \sum_{k=1}^{m} A_k(s)t_k\right] \\ & \cdot \left[\varphi_s + \int_0^{2\pi} f_s(\tau)d\tau\right] - [A(s)]^{-1}\sum_{k=1}^{m} A_k(s)\int_0^{t_k} f_s(\tau)d\tau; \end{aligned} \tag{3.11}$$

for $\mu \neq 1$ correspondingly

$$\begin{aligned} \sum_{k=1}^{m} A_k(s)u_s(t_k) = {} & [B(s)]^{-1}A(s)\left[\varphi_s + \int_0^{2\pi} f_s(\tau)d\tau\right] \\ & + [B(s)]^{-1}(\mu - 1)\sum_{k=1}^{m} A_k(s)\int_0^{t_k} f_s(\tau)d\tau; \end{aligned} \tag{3.12}$$

$$\begin{aligned} u_s(0) = {} & (\mu - 1)^{-1}\{1 - 2\pi[B(s)]^{-1}A(s)\}\left[\varphi_s + \int_0^{2\pi} f_s(\tau)d\tau\right] \\ & - 2\pi[B(s)]^{-1}\sum_{k=1}^{m} A_k(s)\int_0^{t_k} f_s(\tau)d\tau. \end{aligned} \tag{3.13}$$

Thus from equalities (3.10), (3.11), (3.12), (3.13) and (3.7) we obtain the representations (3.5), (3.6) of unique solution of problem (3.2).

According to (2.1) there is the constant C such that

$$|B(s)|^{-1}|A_k(s)| \leq C \text{ for all } k = 1, ..., m, \text{ and } s \in \mathcal{S}^0.$$

Here the constant C is independent on k, s. For $\mu = 1$ we note this condition is identical to the following

$$|A(s)|^{-1}|A_k(s)| \leq C \text{ for all } k = 1, ..., m, \text{ and } s \in \mathcal{S}^0.$$

The desired estimate (3.4) follows from here and (3.5). This finishes the proof of the Lemma 3.2. □

Proof of the Lemma 3.3. At the conditions (2.4), (2.5) only the Problem (3.3) has a unique solution

$$\begin{aligned} u_s(t) = &\int_0^{2\pi} G_s(t,\tau) f_s(\tau)d\tau + [C(s)]^{-1}\varphi_s \exp\{-A_0(s)t\} \\ &- [D(s)]^{-1}\int_0^{2\pi} G_s(t,\tau)d\tau \sum_{k=1}^{m} A_k(s) \int_0^{2\pi} G_s(t_k,\tau) f_s(\tau)d\tau, \end{aligned} \tag{3.14}$$

where $G_s(t,\tau)$ is the Green function

$$G_s(t,\tau) = \begin{cases} \mu[C(s)]^{-1}\exp\{-A_0(s)(t-\tau)\} & \text{if } 0 \le \tau \le t \le 2\pi, \\ [C(s)]^{-1}\exp\{-A_0(s)(2\pi + t - \tau)\} & \text{if } 0 \le t \le \tau \le 2\pi, \end{cases} \tag{3.15}$$

and $C(s), D(s)$ are defined in (2.4), (2.5).

We note that the following equality

$$\int_0^{2\pi} G_s(t,\tau)d\tau = [A_0(s)]^{-1}\{1 - [C(s)]^{-1}(\mu - 1)\exp\{-A_0(s)t\}\} \tag{3.16}$$

is hold.

Considering (3.15) the representation (3.14) is obtained similarly to Lemma 3.2.

It only remains to establish the estimates (3.4) for the solutions (3.14).

The first item of the solution (3.14) estimate similar to ([4], p.120–121)

$$\left\| \int_0^{2\pi} G_s(t,\tau) f_s(\tau)d\tau \right\|_{L_2(0,2\pi)} \le K_1 \|f_s(\tau)\|_{L_2(0,2\pi)}, \tag{3.17}$$

where the constant K_1 doesn't depend on s.

By the equality (3.16) the estimate of the third item of the solution (3.14) follows from the inequalities

- because of (3.17) we obtain

$$\left|\int_0^{2\pi} G_s(t_k,\tau) f_s(\tau) d\tau\right| \le K_2 \|f_s(\tau)\|_{L_2(0,2\pi)} \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0, \ k = 1, ..., m;$$

- because of $D(s) \neq 0$ (2.5) there is the constant $\delta > 0$ such that

$$|D(s)|^{-1} \le \delta^{-1} < +\infty \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0;$$

- by definition $C(s)$ (2.4) we have

$$\left|1 - [C(s)]^{-1}(\mu - 1)\exp\{-A_0(s)t\}\right| \le K_3 \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0, \ t \in (0, 2\pi];$$

- because of (2.1) and (2.3) we have

$$|A_0(s)|^{-1} |A_k(s)| \le K_4 \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0 \ k = 1, ..., m;$$

where the constants δ, K_2, K_3 and K_4 don't depend on s.

By (2.3) the second item of the solution (3.14) yields a estimate

$$\left|[C(s)]^{-1} \exp\{-A_0(s)t\}\varphi_s\right| \le K_5 |\varphi_s| \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0, \ t \in (0, 2\pi].$$

Thus the desired estimates (3.4) follows by the established inequalities.

This finishes the proof of the Lemma 3.3. □

4. Proof of the Theorem 2.1

To finish the proof of the Theorem 2.1 it is necessary to prove the closure of our problem operator (2.2) in the space $\mathcal{H}$, that is to check the correctness of following proposition ([10], p.209).

PROPOSITION 4.1. *The operator $T + E$ is closed if T is closed and E is bounded in the $\mathcal{D}(E)$, $\mathcal{D}(T) \subset \mathcal{D}(E)$.*

Here $\mathcal{D}(X)$ is the domain of a operator X.

Let $\bar{l} \le l_0/2$. Then we define T as the operator L_1 of problem (2.2) without item $\sum_{k=1}^{m} A_k(-iD)u(t_k)$, and $E = \sum_{k=1}^{m} A_k(-iD)u(t_k)$. By the trace theorem ([11], p.33) we obtain

$$\mathcal{D}(T)=\left\{v|v\in L_2\left(0,2\pi;\mathcal{D}(A_0)\right),\ D_t^1 v\in\mathcal{H}\right\}\subset C\left([0,2\pi];[\mathcal{D}(A_0),H]_{1/2}\right)\subseteq\mathcal{D}(E),$$

where $[X,Y]_\theta$ is the interpolating space ([11], p.23), X and Y are the Hilbert spaces, the embedding $X \subset Y$ is dense and continuous, $\theta \in (0,1)$.

Now let $l_0/2 < \bar{l} \le l_0$, $\bar{l} = \text{degree}\{A_{\bar{k}}(s)\}$, where $\bar{k} = \arg\big\{\max\{l_k, k = 1, \dots, m\}\big\}$. Then we define T and E as the operators $Tu = D_t^1 u + A_{\bar{k}}(-iD)u(t_{\bar{k}})$, $Eu = A_0(-iD)u + \sum\limits_{k=1,k\neq\bar{k}}^{m} A_k(-iD)u(t_k)$.

By the trace theorem ([11], p.35–36, 55–56) we obtain

$$\begin{aligned}\mathcal{D}(T) &= \{v|v \in L_2(0,2\pi;\mathcal{D}(A_{\bar{k}}A_{\bar{k}})),\ D_t^1 v \in \mathcal{H}\}\\ &\subset \{C([0,2\pi];\mathcal{D}(A_{\bar{k}})) \cap L_2(0,2\pi;\mathcal{D}(A_0))\} \subseteq \mathcal{D}(E).\end{aligned}$$

It only remains to apply Proposition 4.1. Proof of Theorem 2.1 is completed.

5. Examples

Let $Q = \{x,t|0 < x,t < 2\pi\}$.

(i). For the boundary value problem:

$$(D_t^1 - D_x^2)u + \sum_{k=1}^{m}(\alpha_k + \beta_k D_x^1)u(x,t_k) = f,\ \{x,t\} \in Q, \tag{5.1}$$

$$D_x^j u(0,t) = D_x^j u(2\pi,t),\ j = 0,1;\ \mu u(x,0) = u(x,2\pi), \tag{5.2}$$

the following corollaries take place:

COROLLARY 5.1. *The problem (5.1)–(5.2) for* $\forall f \in L_2(Q)$ *admits a unique strong solution* $u \in L_2(Q)$ *iff*

$$\sum_{k=1}^{m} \alpha_k[2\pi + (\mu-1)t_k] + \mu - 1 \neq 0 \tag{5.3}$$

$$\mu - \exp\{-s^2 2\pi\} \neq 0 \ \forall s \in \mathcal{S} \setminus \{0\}, \tag{5.4}$$

$$\begin{aligned} &1 + s^{-2}\sum_{k=1}^{m}(\alpha_k + is\beta_k)\{1 - [\mu - \exp\{-s^2 2\pi\}]^{-1} \\ &\cdot (\mu-1)\exp\{-s^2 t_k\}\} \neq 0 \ \forall s \in \mathcal{S} \setminus \{0\}. \end{aligned} \tag{5.5}$$

COROLLARY 5.2. *Let* $\mu = 1$. *Then the problem (5.1)–(5.2) for* $\forall f \in L_2(Q)$ *admits a unique strong solution* $u \in L_2(Q)$ *iff*

$$\sum_{k=1}^{m}(\alpha_k + is \cdot \beta_k) + s^2 \neq 0 \ \forall s \in \mathcal{S}. \tag{5.6}$$

COROLLARY 5.3. *Let* $\mu = 1$ *and* $\beta_k = 0, k = 1, ..., m$. *Then the problem (5.1)–(5.2) for* $\forall f \in L_2(Q)$ *admits a unique strong solution* $u \in L_2(Q)$ *iff*

$$\left(-\sum_{k=1}^{m}\alpha_k\right)^{1/2} \notin \mathcal{S}. \tag{5.7}$$

REMARK 5.1. For $\mu = 1$ the correctness of the boundary value problem (5.1)–(5.2) doesn't depend on the points t_k. This follows from corollaries 5.2, 5.3.

(ii). Let $a =$ const. For the boundary value problem:

$$(D_t^1 - D_x^3 + a)u + \sum_{k=1}^{m}\alpha_k u(x, t_k) = f, \ \{x,t\} \in Q, \tag{5.8}$$

$$D_x^j u(0,t) = D_x^j u(2\pi, t), \ j = 0,1,2; \ \mu u(x,0) = u(x, 2\pi), \tag{5.9}$$

the following corollaries take place:

COROLLARY 5.4. *The problem* (5.8)–(5.9) *for* $\forall f \in L_2(Q)$ *admits a unique strong solution* $u \in L_2(Q)$ *iff*

$$2\pi \sum_{k=1}^{m} \alpha_k + (\mu - 1)[1 + \sum_{k=1}^{m} \alpha_k t_k] \neq 0 \tag{5.10}$$

$$\mu \neq \exp\{-2\pi a\}, \tag{5.11}$$

$$\begin{aligned} & is^3 + a + \sum_{k=1}^{m} \alpha_k \{1 - [\mu - \exp\{-2\pi a\}]^{-1} \\ & \cdot (\mu - 1)\exp\{-(is^3 + a)t_k\}\} \neq 0 \ \forall s \in \mathcal{S} \setminus \mathcal{S}^0. \end{aligned} \tag{5.12}$$

COROLLARY 5.5. *Let* $a \neq 0,\ \mu = 1$. *Then the problem* (5.8)–(5.9) *for* $\forall f \in L_2(Q)$ *admits a unique strong solution* $u \in L_2(Q)$ *iff*

$$\sinh(\pi\gamma) \neq 0,\ \frac{\sqrt{3}\gamma}{2} \notin \mathcal{S}, \quad \text{where } \gamma = \left(a + \sum_{k=1}^{m} \alpha_k\right)^{1/3}. \tag{5.13}$$

These corollaries follow from Theorem 2.1 and Corollary 2.1. The conditions (5.3)–(5.12) of the corollaries 5.1–5.4 follow from (2.3)–(2.5) and (2.6)–(2.7) correspondingly.

References

[1] Nakhushev, A. M. *Loaded Equations and their Applications*, Differentsial'nye uravnenija (Russian), **19** (1983), no. 1, 86–94.

[2] ———, *The Equation of mathematical Biology* (Russian), Moscow, 1995, p.301.

[3] Lomov, I. S., *The theorem of absolute basis property of the root vectors for the second-order loaded differential operators*, Differentsial'nye uravnenija (Russian), **27** (1991), no. 9, 1550–1563.

[4] Dezin, A. A., *The general questions of boundary value problems theory*(Russian), Moscow, 1980, p.207.

[5] Dzhenaliev, M. T., *On the quadratic functional of the Cauchy problem for the first-order loaded differential operator equation I, II*, Differentsial'nye uravnenija (Russian), **31** (1995), no. 12, 2029–2037, **32** (1996), no. 4, 518–522.

[6] ———, *On the K.Friedrichs space and the variational principle for the first-order loaded differential equation*, Proc.International conference "The functional space, approximation theory, non-linear analysis", April 27–May 3 , 1995, Moscow, MI RAS, 1995, pp. 112–113.

[7] ———, *On the linear boundary value problems theory for the loaded differential equations* (Russian), Almaty, 1995, p.270.

[8] ———, *Boundary Value Problems for loaded Parabolic and Hyperbolic Equations with Time Derivatives on Boundary Conditions*, Differentsial'nye uravnenija (Russian), **28** (1992), no. 4, 661–666.

[9] ———, *On unique Solvability of Boundary Value Problems for loaded Equations with Time Derivatives on Boundary Conditions*, Reports Ministry of Science–Academy of Sciences, Republic of Kazakhstan, (1996),no. 4, 3–9.

[10] Kato, T. *Perturbation theory for linear operators* (Russian), Moscow, 1972, p.740.

[11] Lions, J. L. and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, (Russian), Moscow, **1** (1971), p.371.

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